Chapter 6

Mathematical Induction

6.1 The Process of Mathematical Induction

6.1.1 Motivating Mathematical Induction

Consider the sum of the first several odd integers. Experimenting a bit, we produce the following:

$$\begin{aligned}
1 &= 1 \\
1+3 &= 4 \\
1+3+5 &= 9 \\
1+3+5+7 &= 16 \\
1+3+5+7+9 &= 25
\end{aligned}$$

Clearly, a pattern is beginning to emerge. In words, it seems that the sum of the first n odd integers is simply the square n^2. In summation notation, we have a strong suspicion that

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

In investigating the above pattern, we begin to notice something interesting. If we consider the first computation, we notice that the sum of the first odd number (1, namely) is equal to 1^2. If we then turn our attention to the fourth computation, for example, we see that $1+3+5+7=16$. Thus, if $n=4$, then this simply says that

$$\sum_{j=1}^{4}(2j-1) = 4^2.$$

If we were to then compute the next (fifth) computation, then we would not add the first four odd numbers again. Instead, we would simply add the fifth odd number (9) to the previous computation and see that $16+9=25$.

Keeping with the above pattern, we can continue this process in a more arbitrary setting. If we already know that, for some k, our suspected sum

$$\sum_{j=1}^{k}(2j-1) = k^2$$

is correct, then to compute the sum for the $(k+1)$-st computation, we would simply add the $(k+1)$-st odd number to the previous computation. Since the

$(k+1)$-st odd number is given by $2(k+1)-1 = 2k+1$, we can add this to both sides of our equation to get that

$$\sum_{j=1}^{k}(2j-1) + (2(k+1)-1) = k^2 + 2k + 1.$$

The entire left-hand side of the above equation is equal to our summation with an ending value of $k+1$ instead of k. The right-hand side, on the other hand, can be simplified to $(k+1)^2$. Thus, our equation becomes

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2,$$

which is the same equation as our suspected equation with $k+1$ in the place of n.

Reviewing our observations, we first noted quite trivially that our desired equation

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

held true for $n = 1$. Then, using some basic algebra, we showed that if we ever knew that our desired equation was true for some particular value of k, then we could easily show that it was true for $k+1$. Thus, if our equation is true for $n = 1$, then, by the above fact, it should also be true for $n = 2$. Since it's now true for $n = 2$, then by the above fact, it is true for $n = 3$. Continuing in this manner, we see that, like dominos, eventually our desired statement will indeed be true for every $n \geq 1$.

This process, called *mathematical induction*, is one of the most important proof techniques and boils down a proof to showing that if a statement is true for k, then it is also true for $k+1$. We devote this chapter to the study of mathematical induction.

6.1.2 Formalizing Mathematical Induction

Mathematical induction is a proof technique most appropriate for proving that a statement $A(n)$ is true for all integers $n \geq n_0$ (where, usually, $n_0 = 0$ or 1). As in the above example, there are two major components of induction: showing that our statement $A(n)$ is true for the beginning value n_0 and showing that if $A(n)$ is true for some k, then it is also true for some $k+1$. Thus, the three steps to mathematical induction.

(1) *Identify the statement $A(n)$ and its starting value n_0.* In our example, we would say $A(n)$ is the statement

$$\sum_{j=1}^{n}(2j-1) = n^2,$$

and we wish to show it is true for all $n \geq 1$ (and thus $n_0 = 1$).

(2) *Show that the base case $A(n_0)$ is true.* In our example, we would have to, by hand, show that $A(1)$ is true. This is straightforward since $A(1)$ is the statement

$$\sum_{j=1}^{1}(2j-1) = 1^2,$$

which is clearly true.

(3) *Show that if $A(k)$ is true for some $k \geq n_0$, then $A(k+1)$ is also true.* This case is usually called the *inductive step* and is the most difficult. In this case, we assume that $A(k)$ is true *for some* $k \geq n_0$. Our job is to show that $A(k+1)$ is true using the above inductive assumption. In our example, we would thus assume that, for some $k \geq n_0$,

$$\sum_{j=1}^{k}(2j-1) = k^2.$$

Using this assumption, we must show that $A(k+1)$ is true; thus, we wish to show that

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2.$$

As our above computation shows us, this step is doable with some basic algebra.

First, we explain why, if manage to successfully complete the three steps of mathematical induction, we can claim that $A(n)$ is true for all $n \geq n_0$. In step (2), we manually verified that the base case $A(n_0)$ is true. Having proven the inductive step in (3), we know that since $A(n_0)$ is true, then it follows that $A(n_0+1)$ is true. Now that we know that $A(n_0+1)$ is true, (3) tells us that $A(n_0+2)$ is true. Continuing, we will eventually show that $A(n)$ is true for every whole number $n \geq n_0$.

Before we formally write up the proof of our example, we will go a bit more in depth in the inductive step (step 3). When first reading this, many students become confused since it sounds like we are assuming what we want to show. That is, why would we assume that $A(k)$ is true? Isn't that what we want to show? What we are doing in the inductive step is actually slightly different. We are showing the following conditional statement to be true: "If $A(k)$ is true, then $A(k+1)$ is true." Just like any other conditional statement, the way that we prove this is to assume the hypothesis and use it to show that the conclusion is true. Thus, we are not assuming that $A(k)$ is true for every k; rather, we are saying that if we ever know that $A(k)$ is true, then we can conclude that $A(k+1)$ is true.

So, we proceed to our first induction proof. Since we have essentially outlined the entire proof above, we will forego a discussion.

Proposition. For any whole number $n \geq 1$,

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

Proof. Consider the statement $A(n)$ given by

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 1$.

First, we confirm that the base case $A(1)$ is true. Computing, we have that

$$\sum_{j=1}^{1}(2j-1) = 1 = 1^2.$$

Thus, $A(1)$ is indeed true.

Next, we perform the inductive step. Thus, we assume that $A(k)$ is true for some $k \geq 1$. So, we assume that

$$\sum_{j=1}^{k}(2j-1) = k^2.$$

We will use this to prove that $A(k+1)$ is true by showing that

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2.$$

Beginning with the left-hand side of the $A(k+1)$ statement and using our inductive assumption, we have that

$$\sum_{j=1}^{k+1}(2j-1) = \left(\sum_{j=1}^{k}(2j-1)\right) + (2(k+1)-1) = k^2 + (2(k+1)-1) =$$

$$k^2 + 2k + 1 = (k+1)^2$$

Thus, using our inductive assumption, we have shown that $A(k+1)$ is true.

By induction, we know that the statement $A(n)$ given by $\sum_{j=1}^{n}(2j-1) = n^2$ is indeed true for all $n \geq 1$.

$\square$

A common mistake in induction proofs occurs during the inductive step. Frequently, a student wishing to show that $A(k+1)$ is true will simply begin with the statement $A(k+1)$ and then proceed logically until a true statement is reached. For example, in our proof, a common mistake would be to start off with

$$\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$$

and then use algebra to arrive at the $A(k)$ statement

$$\sum_{j=1}^{k}(2j-1) = k^2.$$

Then, the student would conclude that since $A(k)$ is true by the inductive assumption, then $A(k+1)$ is also true. This, however, is not logically valid since this student is assuming what he or she wants to prove.

In our proof above, we did not do this; rather, we began with the left-hand side of the equation given in $A(k+1)$ and then used the inductive assumption and algebra to arrive at the right-hand side of $A(k+1)$. When the statement $A(n)$ is an equation, this is usually the best way to proceed because you avoid assuming the entire equation is true from the beginning.

6.1.3 When is Mathematical Induction Useful?

Clearly, mathematical induction can be applied to many statements that we wish to show true; however not every statement can be proven using this new technique. First, since induction requires us to first find our statement $A(n)$ for all integers $n \geq n_0$, it's clear that induction only works for statements indexed by whole numbers. So, statements like "For all $x \in \mathbb{R}$, $x^2 \geq 0$" would not lend itself to induction since the statement is not given in terms of whole numbers.

Next, since we must successfully perform the inductive step in our induction proof, statements where the $A(k+1)$ case can be tackled knowing that the $A(k)$ case is true are particularly well-suited for induction. For example, statements involving sums (like in our above example) are appropriate since it is easy to re-write the left-hand side of our statement $A(k+1)$ in terms of the left-hand side of the statement $A(k)$.

In what follows, we will investigate topics and scenarios where induction serves as an excellent proof technique.

6.2 Examples of Proofs by Induction

In the below sections, we will give a sampling of the swathe of Mathematics in which induction is frequently and successfully used. As you go through the examples, be sure to note what characteristics of the statements make them amenable to the induction proof process.

6.2.1 Induction in Number Theory

We previously studied divisibility in our Sets of Real Numbers chapter. In this section, we will investigate how induction can be used to show divisibility of rather complicated expressions. Before we begin, recall that m divides n (written $m \mid n$) if there exists an integer $k \in \mathbb{Z}$ such that $n = mk$. Below we will prove that $3 \mid (2^{2n} - 1)$ for all $n \geq 0$ using induction. Notice that this statement is well-suited for induction since it is written in terms of all whole numbers $n \geq 0$. Furthermore, we have a hope of writing the $k+1$ case in terms of the k case by using some simple algebra.

Proposition. For all integers $n \geq 0$,

$$3 \mid (2^{2n} - 1).$$

Discussion. We will prove our statement by completing the three steps of mathematical induction.

`Identify` $A(n)$: $A(n)$ is the statement $3 \mid (2^{2n} - 1)$, and we wish to prove it for $n \geq 0$.

`Base Case:` We must, by hand, prove that $A(0)$ is true, which is easy enough since $A(0)$ is the statement $3 \mid (2^0 - 1)$.

`Inductive Step:` We will assume that, for some $k \geq 0$, $3 \mid (2^{2k} - 1)$. Our job is to show that $3 \mid (2^{2(k+1)} - 1)$. Since we know that $3 \mid (2^{2k} - 1)$, we know that we can write $2^{2k} - 1 = 3x$ for some $x \in \mathbb{Z}$ (notice how we use x instead of k because k is already used in the induction process). We will use this to write $2^{2(k+1)} - 1$ as an integer multiple of 3.

Proof. We will use mathematical induction to prove the statement $A(n)$ given by $3 \mid (2^{2n} - 1)$ is true for all $n \geq 0$.

First, we verify that the base case $A(0)$ is true. $A(0)$ states that $3 \mid (2^{2\cdot 0} - 1)$, which is equivalent to $3 \mid 0$. Since $0 = 3 \cdot 0$, then indeed $3 \mid (2^{2\cdot 0} - 1)$. Thus, the base case $A(0)$ has been verified.

Next, we perform the inductive step. Thus, we assume that $A(k)$ is true for some $k \geq 0$. In other words, we assume that $3 \mid (2^{2k} - 1)$. We will use this to show that $A(k+1)$ is true by showing that $3 \mid (2^{2(k+1)} - 1)$. Since $3 \mid (2^{2k} - 1)$, then we know that

$$2^{2k} - 1 = 3x$$

for some $x \in \mathbb{Z}$, and thus $2^{2k} = 3x + 1$. Notice that

$$2^{2(k+1)} - 1 = 2^{2k} \cdot 2^2 - 1 = 4 \cdot 2^{2k} - 1 =$$

$$4(3x + 1) - 1 = 12x + 3 = 3(4x + 1).$$

Since $x \in \mathbb{Z}$, then $4x + 1 \in \mathbb{Z}$ as well. Thus, since $2^{2(k+1)} - 1 = 3(4x + 1)$, then $3 \mid (2^{2(k+1)} - 1)$, as desired. Thus, $A(k + 1)$ is true, and we have completed the inductive step.

By induction, we know that the statement $A(n)$ given by $3 \mid (2^{2n} - 1)$ is true for all integers $n \geq 0$.

$\square$

6.2.2 Induction in Calculus

Many times in calculus, we are asked to find patterns of higher-order derivatives for some particular function. For example, if we are asked to find a general formula for the n-th derivative of $f(x) = \dfrac{1}{x}$ and prove it, we begin by listing off the first few derivatives:

$$\begin{array}{rcl} f(x) & = & x^{-1} \\ f'(x) & = & -1x^{-2} \\ f''(x) & = & 2x^{-3} \\ f'''(x) & = & -6x^{-4} \\ f^{(4)}(x) & = & 24x^{-5} \end{array}$$

Since the derivatives have alternating signs, our general pattern should include a $(-1)^n$ term. Furthermore, the coefficient in front of the term for the n-th derivative seems to be the factorial term $n!$. Lastly, the exponent of x in the n-th derivative seems to be $-(n + 1)$. Thus, we conjecture that if $f(x) = \dfrac{1}{x}$, then

$$f^{(n)}(x) = (-1)^n \, n! \, x^{-(n+1)}.$$

Before we begin the induction proof verifying that this pattern is indeed correct, we will point out why induction is the best proof methodology. First, this statement is indexed by all non-negative integers, so the statement $A(n)$ will be straightforward to write. Furthermore, it will be easy to re-write a statement about its $(k + 1)$-st derivative in terms of the k-th derivative, which is needed when proving that $A(k + 1)$ is true knowing only that $A(k)$ is true. So, we proceed with our induction proof.

Proposition. Let $f(x) = \dfrac{1}{x}$. For all integers $n \geq 0$, its n-th derivative is given by

$$f^{(n)}(x) = (-1)^n \, n! \, x^{-(n+1)}.$$

Discussion. We will prove our statement by completing the three steps of mathematical induction.

`Identify` $A(n)$: Let $A(n)$ be the statement

$$f^{(n)}(x) = (-1)^n \, n! x^{-(n+1)}.$$

We wish to show that $A(n)$ is true for all integers $n \geq 0$.

`Base Case:` We must verify that $A(0)$ is true, which says that

$$f^{(0)}(x) = (-1)^0 \, 0! \, x^{-(0+1)}.$$

Since the 0-th derivative of a function is just the function itself, this is true because the above simplifies to $f(x) = \dfrac{1}{x}$.

`Inductive Step:` We will assume that, for some $k \geq 0$,

$$f^{(k)}(x) = (-1)^k \, k! \, x^{-(k+1)}.$$

Our job will be to show that $A(k+1)$ is also true by showing that

$$f^{(k+1)}(x) = (-1)^{k+1}(k+1)! \, x^{-((k+1)+1)}.$$

We will do this by taking writing the $(k+1)$-st derivative as the derivative of the k-th derivative and using basic calculus and factorial rules.

Proof. We will use mathematical induction to prove that the statement $A(n)$ given by $f^{(n)}(x) = (-1)^n \, n! \, x^{-(n+1)}$ is true for all $n \geq 0$.

First, we verify that the base case $A(0)$ is true. $A(0)$ states that $f^{(0)}(x) = (-1)^0 \, 0! \, x^{-(0+1)}$, which is equivalent to $f(x) = x^{-1}$, which is true. Thus, the base case $A(0)$ has been verified.

Next, we perform the inductive step. Thus, we assume that $A(k)$ is true for some $k \geq 0$. Thus, we assume that $f^{(k)}(x) = (-1)^k \, k! \, x^{-(k+1)}$. We will use this to show that $A(k+1)$ is true by showing that

$$f^{(k+1)}(x) = (-1)^{k+1} \, (k+1)! \, x^{-((k+1)+1)}.$$

We will begin with the left-hand side of our desired equation and use the fact that $(k+1)! = (k+1)k!$ to obtain

$$\begin{aligned}
f^{(k+1)}(x) = \tfrac{d}{dx} f^{(k)}(x) &= \tfrac{d}{dx}(-1)^k \, k! \, x^{-(k+1)} \\
&= (-1)^k \, k! \, \tfrac{d}{dx} x^{-(k+1)} \\
&= (-1)^k \, k! \cdot -(k+1) x^{-(k+1)-1} \\
&= (-1)(-1)^k \, (k+1)k! \, x^{-((k+1)+1)} \\
&= (-1)^{k+1} \, (k+1)! \, x^{-((k+1)+1)}.
\end{aligned}$$

Thus, we have shown our statement $A(k+1)$ to be true and thus our inductive step is complete.

By induction, we know that the statement $A(n)$ given by $f^{(n)}(x) = (-1)^n \, n! \, x^{-(n+1)}$ is true for all $n \geq 0$.

□

6.2.3 Induction in Set Theory

If we are given a finite set S, we can usually find several examples of different subsets of S. However, one might ask if we could determine the total number of subsets of a set S if we know the size of S itself. So, we begin experimenting with a few small cardinality sets:

- When $|S| = 0$, then $S = \varnothing$ and $\varnothing$ is its only subset. Thus, when $|S| = 0$, S has 1 subset.
- When $|S| = 1$, then, for example, $S = \{a\}$. This S has the subsets $\varnothing$ and $\{a\}$. Thus, when $|S| = 1$, S has 2 subsets.
- When $|S| = 2$, then, for example, $S = \{a, b\}$. This S has the following subsets: $\varnothing, \{a\}, \{b\}, \{a, b\}$. Thus, when $|S| = 2$, S has 4 subsets.

- When $|S| = 3$, then, for example, $S = \{a, b, c\}$. This S has the following subsets:
$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$
Thus, when $|S| = 3$, S has 8 subsets.

A pattern begins to emerge. It seems that if S is a set with n elements, then it contains 2^n subsets.

Clearly, the statement that a set of size n has 2^n subset is written in terms of all non-negative whole numbers, which makes it a possible candidate for induction. More important, though, is that it seems that if we know that a set has $k + 1$ elements, then we can look at a set of k elements and find a relationship between the subsets of the k-element set and the subsets of the $(k + 1)$-element set. Notice that all the subsets of the 3-element set $\{a, b, c\}$ come from the subsets of the 2-element set $\{a, b\}$ by either including or not including the element c. Of course, since there are two choices (to include c or not include c) it seems reasonable that the number of subsets doubles when we add one element to the set. So, we proceed with our induction proof.

Proposition. A set with n elements has 2^n subsets.

Discussion. We will prove our statement by completing the three steps of mathematical induction.

Proof.

`Identify` $A(n)$: $A(n)$ is the statement "A set with n elements has 2^n subsets," and we wish to prove it for $n \geq 0$.

`Base Case:` We must verify that $A(0)$ is true, which says that "A set with 0 elements has 2^0 subsets." We already verified this is true since a set with 0 elements is $\varnothing$, which only has itself as a subset.

`Inductive Step:` We will assume that, for some $k \geq 0$, $A(k)$ is true. Thus, we will assume that "A set with k elements has 2^k subsets." We will show that $A(k + 1)$ is true by showing that "a set with $k + 1$ elements has 2^{k+1} subsets." We will do this by letting S be a set with $k + 1$ elements. Then, we can choose any point $a \in S$ and notice that any subset of S either contains a or does not contain a (but not both). In each case, we will use the inductive assumption to show that there are 2^k subsets and thus $2 \cdot 2^k = 2^{k+1}$ subsets overall.

Proof. We will use mathematical induction to prove that the statement $A(n)$ given by "A set with n elements has 2^n subsets" is true for all integers $n \geq 0$.

First, we verify the base case $A(0)$, which states that a set with 0 elements has 2^0 subsets. If a set has 0 elements, then it is the empty set $\varnothing$. The empty set only has a single subset, which is itself $\varnothing$. Thus, if a set has 0 elements, then it has $2^0 = 1$ subsets.

Next, we perform the inductive step. Thus, we assume that the $A(k)$ is true for some $k \geq 0$. In other words, we will assume that any set with k elements has 2^k subsets. We will show that $A(k+1)$ is true by showing that any set with $k + 1$ elements has 2^{k+1} subsets. To do so, let S be a set with $k + 1$ elements. Let $a \in S$ be any element of S. Then, we can write S as the disjoint union $S = \{a\} \cup T$, where T is a set with k elements. Notice that any subset $A \subset S$ falls into exactly one of two categories: A does not contain a or A does contain a. If A does not contain a, then A is a subset of T. Since T has k elements, the inductive assumption tells us that T has 2^k subsets, and thus there are 2^k such subsets A that fall into this category. On the other hand, if A does contain a, then we can uniquely write $A = \{a\} \cup A'$, where A' is a subset of T. Since A'

is a subset of T, there are 2^k such subsets by the inductive assumption. Thus, there are 2^k subset A of S that do contain a. Since all subsets of S fall into exactly one of the above two categories, the total number of subsets is just the sum

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Thus, the set S with $k+1$ elements has 2^{k+1} subsets and $A(k+1)$ is verified to be true.

By induction, we know that the statement "A set with n elements has 2^n subsets" is true for all integers $n \geq 0$.