THE CATENARY DEGREE OF ELEMENTS IN NUMERICAL MONOIDS

M. Corrales, A. Miller, C. Miller, D. Patel

August 8, 2013
Definitions/Motivation
1 Definitions/Motivation
2 The catenary degree of elements in:
   - 2-generated numerical monoids
1 Definitions/Motivation

2 The catenary degree of elements in:
   - 2-generated numerical monoids
   - a special case of 3-generated numerical monoids
1 Definitions/Motivation

2 The catenary degree of elements in:
   - 2-generated numerical monoids
   - a special case of 3-generated numerical monoids
   - numerical monoids generated by an arithmetic sequence
Outline

1 Definitions/Motivation
2 The catenary degree of elements in:
   - 2-generated numerical monoids
   - a special case of 3-generated numerical monoids
   - numerical monoids generated by an arithmetic sequence
3 Periodicity of the catenary degree, tame degree, and monotone catenary degree
 Definitions/Motivation

2 The catenary degree of elements in:
   - 2-generated numerical monoids
   - a special case of 3-generated numerical monoids
   - numerical monoids generated by an arithmetic sequence

3 Periodicity of the catenary degree, tame degree, and monotone catenary degree

4 Possibilities for further research
**Definition**

A **commutative monoid** is a set equipped with a commutative, associative binary operation and an identity element.

**Example**

$\mathbb{N}_0$ (the natural numbers adjoin 0) is a commutative monoid.
**Definition**

A **commutative monoid** is a set equipped with a commutative, associative binary operation and an identity element.

**Example**

$\mathbb{N}_0$ (the natural numbers adjoin 0) is a commutative monoid.

**Definition**

A subset $S \subseteq \mathbb{N}_0$ is a **numerical monoid** if $\mathbb{N}_0 \setminus S$ is finite.
**Some Important Preliminary Results**

**Theorem**

Every numerical monoid has a unique minimal set of generators, \( \langle n_1, \ldots, n_k \rangle \).

**Theorem**

\( S \subseteq \mathbb{N}_0 \) is a numerical monoid if and only if \( \gcd(n_1, \ldots, n_k) = 1 \).
Some Important Preliminary Results

**Theorem**

Every numerical monoid has a unique minimal set of generators, \( \langle n_1, \ldots, n_k \rangle \).

**Theorem**

\( S \subseteq \mathbb{N}_0 \) is a numerical monoid if and only if \( \gcd(n_1, \ldots, n_k) = 1 \).

We will be looking at monoids of the form \( \langle n_1, \ldots, n_k \rangle \), with \( \gcd(n_1, \ldots, n_k) = 1 \), throughout our presentation.
For any $n \in \mathbb{Z}$, The Fundamental Theorem of Arithmetic tells us that $n$ has a unique factorization in terms of the irreducible elements of $\mathbb{Z}$ (i.e. the prime numbers).
For any $n \in \mathbb{Z}$, The Fundamental Theorem of Arithmetic tells us that $n$ has a unique factorization in terms of the irreducible elements of $\mathbb{Z}$ (i.e. the prime numbers). This is NOT true for numerical monoids.

**Example**

Let $S = \langle 4, 5 \rangle$, and let $s = 20 \in S$. 20 can be factored as 4 copies of 5 OR as 5 copies of 4.
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are:

- $(0,0,7)$
- $(3,2,5)$
- $(2,5,4)$
- $(7,1,4)$
- $(1,8,3)$
- $(6,4,3)$
- $(11,0,3)$
- $(0,11,2)$
- $(5,7,2)$
- $(10,3,2)$
- $(4,10,1)$
- $(9,6,1)$
- $(14,2,1)$
- $(3,13,0)$
- $(8,9,0)$
- $(13,5,0)$
- $(18,1,0)$

As you can see, factorizations can accumulate very quickly.
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are

(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2), (4,10,1), (9,6,1), (14,2,1), (3,13,0), (8,9,0), (13,5,0), and (18,1,0).

As you can see, factorizations can accumulate very quickly.
A More Complicated Example of Non-UniQue Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7)$
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are (0,0,7), (3,2,5)
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are 
$(0,0,7), (3,2,5), (2,5,4)$
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are

$(0,0,7), (3,2,5), (2,5,4), (7,1,4)$
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3)$.
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3)$.
A More Complicated Example of Non-UniQue Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3)$.
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are
\[(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2)\]
A More Complicated Example of Non-UniQue Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are (0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2)
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2)$. As you can see, factorizations can accumulate very quickly.
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are
(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2),
(5,7,2), (10,3,2), (4,10,1).
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are

$(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2), (4,10,1), (9,6,1)$
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are:

- (0,0,7)
- (3,2,5)
- (2,5,4)
- (7,1,4)
- (1,8,3)
- (6,4,3)
- (11,0,3)
- (0,11,2)
- (5,7,2)
- (10,3,2)
- (4,10,1)
- (9,6,1)
- (14,2,1)

As you can see, factorizations can accumulate very quickly.
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are

$$(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2), (4,10,1), (9,6,1), (14,2,1), (3,13,0)$$

As you can see, factorizations can accumulate very quickly.
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are
(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2), (4,10,1), (9,6,1), (14,2,1), (3,13,0), (8,9,0)
Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are
$(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), $(
$5,7,2), (10,3,2), (4,10,1), (9,6,1), (14,2,1), (3,13,0), (8,9,0), (13,5,0)$
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are $(0,0,7), (3,2,5), (2,5,4), (7,1,4), (1,8,3), (6,4,3), (11,0,3), (0,11,2), (5,7,2), (10,3,2), (4,10,1), (9,6,1), (14,2,1), (3,13,0), (8,9,0), (13,5,0), and (18,1,0).
A More Complicated Example of Non-Unique Factorization

Let $S = \langle 4, 5, 11 \rangle$, and let $s = 77 \in S$. The factorizations of 77 are 
$(0,0,7)$, $(3,2,5)$, $(2,5,4)$, $(7,1,4)$, $(1,8,3)$, $(6,4,3)$, $(11,0,3)$, $(0,11,2)$, 
$(5,7,2)$, $(10,3,2)$, $(4,10,1)$, $(9,6,1)$, $(14,2,1)$, $(3,13,0)$, $(8,9,0)$, $(13,5,0)$, and $(18,1,0)$. As you can see, factorizations can accumulate very quickly.
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. 
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. We denote the set of factorizations of $s$ by $Z(s)$.
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. We denote the set of factorizations of $s$ by $Z(s)$. We denote an arbitrary element $z \in Z(s)$ with a $k$-tuple (i.e. as $z = (a_1, \ldots, a_k)$),...
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. We denote the set of factorizations of $s$ by $Z(s)$. We denote an arbitrary element $z \in Z(s)$ with a $k$-tuple (i.e. as $z = (a_1, \ldots, a_k)$), and we define the length of $z$ to be $|z| = a_1 + \cdots + a_k$. 

**Definition** Let $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k) \in Z(s)$. We define the greatest common divisor of $z$ and $z'$ to be $\gcd(z, z') = (\min\{a_1, b_1\}, \ldots, \min\{a_k, b_k\})$.

**Definition** Let $z$ and $z'$ be as above. We define the distance between $z$ and $z'$ to be $d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\}$. 

M. Corrales, A. Miller, C. Miller, D. Patel (2013) 
*The Catenary Degree of Elements in $\mathbb{N}$*
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. We denote the set of factorizations of $s$ by $Z(s)$. We denote an arbitrary element $z \in Z(s)$ with a $k$-tuple (i.e. as $z = (a_1, \ldots, a_k)$), and we define the length of $z$ to be $|z| = a_1 + \cdots + a_k$.

**Definition**

Let $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k) \in Z(s)$. We define the GCD of $z$ and $z'$ to be

$$\gcd(z, z') = (\min\{a_1, b_1\}, \ldots, \min\{a_k, b_k\}).$$
Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid, and let $s \in S$. We denote the set of factorizations of $s$ by $Z(s)$. We denote an arbitrary element $z \in Z(s)$ with a $k$-tuple (i.e. as $z = (a_1, \ldots, a_k)$), and we define the length of $z$ to be $|z| = a_1 + \cdots + a_k$.

**Definition**

Let $z = (a_1, \ldots, a_k)$ and $z' = (b_1, \ldots, b_k) \in Z(s)$. We define the GCD of $z$ and $z'$ to be

$$\text{gcd}(z, z') = (\min\{a_1, b_1\}, \ldots, \min\{a_k, b_k\}).$$

**Definition**

Let $z$ and $z'$ be as above. We define the distance between $z$ and $z'$ to be

$$d(z, z') = \max\{|z - \text{gcd}(z, z')|, |z' - \text{gcd}(z, z')|\}.$$
Consider $77 \in \langle 4, 5, 11 \rangle$ again, and in particular consider the factorizations $z = (5, 7, 2)$ and $z' = (6, 4, 3)$.
Consider $77 \in \langle 4, 5, 11 \rangle$ again, and in particular consider the factorizations $z = (5, 7, 2)$ and $z' = (6, 4, 3)$. Then $\gcd(z, z') = (5, 4, 2)$, $z - \gcd(z, z') = (0, 3, 0)$, and $z' - \gcd(z, z') = (1, 0, 1)$,
Consider 77 ∈ ⟨4, 5, 11⟩ again, and in particular consider the factorizations $z = (5, 7, 2)$ and $z' = (6, 4, 3)$.
Then $\gcd(z, z') = (5, 4, 2)$,
$z - \gcd(z, z') = (0, 3, 0)$, and $z' - \gcd(z, z') = (1, 0, 1)$,
so $|z - \gcd(z, z')| = 3$ and $|z' - \gcd(z, z')| = 2$. 

Consider $77 \in \langle 4, 5, 11 \rangle$ again, and in particular consider the factorizations $z = (5, 7, 2)$ and $z' = (6, 4, 3)$. Then $\gcd(z, z') = (5, 4, 2)$, 
$z - \gcd(z, z') = (0, 3, 0)$, and $z' - \gcd(z, z') = (1, 0, 1)$, 
so $|z - \gcd(z, z')| = 3$ and $|z' - \gcd(z, z')| = 2$. 
Finally, $d(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\} = \max\{3, 2\} = 3$. 
Let $s \in \langle n_1, \ldots, n_k \rangle$, and let $z, z' \in Z(s)$. 

What is the "easiest" (i.e. with minimal distance) way to turn $z$ into $z'$ by "moving between" only other factorizations of $s$?

Construct a chain of factorizations $z = z_0, z_1, \ldots, z_l = z'$ from $z$ to $z'$. 

Definition We call this chain of factorizations an $N$-chain if $N$ is the minimal $n \in \mathbb{N}$ such that $d(z_i, z_{i+1}) \leq N$ for $i = 0, \ldots, l-1$. 

We are (finally) ready to define our main concept. 

Definition The catenary degree of $s$ (denoted $c(s)$) is the minimal $N \in \mathbb{N}$ such that, for any $z, z' \in Z(s)$, there exists an $N$-chain between $z$ and $z'$. 

Definition The catenary degree of $S$ is $c(S) = \sup \{ c(s) | s \in S \}$. 

M. Corrales, A. Miller, C. Miller, D. Patel
Chains of Factorizations

Let \( s \in \langle n_1, \ldots, n_k \rangle \), and let \( z, z' \in \mathbb{Z}(s) \). What is the “easiest” (i.e. with minimal distance) way to turn \( z \) into \( z' \) by “moving between” only other factorizations of \( s \)?

We call this chain of factorizations an \( N \)-chain if \( N \) is the minimal \( n \in \mathbb{N} \) such that \( d(z_i, z_{i+1}) \leq N \) for \( i = 0, \ldots, l-1 \).

We are (finally) ready to define our main concept.

Definition The catenary degree of \( s \) (denoted \( c(s) \)) is the minimal \( N \in \mathbb{N} \) such that, for any \( z, z' \in \mathbb{Z}(s) \), there exists an \( N \)-chain between \( z \) and \( z' \).

Definition The catenary degree of \( S \) is \( c(S) = \sup \{ c(s) | s \in S \} \).
Chains of Factorizations

Let $s \in \langle n_1, \ldots, n_k \rangle$, and let $z, z' \in Z(s)$. What is the “easiest” (i.e. with minimal distance) way to turn $z$ into $z'$ by “moving between” only other factorizations of $s$? Construct a chain of factorizations $z = z_0, z_1, \ldots, z_{l-1}, z_l = z'$ from $z$ to $z'$. 

---

**Definition**

We call this chain of factorizations an $N$-chain if $N$ is the minimal $n \in \mathbb{N}$ such that $d(z_i, z_{i+1}) \leq N$ for $i = 0, \ldots, l-1$. 

We are (finally) ready to define our main concept.

**Definition**

The catenary degree of $s$ (denoted $c(s)$) is the minimal $N \in \mathbb{N}$ such that, for any $z, z' \in Z(s)$, there exists an $N$-chain between $z$ and $z'$. 

**Definition**

The catenary degree of $S$ is $c(S) = \sup \{c(s) | s \in S\}$. 

M. Corrales, A. Miller, C. Miller, D. Patel

The Catenary Degree of Elements in $\mathbb{N}$

August 8, 2013 9 / 63
Chains of Factorizations

Let \( s \in \langle n_1, \ldots, n_k \rangle \), and let \( z, z' \in Z(s) \). What is the “easiest” (i.e. with minimal distance) way to turn \( z \) into \( z' \) by “moving between” only other factorizations of \( s \)? Construct a chain of factorizations
\[
z = z_0, z_1, \ldots, z_{l-1}, z_l = z'
\]
from \( z \) to \( z' \).

Definition

We call this chain of factorizations an \( N \)-chain if \( N \) is the minimal \( n \in \mathbb{N} \) such that \( d(z_i, z_{i+1}) \leq N \) for \( i = 0, \ldots, l - 1 \).
**Chains of Factorizations**

Let \( s \in \langle n_1, \ldots, n_k \rangle \), and let \( z, z' \in Z(s) \). What is the “easiest” (i.e. with minimal distance) way to turn \( z \) into \( z' \) by “moving between” only other factorizations of \( s \)? Construct a **chain of factorizations**
\[
z = z_0, z_1, \ldots, z_{l-1}, z_l = z'
\]
from \( z \) to \( z' \).

**Definition**

We call this chain of factorizations an **\( N \)-chain** if \( N \) is the minimal \( n \in \mathbb{N} \) such that \( d(z_i, z_{i+1}) \leq N \) for \( i = 0, \ldots, l - 1 \).

We are (finally) ready to define our main concept.
Chains of Factorizations

Let \( s = \langle n_1, \ldots, n_k \rangle \), and let \( z, z' \in Z(s) \). What is the “easiest” (i.e. with minimal distance) way to turn \( z \) into \( z' \) by “moving between” only other factorizations of \( s \)? Construct a chain of factorizations
\[
z = z_0, z_1, \ldots, z_{l-1}, z_l = z'
\]
from \( z \) to \( z' \).

**Definition**

We call this chain of factorizations an \( N \)-chain if \( N \) is the minimal \( n \in \mathbb{N} \) such that \( d(z_i, z_{i+1}) \leq N \) for \( i = 0, \ldots, l-1 \).

We are (finally) ready to define our main concept.

**Definition**

The catenary degree of \( s \) (denoted \( c(s) \)) is the minimal \( N \in \mathbb{N} \) such that, for any \( z, z' \in Z(s) \), there exists an \( N \)-chain between \( z \) and \( z' \).
Chains of Factorizations

Let $s \in \langle n_1, \ldots, n_k \rangle$, and let $z, z' \in Z(s)$. What is the “easiest” (i.e. with minimal distance) way to turn $z$ into $z'$ by “moving between” only other factorizations of $s$? Construct a chain of factorizations $z = z_0, z_1, \ldots, z_{l-1}, z_l = z'$ from $z$ to $z'$.

**Definition**

We call this chain of factorizations an $N$-chain if $N$ is the minimal $n \in \mathbb{N}$ such that $d(z_i, z_{i+1}) \leq N$ for $i = 0, \ldots, l - 1$.

We are (finally) ready to define our main concept.

**Definition**

The catenary degree of $s$ (denoted $c(s)$) is the minimal $N \in \mathbb{N}$ such that, for any $z, z' \in Z(s)$, there exists an $N$-chain between $z$ and $z'$.

**Definition**

The catenary degree of $S$ is $c(S) = \sup \{ c(s) \mid s \in S \}$. 

Definitions/Motivation
1 Definitions/Motivation

2 The catenary degree of elements in:
   - 2-generated numerical monoids
Finding the Catenary Degree

Consider, again, $77 \in \langle 4, 5, 11 \rangle$. 
Finding the Catenary Degree

Consider, again, $77 \in \langle 4, 5, 11 \rangle$. 
Now let $S = \langle n_1, n_2 \rangle$. Without loss of generality, let $n_2 > n_1$. 
Now let $S = \langle n_1, n_2 \rangle$. Without loss of generality, let $n_2 > n_1$. Our main results are:

**Theorem**

For every $s \in S$, $c(s) \in \{0, n_2\}$

and

**Theorem**

$c(s) = n_2$ iff $s - n_1 n_2 \in \langle n_1, n_2 \rangle$. 
The Catenary Degree of 2-Generated Numerical Monoids

Clearly, if $s \in S$ has a unique factorization, then $c(s) = 0$. 

Example
Consider $77 \in \langle 4, 5 \rangle$.

$Z(s) = \{(3, 13), (8, 9), (13, 5), (18, 1)\}$. 
The Catenary Degree of 2-Generated Numerical Monoids

Clearly, if $s \in S$ has a unique factorization, then $c(s) = 0$. Let’s examine an element which has multiple factorizations:

**Example**

Consider $77 \in \langle 4, 5 \rangle$. $Z(s) = \{(3, 13), (8, 9), (13, 5), (18, 1)\}$. 
The Catenary Degree of 2-Generated Numerical Monoids

Clearly, if \( s \in S \) has a unique factorization, then \( c(s) = 0 \). Let’s examine an element which has multiple factorizations:

**Example**

Consider \( 77 \in \langle 4, 5 \rangle \). \( Z(s) = \{(3, 13), (8, 9), (13, 5), (18, 1)\} \).
The Catenary Degree of 2-Generated Numerical Monoids

Clearly, if \( s \in S \) has a unique factorization, then \( c(s) = 0 \). Let’s examine an element which has multiple factorizations:

Example

Consider \( 77 \in \langle 4, 5 \rangle \). \( Z(s) = \{(3, 13), (8, 9), (13, 5), (18, 1)\} \).
Clearly, if $s \in S$ has a unique factorization, then $c(s) = 0$. Let’s examine an element which has multiple factorizations:

**Example**

Consider $77 \in \langle 4, 5 \rangle$. $Z(s) = \{(3, 13), (8, 9), (13, 5), (18, 1)\}$. 
It is not hard to show that if \( s - n_1 n_2 \in \langle n_1, n_2 \rangle \), then \( c(s) = n_2 \).
It is not hard to show that if $s - n_1 n_2 \in \langle n_1, n_2 \rangle$, then $c(s) = n_2$.

**Proof.**

It suffices to show that $s$ has multiple factorizations in $\langle n_1, n_2 \rangle$. 
It is not hard to show that if \( s - n_1 n_2 \in \langle n_1, n_2 \rangle \), then \( c(s) = n_2 \).

**Proof.**

It suffices to show that \( s \) has multiple factorizations in \( \langle n_1, n_2 \rangle \). If \( s - n_1 n_2 \in \langle n_1, n_2 \rangle \), we can write \( s - n_1 n_2 = an_1 + bn_2 \).
It is not hard to show that if $s - n_1 n_2 \in \langle n_1, n_2 \rangle$, then $c(s) = n_2$.

**Proof.**

It suffices to show that $s$ has multiple factorizations in $\langle n_1, n_2 \rangle$. If $s - n_1 n_2 \in \langle n_1, n_2 \rangle$, we can write $s - n_1 n_2 = an_1 + bn_2$. Then $s = an_1 + bn_2 + n_1 n_2 = (a + n_2)n_1 + bn_2 = an_1 + (b + n_1)n_2$, and the latter two expressions are distinct factorizations of $s$. 

\[ \square \]
The 2-generator case is summarized as follows:

\[
c(s) = \begin{cases} 
0 & : s < n_1 n_2 \\
n_2 & : s = n_1 n_2 \\
n_2 & : n_1 n_2 < s < 2n_1 n_2 - n_1 - n_2 \text{ and } s - n_1 n_2 \notin \langle n_1, n_2 \rangle \\
0 & : s = 2n_1 n_2 - n_1 - n_2 \\
n_2 & : 2n_1 n_2 - n_1 - n_2 < s
\end{cases}
\]
The Catenary Degree of 2-Generated Numerical Monoids

plot(points(catspairs), title="s vs. c(s) (S=<4,5>)")

s vs. c(s) (S=<4,5>)
The Frobenius Number

In the 2-generator case, $c(s)$ becomes constant because, by definition, $\mathbb{N}_0 \setminus S$ is finite. There is a name for the number after which every $n \in \mathbb{N}_0$ is in $S$. 

**Definition**

The Frobenius number of a numerical monoid is defined as $F(S) = \max\{n \in \mathbb{N}_0 | n \not\in S\}$. If $S = \langle n_1, n_2 \rangle$, then $F(S) = n_1 n_2 - n_1 - n_2$. If we add a third generator to $S$, and let $n_3 = F(S) = n_1 n_2 - n_1 - n_2$, we get some nice results.
In the 2-generator case, $c(s)$ becomes constant because, by definition, $\mathbb{N}_0 \setminus S$ is finite. There is a name for the number after which every $n \in \mathbb{N}_0$ is in $S$.

**Definition**

The **Frobenius number** of a numerical monoid is defined as

$$\mathcal{F}(S) = \max\{n \in \mathbb{N}_0 \mid n \notin S\}.$$
In the 2-generator case, \( c(s) \) becomes constant because, by definition, \( \mathbb{N}_0 \setminus S \) is finite. There is a name for the number after which every \( n \in \mathbb{N}_0 \) is in \( S \).

**Definition**

The **Frobenius number** of a numerical monoid is defined as

\[
\mathcal{F}(S) = \max\{ n \in \mathbb{N}_0 \mid n \not\in S \}.
\]

If \( S = \langle n_1, n_2 \rangle \), then \( \mathcal{F}(S) = n_1 n_2 - n_1 - n_2 \).
The Frobenius Number

In the 2-generator case, \(c(s)\) becomes constant because, by definition, \(\mathbb{N}_0 \setminus S\) is finite. There is a name for the number after which every \(n \in \mathbb{N}_0\) is in \(S\).

**Definition**

The **Frobenius number** of a numerical monoid is defined as

\[
F(S) = \max\{n \in \mathbb{N}_0 \mid n \not\in S\}.
\]

If \(S = \langle n_1, n_2 \rangle\), then \(F(S) = n_1n_2 - n_1 - n_2\).

If we add a third generator to \(S\), and let \(n_3 = F(S) = n_1n_2 - n_1 - n_2\), we get some nice results.
1 Definitions/Motivation
1. Definitions/Motivation
2. The catenary degree of elements in:
   - 2-generated numerical monoids
1 Definitions/Motivation

2 The catenary degree of elements in:
   - 2-generated numerical monoids
   - a special case of 3-generated numerical monoids
Let $S = \langle n_1, n_2, n_3 \rangle$, where $n_3 = n_1 n_2 - n_1 - n_2$. 
Let $S = \langle n_1, n_2, n_3 \rangle$, where $n_3 = n_1n_2 - n_1 - n_2$.

**Theorem**

For $s \in S$, write $s = kn_3 + r$. If $r = 0$, then $c(s) = n_1 + n_2 - 4$. If $r \neq 0$, then $c(s) = n_2 - 1$, for all $s > n_3 + (n_2 - 2)n_1$. 
Let $S = \langle n_1, n_2, n_3 \rangle$, where $n_3 = n_1 n_2 - n_1 - n_2$.

**Theorem**

For $s \in S$, write $s = kn_3 + r$. If $r = 0$, then $c(s) = n_1 + n_2 - 4$. If $r \neq 0$, then $c(s) = n_2 - 1$, for all $s > n_3 + (n_2 - 2)n_1$.

**Corollary**

For $s > n_3 + (n_2 - 2)n_1$, $c(s) = c(s + n_3)$. If $s = n_3 + (n_2 - 2)n_1$, $c(s) \neq c(s + n_3)$. 
A Special 3-Generated Numerical Monoid

Let $S = \langle 4, 5, 11 \rangle$. 
Let $S = \langle 4, 5, 11 \rangle$. Consider $76 \in S$. Note that $r \neq 0$ in this case.
A Special 3-Generated Numerical Monoid

Let $S = \langle 4, 5, 11 \rangle$. Consider $76 \in S$. Note that $r \neq 0$ in this case.
Let $S = \langle 4, 5, 11 \rangle$. Consider $76 \in S$. Note that $r \neq 0$ in this case.
Now consider (again) $77 \in S$. Note that $r = 0$ in this case.
Now consider (again) $77 \in S$. Note that $r = 0$ in this case.
A Special 3-Generated Numerical Monoid

plot(points(catspairs3), title="s vs. c(s) (S=\langle4,5,11\rangle)")

s vs. c(s) (S=\langle4,5,11\rangle)
Numerical Monoids Generated By Arithmetic Sequences: 3-generators
2-Generators

$S = \langle n_1, n_2 \rangle = \langle a, a + d \rangle$, where $\gcd(a, d) = 1$ and $d < a$. 
2-GENERATORS

S = ⟨n₁, n₂⟩ = ⟨a, a + d⟩, where gcd(a, d) = 1 and d < a.

EXAMPLE

S = ⟨5, 6⟩
2-Generators

$S = \langle n_1, n_2 \rangle = \langle a, a + d \rangle$, where $\gcd(a, d) = 1$ and $d < a$.

Example

$S = \langle 5, 6 \rangle$
2-GENERATORS

$S = \langle n_1, n_2 \rangle = \langle a, a + d \rangle$, where $\gcd(a, d) = 1$ and $d < a$.

EXAMPLE

$S = \langle 5, 6 \rangle$

Observations:
2-Generators

\[ S = \langle n_1, n_2 \rangle = \langle a, a+d \rangle, \text{ where } \gcd(a, d) = 1 \text{ and } d < a. \]

Example

\[ S = \langle 5, 6 \rangle \]

Observations:

- \( c(S) = 6 \)
2-Generators

\( S = \langle n_1, n_2 \rangle = \langle a, a + d \rangle \), where \( \gcd(a, d) = 1 \) and \( d < a \).

Example

\( S = \langle 5, 6 \rangle \)

Observations:
- \( c(S) = 6 \)
- \( c(x) \in \{0, c(S)\} \)
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

\[ S = \langle a, a + d, a + 2d \rangle \] is a numerical monoid, \( \gcd(a, d) = 1 \) and \( a < 2d \).

Example

\[ S = \langle 5, 6, 7 \rangle \]

Observations:

- \( c(S) = 4 \)
- \( c(x) \in \{0, 2, 4\} \)
Natural extension to numerical monoids with 3-generators.

**3-GENERATORS**

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\gcd(a, d) = 1$ and $2, d < a$. 
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

3-Generators

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\gcd(a, d) = 1$ and $2, d < a$.

Example

$S = \langle 5, 6, 7 \rangle$
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

3-Generators

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\gcd(a, d) = 1$ and $2, d < a$.

Example

$$S = \langle 5, 6, 7 \rangle$$
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

3-generators

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\gcd(a, d) = 1$ and $2, d < a$.

Example

$S = \langle 5, 6, 7 \rangle$

Observations:
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

3-Generators

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\gcd(a, d) = 1$ and $2, d < a$.

Example

$S = \langle 5, 6, 7 \rangle$

Observations:

- $c(S) = 4$
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Natural extension to numerical monoids with 3-generators.

3-Generators

$S = \langle a, a + d, a + 2d \rangle$ is a numerical monoid, $\text{gcd}(a, d) = 1$ and $2, d < a$.

Example

$S = \langle 5, 6, 7 \rangle$

Observations:

- $c(S) = 4$
- $c(x) \in \{0, 2, c(S)\}$
Example

\[ S = \langle 5, 7, 9 \rangle \]
Example

\[ S = \langle 5, 7, 9 \rangle \]
Example

\[ S = \langle 5, 7, 9 \rangle \]

Observations:
Example

\[ S = \langle 5, 7, 9 \rangle \]

Observations:
- \( c(S) = 5 \)
Example

$$S = \langle 5, 7, 9 \rangle$$

Observations:

- $$c(S) = 5$$
- $$c(x) \in \{0, 2, c(S)\}$$
Proposition

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, d < a$. Take $x \in S$.

- $c(x) \in \{0, 2, c(S)\}$
**Proposition**

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, d < a$. Take $x \in S$.

- $c(x) \in \{0, 2, c(S)\}$
- If $x = 2(a + d)$, then $c(x) = 2$. Idea of the proof:
  
  $2(a + d) = a + (a + 2d)$
**Proposition**

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, d < a$. Take $x \in S$.

- $c(x) \in \{0, 2, c(S)\}$
- If $x = 2(a + d)$, then $c(x) = 2$. Idea of the proof:
  $2(a + d) = a + (a + 2d)$
- If $x = a \cdot c(S)$, then $c(x) = c(S)$. Idea of the proof:
  $a \cdot c(S) \in \langle a + d, a + 2d \rangle$
We characterized the catenary degree of each elements in terms of $\mathcal{L}(x)$. 

Proposition

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2 < c < a$. Take $x \in S$.

$c(x) = 0$ if and only if $x$ has just one factorization

$c(x) = 2$ if and only if $x$ has more than one factorization

$|\mathcal{L}(x)| = 1$ if and only if $c(x) = c(S)$

$|\mathcal{L}(x)| > 1$ if and only if $c(x) = c(S)$.
We characterized the catenary degree of each elements in terms of $\mathcal{L}(x)$.

**Proposition**

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, c < a$. Take $x \in S$.

- $c(x) = 0$ if and only if $x$ has just one factorization
We characterized the catenary degree of each elements in terms of $\mathcal{L}(x)$.

**Proposition**

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, c < a$. Take $x \in S$.

- $c(x) = 0$ if and only if $x$ has just one factorization
- $c(x) = 2$ if and only if $x$ has more than one factorization and $|\mathcal{L}(x)| = 1$
We characterized the catenary degree of each elements in terms of $\mathcal{L}(x)$.

**Proposition**

Let $S = \langle a, a + d, a + 2d \rangle$, where $\gcd(a, d) = 1$ and $2, c < a$. Take $x \in S$.

- $c(x) = 0$ if and only if $x$ has just one factorization
- $c(x) = 2$ if and only if $x$ has more than one factorization and $|\mathcal{L}(x)| = 1$
- $c(x) = c(S)$ if and only if $|\mathcal{L}(x)| > 1$
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Idea of the proof:
Dissonance

\[ S = \langle 5, 6, 7 \rangle, \quad c(S) = 4, \quad \mathcal{F}(S) = 9 \]
Numerical Monoids Generated By Arithmetic Sequences

3-generators case

Dissonance

\( S = \langle 5, 6, 7 \rangle, \ c(S) = 4, \ \mathcal{F}(S) = 9 \)

\( S = \langle 5, 7, 9 \rangle, \ c(S) = 5, \ \mathcal{F}(S) = 13 \)
**Dissonance**

\[ S = \langle 5, 6, 7 \rangle, \ c(S) = 4, \ \mathcal{F}(S) = 9 \]

\[ S = \langle 5, 7, 9 \rangle, \ c(S) = 5, \ \mathcal{F}(S) = 13 \]

For \( S = \langle 5, 6, 7 \rangle \), \( \text{dis} = a \cdot c(S) + \mathcal{F}(S) = 29 \)
Dissonance

\[ S = \langle 5, 6, 7 \rangle, \quad c(S) = 4, \quad F(S) = 9 \]

\[ S = \langle 5, 7, 9 \rangle, \quad c(S) = 5, \quad F(S) = 13 \]

For \( S = \langle 5, 6, 7 \rangle \), \( dis = a \cdot c(S) + F(S) = 29 \)

For \( S = \langle 5, 7, 9 \rangle \), \( dis = a \cdot c(S) + F(S) = 38 \)
**Dissonance**

Dissonance of $S$ is the biggest element in $S$ with catenary degree different than $c(S)$.

**Proposition**

$\text{dis} = a \cdot c(S) + \mathcal{F}(S)$. Idea of the proof: $c(\text{dis}) = 2$ and $c(\text{dis} + n) = c(s)$, for $n \in \mathbb{N}$.
Numerical Monoids Generated By Arithmetic Sequences: n-generators
**Numerical Monoids Generated By Arithmetic Sequences**

**N-generator case**

\[ S = \langle a, a + d, \ldots, a + kd \rangle \]

**Question**

Is it still true that if \( x \in S \), then \( c(x) \in \{0, 2, c(S)\} \)?

**Comment**

In fact, all the major theorems about catenary degrees that we proved for 3-generator case, holds for n-generator case, that is:

- \( c(x) = 2 \) if and only if \( |\mathbb{Z}(x)| > 1 \) and all the factorizations of \( x \) have the same length.
- \( c(x) = c(S) \) if and only if \( x \) has at least two factorizations with different lengths.
Numerical Monoids Generated By Arithmetic Sequences

N-generator case

$$S = \langle a, a + d, \ldots, a + kd \rangle$$

Question

Is it still true that if $x \in S$, then $c(x) \in \{0, 2, c(S)\}$? YES!
Numerical Monoids Generated by Arithmetic Sequences

N-generator case

\[ S = \langle a, a + d, \ldots, a + kd \rangle \]

**Question**

Is it still true that if \( x \in S \), then \( c(x) \in \{0, 2, c(S)\} \)? **YES!**

**Comment**

In fact, all the major theorems about catenary degrees that we proved for 3-generator case, holds for n-generator case, that is:

- \( c(x) = 2 \) if and only if \( |Z(x)| > 1 \) and all the factorizations of \( x \) have the same length.
Numerical Monoids Generated By Arithmetic Sequences
N-generator Case

\[ S = \langle a, a + d, \ldots, a + kd \rangle \]

**Question**

Is it still true that if \( x \in S \), then \( c(x) \in \{0, 2, c(S)\} \)? YES!

**Comment**

In fact, all the major theorems about catenary degrees that we proved for 3-generator case, holds for n-generator case, that is:

- \( c(x) = 2 \) if and only if \( |Z(x)| > 1 \) and all the factorizations of \( x \) have the same length.
- \( c(x) = c(S) \) if and only if \( x \) has at least two factorizations with different lengths.
 Movements

What about the movement between factorizations if catenary degree is 2?

- It gets a little complicated:

Example

Let 

\[ S = \langle 5, 8, 11, 14 \rangle \]

Consider, the element

\[ 24 = 0(5) + 3(8) + 0(11) + 0(14) \]
Movements

What about the movement between factorizations if catenary degree is 2?

-It gets a LITTLE complicated:

Example

Let \( S = \langle 5, 8, 11, 14 \rangle \)

Consider, the element

\[ 24 = 0(5) + 3(8) + 0(11) + 0(14) \]
What about the movement between factorizations if catenary degree is 2?

-It gets a LITTLE complicated:

**Example**

Let $S = \langle 5, 8, 11, 14 \rangle$ Consider, the element

$24 = 0(5) + 3(8) + 0(11) + 0(14)$

Let us write the number of copies of each generator in the form below:
Dissonance number for more than 3 generator

It turns out that something is going on with the dissonance for more than 3-generator case.
Dissonance number for more than 3 generator

It turns out that something is going on with the dissonance for more than 3-generator case.

There are two possibilities of dissonance:

- \( a \cdot c(S) + \mathcal{F}(S) \)
- \( a \cdot c(S) + \mathcal{F}(S) - a \)
Dissonance number for more than 3 generator

It turns out that something is going on with the dissonance for more than 3-generator case. There are two possibilities of dissonance:

- \( a \cdot c(S) + \mathcal{F}(S) \)
- \( a \cdot c(S) + \mathcal{F}(S) - a \)

Question

So how do we classify dissonance numbers?
**Classification of dissonance number**

**Theorem**

\[ k \geq 2 + (a - 1) \pmod{k} + (a - 2) \pmod{k}, \text{ if and only if } \]
\[ \text{dis}(S) = a \cdot c(S) + \mathcal{F}(S) - a \]
**Classification of dissonance number**

**Theorem**

\[ k \geq 2 + (a - 1) \mod k + (a - 2) \mod k, \text{ if and only if } \]

\[ dis(S) = a \cdot c(S) + \mathcal{F}(S) - a \]

**Proof**

First, we looked at in which cases does the number \( a \cdot c(S) + \mathcal{F}(S) \) have factorization of different length. This told us that in those cases its catenary degree is \( c(S) \).
**Theorem**

\[ k \geq 2 + (a - 1) \pmod{k} + (a - 2) \pmod{k}, \text{ if and only if } \]

\[ \text{dis}(S) = a \cdot c(S) + \mathcal{F}(S) - a \]

**Proof**

First, we looked at in which cases does the number \( a \cdot c(S) + \mathcal{F}(S) \) has factorization of different length. This told us that in that cases its catenary degree is \( c(S) \).

Next, we proved that if \( c(a \cdot c(S) + \mathcal{F}(S)) = c(S) \) then all the elements in between \( a \cdot c(S) + \mathcal{F}(S) - a \) and \( a \cdot c(S) + \mathcal{F}(S) \) have catenary degree \( c(S) \).
THEOREM

\[ k \geq 2 + (a - 1) \mod k + (a - 2) \mod k, \text{ if and only if } \]
\[ dis(S) = a \cdot c(S) + \mathcal{F}(S) - a \]

PROOF

First, we looked at in which cases does the number \( a \cdot c(S) + \mathcal{F}(S) \) has factorization of different length. This told as that in that cases its catenary degree is \( c(S) \).

Next, we proved that if \( c(a \cdot c(S) + \mathcal{F}(S)) = c(S) \) then all the elements in between \( a \cdot c(S) + \mathcal{F}(S) - a \) and \( a \cdot c(S) + \mathcal{F}(S) \) have catenary degree \( c(S) \).

Finally we proved that \( c(a \cdot c(S) + \mathcal{F}(S) - a) \neq c(S) \).

In the end we obtain the above result.
Catenary degree of elements generated by arithmetic sequence is eventually constant

Recall (Theorem)

\[ c(x) = c(S) \] if and only if there are factorizations of different length
Recall (Theorem)

\[ c(x) = c(S) \] if and only if there are factorizations of different length

Corollary

For \( x > a \cdot c(S) + \mathcal{F}(S) \), \( c(x) = c(S) \)
Recall (Theorem)

\[ c(x) = c(S) \] if and only if there are factorizations of different length

Corollary

For \( x > a \cdot c(S) + \mathcal{F}(S) \), \( c(x) = c(S) \)

Proof

\( x > a \cdot c(S) + \mathcal{F}(S) \implies x = a \cdot c(S) + \mathcal{F}(S) + v \) for some \( v \in \mathbb{N} \)
Recall (Theorem)
\[ c(x) = c(S) \text{ if and only if there are factorizations of different length} \]

Corollary
For \( x > a \cdot c(S) + F(S) \), \( c(x) = c(S) \)

Proof
\[ x > a \cdot c(S) + F(S) \implies x = a \cdot c(S) + F(S) + \nu \text{ for some } \nu \in \mathbb{N} \]
Observe that \( F(S) + \nu \) has at least one factorization since it is in \( S \). Since \( a \cdot c(S) \) is a multiple of first generator, we get the first factorization.
Recall (Theorem)

$c(x) = c(S)$ if and only if there are factorizations of different length.

Corollary

For $x > a \cdot c(S) + \mathcal{F}(S)$, $c(x) = c(S)$.

Proof

$x > a \cdot c(S) + \mathcal{F}(S) \implies x = a \cdot c(S) + \mathcal{F}(S) + v$ for some $v \in \mathbb{N}$.

Observe that $\mathcal{F}(S) + v$ has at least one factorization since it is in $S$. Since $a \cdot c(S)$ is a multiple of first generator, we get the first factorization.

To find the second factorization we use the result, $a \cdot c(S) \in \langle a + d, a + 2d \ldots, a + kd \rangle$. Hence, we can write $a \cdot c(S)$ in terms of rest of the generators.
Recall (Theorem)

\[ c(x) = c(S) \] if and only if there are factorizations of different length

Corollary

For \( x > a \cdot c(S) + \mathcal{F}(S) \), \( c(x) = c(S) \)

Proof

\[ x > a \cdot c(S) + \mathcal{F}(S) \implies x = a \cdot c(S) + \mathcal{F}(S) + v \] for some \( v \in \mathbb{N} \)

Observe that \( \mathcal{F}(S) + v \) has at least one factorization since it is in \( S \). Since \( a \cdot c(S) \) is a multiple of first generator, we get the first factorization. To find the second factorization we use the result, \( a \cdot c(S) \in \langle a + d, a + 2d \ldots, a + kd \rangle \). Hence, we can write \( a \cdot c(S) \) in terms of rest of the generators. From this we obtain a factorization of different length.
Periodicity of Catenary Degree and other Combinatorial Invariants
**Definition**

Given a numerical monoid $\langle n_1, \ldots, n_k \rangle$, we define $L = \text{lcm}(n_1, \ldots, n_k)$

**Definition**

Distance-preserving maps are maps $\phi : \mathbb{Z}(s - L) \rightarrow \mathbb{Z}(s)$ such that $d(x, y) = d(\phi(x), \phi(y))$. We most often use the maps $\{\phi_i\}_{i=1}^k$, where

$$\phi_i : z \rightarrow z + (0, \ldots, \frac{L}{n_i}, 0, \ldots, 0)$$

Written another way (informally),

$$\phi_i : a_1 n_1 + \ldots + a_i n_i + \ldots + a_k n_k \rightarrow a_1 n_1 + \ldots + (a_i + \frac{L}{n_i}) n_i + \ldots a_k n_k$$
**Example**

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. 

Z(35) = \{(7, 0), (0, 5)\}. 

$\phi_1: z \mapsto z + (7, 0)$ 

$\phi_1((7, 0)) = (14, 0)$ 

$\phi_1((0, 5)) = (7, 5)$ 

$d(\phi_1((7, 0)), \phi_1((0, 5))) = \max\{|(14, 0) - (7, 0)|, |(7, 5) - (7, 0)|\} = d((7, 0), (0, 5))$

So the distance between these factorizations is preserved.
**Example**

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. $Z(35) = \{(7, 0), (0, 5)\}$. 
Example

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. $Z(35) = \{(7, 0), (0, 5)\}$.

$$\phi_1 : z \mapsto z + (7, 0)$$
**Example**

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. $Z(35) = \{(7, 0), (0, 5)\}$.

$$
\begin{align*}
\phi_1 : z &\rightarrow z + (7, 0) \\
\phi_1((7, 0)) &= (14, 0) \\
\phi_1((0, 5)) &= (7, 5)
\end{align*}
$$
Example

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. $Z(35) = \{(7, 0), (0, 5)\}$.

\[
\phi_1 : z \rightarrow z + (7, 0)
\]

\[
\phi_1((7, 0)) = (14, 0)
\]

\[
\phi_1((0, 5)) = (7, 5)
\]

\[
d(\phi_1((7, 0)), \phi_1((0, 5))) = d((14, 0), (7, 5))
\]
**Example**

Let \( S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \} \). Let \( s = 35 \).

\( Z(35) = \{(7, 0), (0, 5)\} \).

\[ \phi_1 : z \rightarrow z + (7, 0) \]

\[ \phi_1((7, 0)) = (14, 0) \]

\[ \phi_1((0, 5)) = (7, 5) \]

\[ d(\phi_1((7, 0)), \phi_1((0, 5))) = d((14, 0), (7, 5)) \]

\[ = \max \{|(14, 0) - (7, 0)|, |(7, 5) - (7, 0)|\} = \max \{|(7, 0)|, |(0, 5)|\} \]
**Example**

Let $S = \langle 5, 7 \rangle = \{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \ldots \}$. Let $s = 35$. $Z(35) = \{(7, 0), (0, 5)\}$.

\[ \phi_1 : z \rightarrow z + (7, 0) \]

\[ \phi_1((7, 0)) = (14, 0) \]

\[ \phi_1((0, 5)) = (7, 5) \]

\[ d(\phi_1((7, 0)), \phi_1((0, 5))) = d((14, 0), (7, 5)) \]

\[ = \max \{|(14, 0) - (7, 0)|, |(7, 5) - (7, 0)|\} = \max \{|(7, 0)|, |(0, 5)|\} \]

\[ = d((7, 0), (0, 5)) \]

So the distance between these factorizations is preserved.
Theorem

\[ c(s - L) \geq c(s) \text{ for } s \geq \max \{ L \cdot k \cdot n_k, \mathcal{F} + 2L + 1 \}. \]
Theorem
\[ c(s - L) \geq c(s) \text{ for } s \geq \max \{ L \cdot k \cdot n_k, F + 2L + 1 \}. \]

Corollary
This is sufficient to prove that the catenary degree is eventually periodic.
Typically, factorizations of larger numbers are more complicated than those of smaller ones.
Note that \((a_1, \ldots a_k) \in \phi_i(Z(s - L))\) is equivalent to saying that \(a_i \geq \frac{L}{n_i}\).
Every single factorization is colored. So every factorization is in the image of some distance preserving map $\phi_i$. 
Theorem

\[ \bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k. \]
Theorem

\[ \bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k. \]

Proof

That \( \phi_i(Z(s - L)) \subseteq Z(s) \) can be seen from the picture.
Theorem

$$\bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k.$$ 

Proof

- That $\phi_i(Z(s - L)) \subseteq Z(s)$ can be seen from the picture.
- Pick a factorization $(a_1, \ldots, a_k) \in Z(s)$. 
**Theorem**

\[ \bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k. \]

**Proof**

- That \( \phi_i(Z(s - L)) \subseteq Z(s) \) can be seen from the picture.
- Pick a factorization \((a_1, \ldots, a_k) \in Z(s)\). Then

\[
\sum_{i=1}^{k} a_i n_i = s
\]
**Theorem**

\[ \bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k. \]

**Proof**

- That \( \phi_i(Z(s - L)) \subseteq Z(s) \) can be seen from the picture.
- Pick a factorization \((a_1, \ldots, a_k) \in Z(s)\). Then

\[
\sum_{i=1}^{k} a_i n_i = s
\]

\[
\max_{i \leq k} a_i \cdot k \cdot n_k = \sum_{i=1}^{k} \max_{i \leq k} a_i \cdot n_k \geq \sum_{i=1}^{k} a_i n_i = s \geq L \cdot k \cdot n_k
\]
Theorem

\[ \bigcup_{i \leq k} \phi_i(Z(s - L)) = Z(s) \text{ for } s \geq L \cdot k \cdot n_k. \]

Proof

- That \( \phi_i(Z(s - L)) \subseteq Z(s) \) can be seen from the picture.
- Pick a factorization \((a_1, \ldots, a_k) \in Z(s)\). Then

\[
\sum_{i=1}^{k} a_i n_i = s
\]

\[
\max_{i \leq k} a_i \cdot k \cdot n_k = \sum_{i=1}^{k} \max_{i \leq k} a_i \cdot n_k \geq \sum_{i=1}^{k} a_i n_i = s \geq L \cdot k \cdot n_k
\]

Denote \( \max_{i \leq k} a_i = a_j \). Then \( a_j \geq L > \frac{L}{n_j} \)

We conclude that \((a_1, \ldots, a_k) \in \phi_j(Z(s - L))\)
**Theorem**

\[ \phi_i(Z(s - L)) \cap \phi_j(Z(s - L)) \neq \emptyset \text{ for } s \geq \mathcal{F} + 2L + 1 \]

**Proof**

- We shall construct an appropriate element. Namely, one such that the 
  \(i\)th component is larger than \(\frac{L}{n_i}\) and the \(j\)th component is greater 
  than \(\frac{L}{n_j}\). Write \(s = \mathcal{F} + 2L + 1 + r = (\mathcal{F} + 1 + r) + 2L\), where \(r \geq 0\).
- Pick \((a_1, \ldots, a_k) \in Z(\mathcal{F} + 1 + r) \neq \emptyset\).
- Then \(a_1n_1 + \ldots + a_kn_k + \frac{L}{n_i}n_i + \frac{L}{n_j}n_j = s\).
- But another representation for this is 
  \((a_1, \ldots, a_i + \frac{L}{n_i}, \ldots, a_j + \frac{L}{n_j}, \ldots, a_k) \in \phi_i(Z(s - L)) \cap \phi_j(Z(s - L))\).
\[
\phi_1: Z(s-L) \to Z(s)
\]
\[
\phi_2: \text{details of the diagram}
\]
**Result**

We conclude that $c(s - L) \geq c(s)$.

**Comment**

Why is this sufficient to prove that the catenary degree is periodic?

We can prove that $\{c(s + iL)\}_{i=1}^{\infty}$ is eventually constant, since it is nonincreasing and a subset of $\mathbb{N}$. So for sufficiently large $s$, $c(s) = c(s + L)$. 
**Result**

We conclude that $c(s - L) \geq c(s)$.

**Comment**

- Why is this sufficient to prove that the catenary degree is periodic?
Result

We conclude that $c(s - L) \geq c(s)$.

Comment

- Why is this sufficient to prove that the catenary degree is periodic?
- We can prove that $\{c(s + iL)\}_{i=1}^{\infty}$ is eventually constant, since it is nonincreasing and a subset of $\mathbb{N}$. So for sufficiently large $s$, $c(s) = c(s + L)$. 
A couple of other things that are periodic.

**Definition**

Given $i \leq k$, let $Z^i(s) = \{(a_1, \ldots a_k) \in Z(s) \mid a_i \neq 0\}$

Let $t_i(s) = \max_{z \in Z(s)} d(z, Z^i)$

Let $t(s) = \max_{i \leq k} t_i(s)$

Then $t(s)$ is the **tame degree** of $s \in S$

**Alternative Definition**

Given $i \leq k$, let $Z^i(s) = \{(a_1, \ldots a_k) \in Z(s) \mid a_i \neq 0\}$

the **tame degree** is the minimal number such that any factorization $z \in Z(s)$, $d(z, Z^i(s)) \leq t(s)$ for all $z \in Z(s)$ for all $i \leq k$. 

Theorem

The tame degree is eventually periodic
A couple of other things that are periodic.

**Definition**

Given $i \leq k$, let $Z^i(s) = \{(a_1, \ldots, a_k) \in Z(s) \mid a_i \neq 0\}$

Let $t_i(s) = \max_{z \in Z(s)} d(z, Z^i)$

Let $t(s) = \max_{i \leq k} t_i(s)$

Then $t(s)$ is the **tame degree** of $s \in S$.

**Alternative Definition**

Given $i \leq k$, let $Z^i(s) = \{(a_1, \ldots, a_k) \in Z(s) \mid a_i \neq 0\}$

the **tame degree** is the minimal number such that any factorization $z \in Z(s)$, $d(z, Z^i(s)) \leq t(s)$ for all $z \in Z(s)$ for all $i \leq k$.

**Theorem**

*The tame degree is eventually periodic*
**Definition**

The **monotone catenary degree** is the minimal number such that given any two factorizations $z, z' \in Z(s)$, with $|z| \leq |z'|$, there is a path $z = z_1, \ldots, z_k = z'$ such that $|z_i| \leq |z_{i+1}|$ and $d(z_i, z_{i+1}) \leq c_{\text{mon}}(s)$.

The **monotone catenary degree** of a monoid is $c_{\text{mon}}(S) = \sup_{s \in S} c_{\text{mon}}(s)$.
**Definition**

The **monotone catenary degree** is the minimal number such that given any two factorizations $z, z' \in Z(s)$, with $|z| \leq |z'|$, there is a path $z = z_1, \ldots, z_k = z'$ such that $|z_i| \leq |z_{i+1}|$ and $d(z_i, z_{i+1}) \leq c_{\text{mon}}(s)$

The **monotone catenary degree** of a monoid is $c_{\text{mon}}(S) = \sup_{s \in S} c_{\text{mon}}(s)$

**Example**

The monotone catenary degree is distinct from the catenary degree. For example $c(\langle 6, 10, 15 \rangle) = 5$, and $c_{\text{mon}}(\langle 6, 10, 15 \rangle) \geq 9$
**Definition**

The **monotone catenary degree** is the minimal number such that given any two factorizations $z, z' \in Z(s)$, with $|z| \leq |z'|$, there is a path $z = z_1, \ldots z_k = z'$ such that $|z_i| \leq |z_{i+1}|$ and $d(z_i, z_{i+1}) \leq c_{\text{mon}}(s)$.

The **monotone catenary degree** of a monoid is $c_{\text{mon}}(S) = \sup_{s \in S} c_{\text{mon}}(s)$.

**Example**

The monotone catenary degree is distinct from the catenary degree. For example $c(\langle 6, 10, 15 \rangle) = 5$, and $c_{\text{mon}}(\langle 6, 10, 15 \rangle) \geq 9$.

**Theorem**

If $S = \langle a, a + d, \ldots a + kd \rangle$, then $c_{\text{mon}}(s) = c(s)$ for all $s \in S$. 
Theorem

The monotone catenary degree is eventually periodic.

Proof

The proof relies on the fact that the monotone catenary degree can been written in terms of other invariants: the adjacent catenary degree and equivalent catenary degree. These are both periodic as well.
Further Research

1. Examine the catenary degree of 3-generated numerical monoids in general.
2. Analyze the catenary degree of elements in monoids generated by a generalized arithmetic sequence or a geometric sequence.
3. Improve the period of the catenary degree. Data shows that it is one of the generators or becomes constant.
4. Determine where the catenary degree starts to be periodic.
Bibliography

Acknowledgements

- We would like to thank Dr. Scott Chapman, Dr. Roberto Pelayo (Bob), and Chris O’Neill for guiding us in our project.
- We would also like to Dr. Rebecca Garcia and Ashlee for their organizational prowess!
- Last but not the least, we would like to thank the NSA, NSF, Sam Houston State University, and University of Hawaii-Hilo for organizing the program and encouraging undergraduate research.