Riemann Problems of the Shallow Water Equations
Nonlinear Systems of Conservation Laws

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What happens if a dam breaks?
One-Dimensional Shallow Water Equations

\[
\begin{bmatrix}
  h \\
  hu
\end{bmatrix}_t + \begin{bmatrix}
  hu \\
  hu^2 + \frac{1}{2}gh^2
\end{bmatrix}_x = 0
\]

Using conserved quantities, \( q(x,t) = \begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \)

The Shallow Water Equations can be rewritten as,

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\begin{bmatrix}
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  q_2
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The Shallow Water Equations can be rewritten as,

\[
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  q_2
\end{bmatrix}
_t + \begin{bmatrix}
  \frac{q_2^2}{q_1} + \frac{1}{2}gq_1^2 \\
  q_2
\end{bmatrix}
_x = 0
\]
What is a Riemann Problem?

A Riemann Problem is an initial boundary value problem for the conservation law with a piecewise constant initial condition.
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Riemann problems for a single nonlinear equation:
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Riemann problems for a nonlinear system of equations:
Rarefaction Waves

What is a Rarefaction Wave?

It is a continuous solution to a Riemann Problem.

Conditions of a Rarefaction Wave:

\[ F'(u_r) > F'(u_l) \]
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It is a *continuous* solution to a Riemann Problem.
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A rarefaction wave is solved in a **single nonlinear equation** by using a solution that is self-similar,

\[ u(x, t) = u \left( \frac{x}{t} \right) \]
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Consider Burgers’ equation as an example,

\[ F(u) = \frac{1}{2}u^2 \]
A rarefaction wave is solved in a **single nonlinear equation** by using a solution that is self-similar,

\[
u(x, t) = u\left(\frac{x}{t}\right)\]

Consider Burgers’ equation as an example,

\[
F(u) = \frac{1}{2}u^2
\]

Plugged into the conservation law,

\[
\frac{\partial \left( u \left( \frac{x}{t} \right) \right)}{\partial t} + \frac{\partial \left( \frac{1}{2} u \left( \frac{x}{t} \right)^2 \right)}{\partial x} = 0
\]
Rarefaction Wave Solution

\[ u(x, t) = (F')^{-1} \left( \frac{x}{t} \right) \]

where \( F' \) is the speed of the characteristic paths \( u \) in the structure of the rarefaction wave.
Rarefaction Waves

Rarefaction Wave Solution

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where \( F' \) is the speed of the characteristic paths \( u \) in the structure of the rarefaction wave.

A similar process can be done to solve for the structure inside a nonlinear system of equations by choosing a self-similar solution,

Self-Similar Solution for a System of Nonlinear Equations

\[ q(x, t) = q \left( \frac{x}{t} \right) \]
Plugged into the conservation law,

\[ DF(q) \cdot q' \left( \frac{x}{t} \right) = \left( \frac{x}{t} \right) \cdot q' \left( \frac{x}{t} \right) \]
Plugged into the conservation law,

\[ DF(q) \cdot q' \left( \frac{x}{t} \right) = \left( \frac{x}{t} \right) \cdot q' \left( \frac{x}{t} \right) \]

where

\[
DF(q) = \begin{bmatrix}
0 & 1 \\
- \left( \frac{q_2}{q_1} \right)^2 + gq_1 & \frac{2q_2}{q_1}
\end{bmatrix}
\]
Plugged into the conservation law,

\[ DF(q) \cdot q' \left( \frac{x}{t} \right) = \left( \frac{x}{t} \right) \cdot q' \left( \frac{x}{t} \right) \]

where

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DF(q) = \begin{bmatrix}
0 & 1 \\
- \left( \frac{q_2}{q_1} \right)^2 + gq_1 & 1 - \frac{2q_2}{q_1}
\end{bmatrix}
\]

Similar to the eigenvector equation,

\[ Ax = \lambda x \]

Therefore \( q' \left( \frac{x}{t} \right) \) is an eigenvector with corresponding eigenvalue \( \lambda \).
Finding the eigenvalues and eigenvectors of $DF(q)$,
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**Eigenvalues**

$$\lambda_1 = u - \sqrt{gh}$$

$$\lambda_2 = u + \sqrt{gh}$$
Finding the eigenvalues and eigenvectors of $DF(q)$,

**Eigenvalues**

$$\lambda_1 = u - \sqrt{gh}$$

$$\lambda_2 = u + \sqrt{gh}$$

**Eigenvectors**

$$\vec{r}_1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}$$

$$\vec{r}_2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}$$
Rarefaction Waves

Going back to the equation,

$$q\left(\frac{x}{t}\right) = (\lambda_p)^{-1} \left(\frac{x}{t}\right)$$

where $q\left(\frac{x}{t}\right)$ is the state on a curve corresponding to $\lambda_p$. 

**Self-Similar Equation**
Going back to the equation,

\[
q \left( \frac{x}{t} \right) = (\lambda_p)^{-1} \left( \frac{x}{t} \right)
\]

where \( q \left( \frac{x}{t} \right) \) is the state on a curve corresponding to \( \lambda_p \).

What is an integral curve?
Rarefaction Waves

Going back to the equation,

\[
q \left( \frac{x}{t} \right) = (\lambda_p)^{-1} \left( \frac{x}{t} \right)
\]

where \( q \left( \frac{x}{t} \right) \) is the state on a curve corresponding to \( \lambda_p \).

What is an integral curve?

Integral Curve

A curve of the vector field \( \vec{r}_p \) that has a tangent vector at each point \( q \) that is an eigenvector of \( DF(q) \) corresponding to the eigenvalue \( \lambda_p(q) \).
Integral Curves

Finding the integral curves by plotting the points \((h, u)\) on the vector field by solving for the system of differential equations obtained by the eigenvectors \(\vec{r}_1\) and \(\vec{r}_2\).
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Differential Equations of Eigenvector \(\vec{r}_1\)

\[
\frac{dq_1}{dt} = 1
\]

\[
\frac{dq_2}{dt} = \frac{q_2}{q_1} - \sqrt{g \cdot q_1}
\]
Integral Curves

Finding the integral curves by plotting the points \((h, u)\) on the vector field by solving for the system of differential equations obtained by the eigenvectors \(\vec{r}_1\) and \(\vec{r}_2\).

Integral Curves of \(\vec{r}_1\)

\[
h = h_* \\
hu = hu_* + 2h \left( \sqrt{gh_*} - \sqrt{gh} \right)
\]
Finding the integral curves by plotting the points \((h, u)\) on the vector field by solving for the system of differential equations obtained by the eigenvectors \(\vec{r}_1\) and \(\vec{r}_2\).

**Differential Equations of Eigenvector \(\vec{r}_2\)**

\[
\frac{dq_1}{dt} = 1 \\
\frac{dq_2}{dt} = \frac{q_2}{q_1} - \sqrt{g \cdot q_1}
\]
Finding the integral curves by plotting the points \((h, u)\) on the vector field by solving for the system of differential equations obtained by the eigenvectors \(\vec{r}_1\) and \(\vec{r}_2\).

Integral Curves of \(\vec{r}_2\)

\[
h = h_*
\]

\[
hu = hu_* + 2h \left( \sqrt{gh_*} - \sqrt{gh} \right)
\]
Directional Derivatives

Need to find the section of the integral curve that is increasing when dealing with rarefactions waves.
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Directional Derivatives

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Recall the solution in the beginning,

**Speed of Rarefaction Waves Equation**

\[ q \left( \frac{x}{t} \right) = (\lambda_1)^{-1} \left( \frac{x}{t} \right) \]
Recall the solution in the beginning,

**Speed of Rarefaction Waves Equation**

\[ q \left( \frac{x}{t} \right) = \left( \lambda_1 \right)^{-1} \left( \frac{x}{t} \right) \]
Once the speeds of the characteristic paths inside the rarefaction wave are found and the states corresponding to the speeds on the integral curve, the structure inside the rarefaction wave can be displayed.
All-Rarefaction Riemann Problem
All-Rarefaction Riemann Problem

![Graph showing the relationship between $hu$ and $h$ with points $q_l$ and $q_r$.]
All-Rarefaction Riemann Problem

$h_u$

$q_r$

$q_m$

$q_l$

$R_1(q_l)$
All-Rarefaction Riemann Problem

$h u$

$h$

$q_l$

$q_m$

$q_r$

$R_1(q_l)$
All-Rarefaction Riemann Problem

$h u$

$R_2(q_m)$

$q_m$

$q_l$

$h$

$R_1(q_l)$

$q_r$

$h u$

$q_l$

$q_m$

$q_r$
What is a shockwave?

It is a discontinuous solution to a Riemann Problem.
What is a shockwave?

It is a **discontinuous** solution to a Riemann Problem.
What is a shockwave?

It is a **discontinuous** solution to a Riemann Problem.

A shockwave forms in a **single nonlinear equation**:

\[ s(u_l - u_r) = F(u_l) - F(u_r) \]
What is a shockwave?

**Shockwave**

It is a **discontinuous** solution to a Riemann Problem.

A shockwave forms in a **single nonlinear equation**:

**Lax Entropy Condition**

\[ F'(u_l) > F'(u_r) \]
A shockwave forms in a non-linear system of equations:

\[ s(q_* - q) = f(q_*) - f(q) \]
A shockwave forms in a non-linear system of equations:

\[ s(q_* - q) = f(q_*) - f(q) \]
A shockwave forms in a non-linear system of equations:

\[ \lambda_p(q_L) > \lambda_p(q_R) \]
A shockwave forms in a non-linear system of equations:

\[ \lambda_{p}(q_{L}) > \lambda_{p}(q_{R}) \]
State Space

Note: Integral curves of the state space for an arbitrary state $q$, share the same tangent vector at the base state but are not equal curves.
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Shockwaves

Produces a system of algebraic equations when plugged into the jump conditions,

**Shallow Water Equations**

\[
\begin{align*}
    s(h_\ast - h) &= h_\ast u_\ast - hu \\
    s(h_\ast q_\ast - hu) &= h_\ast u_\ast^2 - hu^2 + \frac{1}{2}g(h_\ast^2 - h^2)
\end{align*}
\]
Shockwaves

Produces a system of algebraic equations when plugged into the jump conditions,

**Shallow Water Equations**

\[
s(h_* - h) = h_*u_* - hu
\]
\[
s(h_*q_* - hu) = h_*u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2)
\]

After solving for the 3 unknowns,

**Equation of the Shockwave Solution**

\[
u(h) = u_* \pm (h_* - h)\sqrt{\frac{g}{2} \left( \frac{1}{h} + \frac{1}{h_*} \right)}
\]
Shockwaves

Hugoniot Loci

“−” corresponds to the 1-Shockwave

“+” corresponds to the 2-Shockwave
Shockwaves

Hugoniot Loci

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“+” corresponds to the 2-Shockwave
The following diagram illustrates the relationship between $h$ and $hu$ for Riemann problems of the shallow water equations. The points $q_l$ and $q_r$ represent specific states in the phase space.
$S_1(q_l) \rightarrow q_m \rightarrow S_2(q_m)$
\[
\phi_l(h) = \begin{cases} 
  u_l + 2 \left( \sqrt{gh_l} - \sqrt{gh} \right), & \text{if } h < h_l, \\
  u_l - (h - h_l) \sqrt{\frac{g}{2} \left( \frac{1}{h} + \frac{1}{h_l} \right)}, & \text{if } h > h_l 
\end{cases}
\]

\[
\phi_r(h) = \begin{cases} 
  u_r - 2 \left( \sqrt{gh_r} - \sqrt{gh} \right), & \text{if } h < h_r, \\
  u_r + (h - h_r) \sqrt{\frac{g}{2} \left( \frac{1}{h} + \frac{1}{h_r} \right)}, & \text{if } h > h_r 
\end{cases}
\]
Dam-Break Solution

\[ h \]

\[ q_l \]

\[ q_r \]

\[ h u \]

\[ q_r \]

\[ q_l \]
Dam-Break Solution

\[ h \]

\[ q_l \]

\[ q_r \]

\[ hu \]

\[ q_l \]

\[ q_r \]

\[ R_1(q_l) \]

\[ q_m \]
Dam-Break Solution

\[ q_l \quad h \quad q_m \quad hu \quad q_r \]
Theorem

Given \( q_l = \begin{pmatrix} q_{1l} \\ q_{2l} \end{pmatrix} \) and \( q_r = \begin{pmatrix} q_{1r} \\ q_{2r} \end{pmatrix} \), where \( q_{1l}, q_{1r} > 0 \), there exists a solution to determine the intermediate state \( q_m \) to the Riemann Problem of the Shallow Water Equations.
