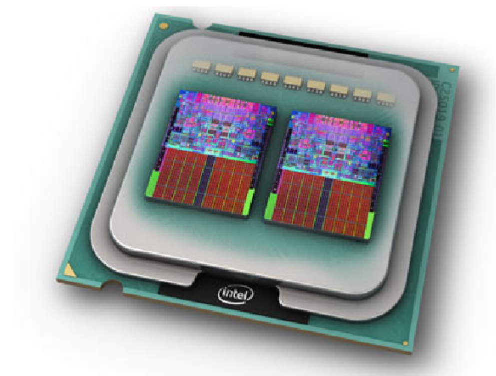
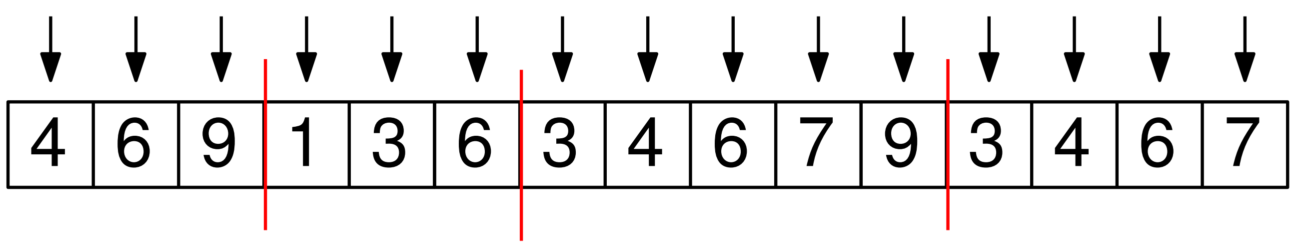
INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

AlgoPARC

ICS 443: Parallel Algorithms

Prof. Nodari Sitchinava

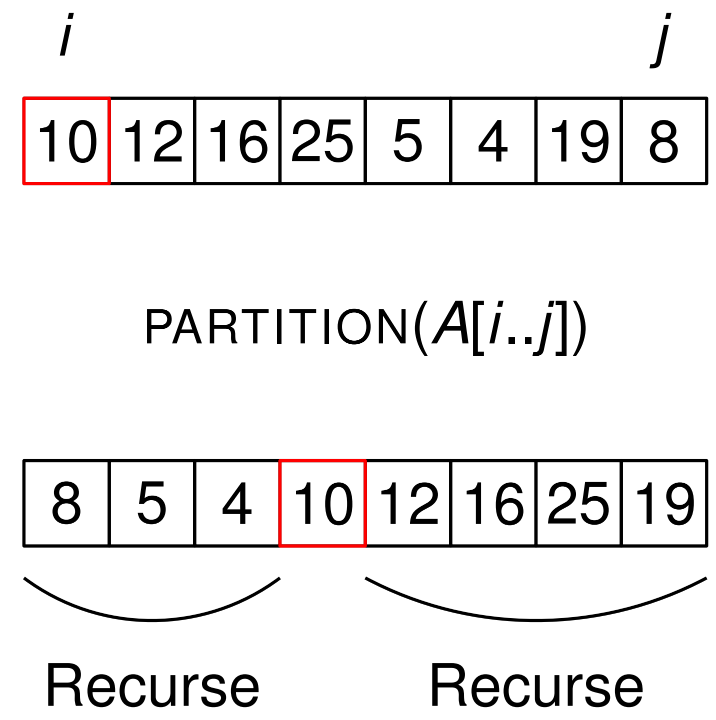




Lecture 6: Segmented Prefix Sums

QUICKSORT Review

procedure QUICKSORT($A[i..j]$)
  if $i < j$ **then**
    $pivot$ = RANDOM(i, j)
    SWAP($A[i], A[pivot]$)
    k = PARTITION($A[i..j]$)
    in parallel do
      QUICKSORT($A[i..k-1]$)
      QUICKSORT($A[k+1..j]$)



QUICKSORT Review

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  if i < j then
      pivot = RANDOM(i, j)
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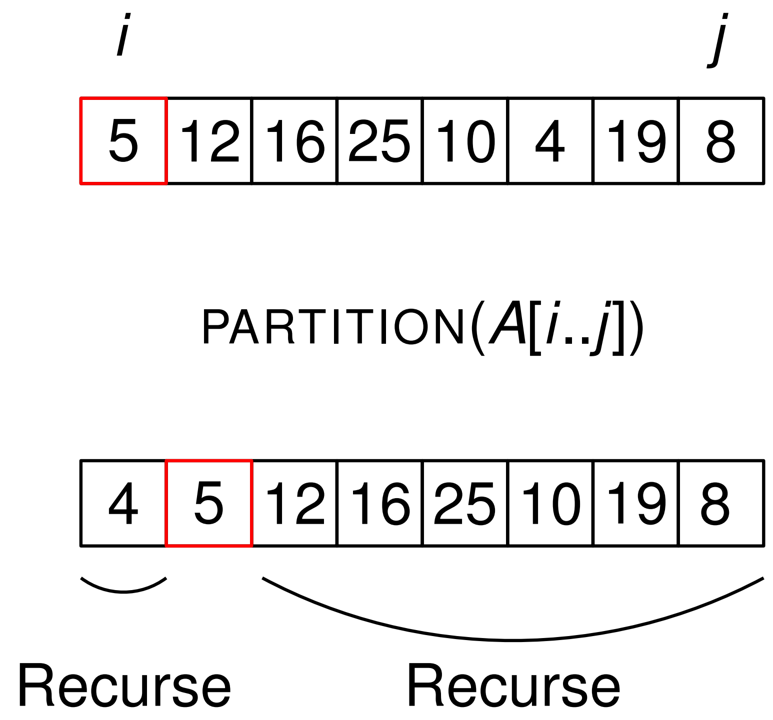
      k = PARTITION(A[i..j])
      in parallel do
          QUICKSORT(A[i..k − 1])
          QUICKSORT(A[k + 1..j])
```



QUICKSORT Review

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procedure QUICKSORT(A[i..j])
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      pivot = RANDOM(i, j)
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```
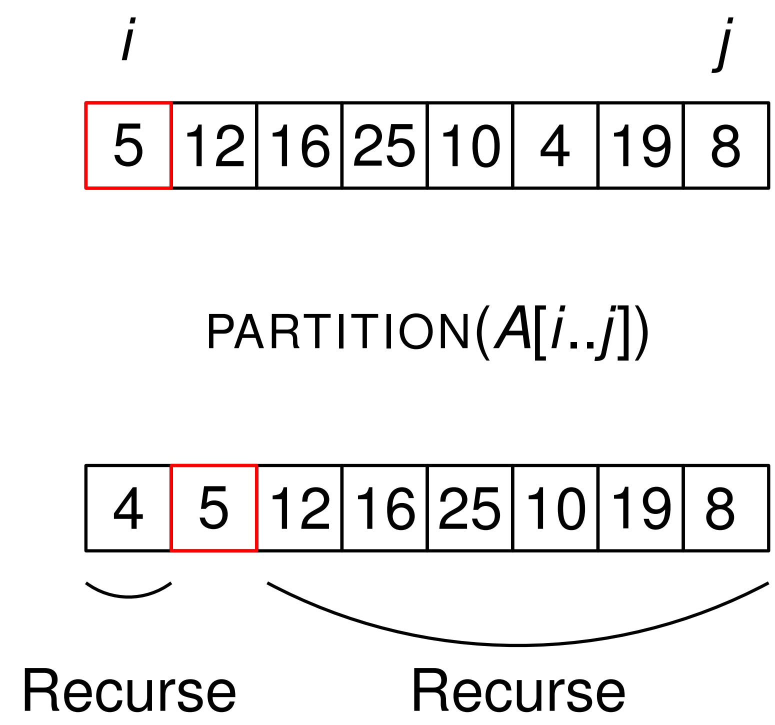



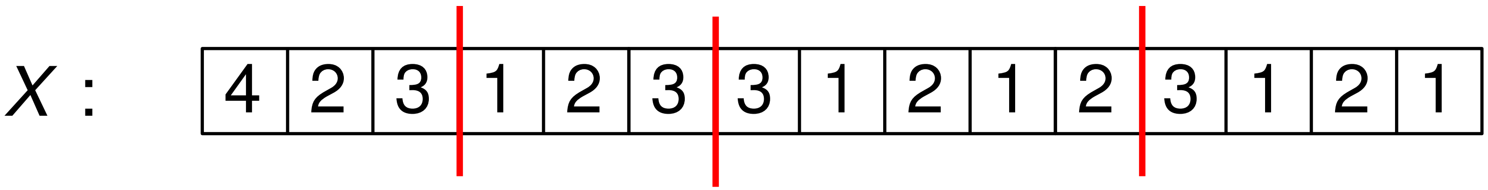
Allocating processors to recursive calls?

Segmented Prefix Sums

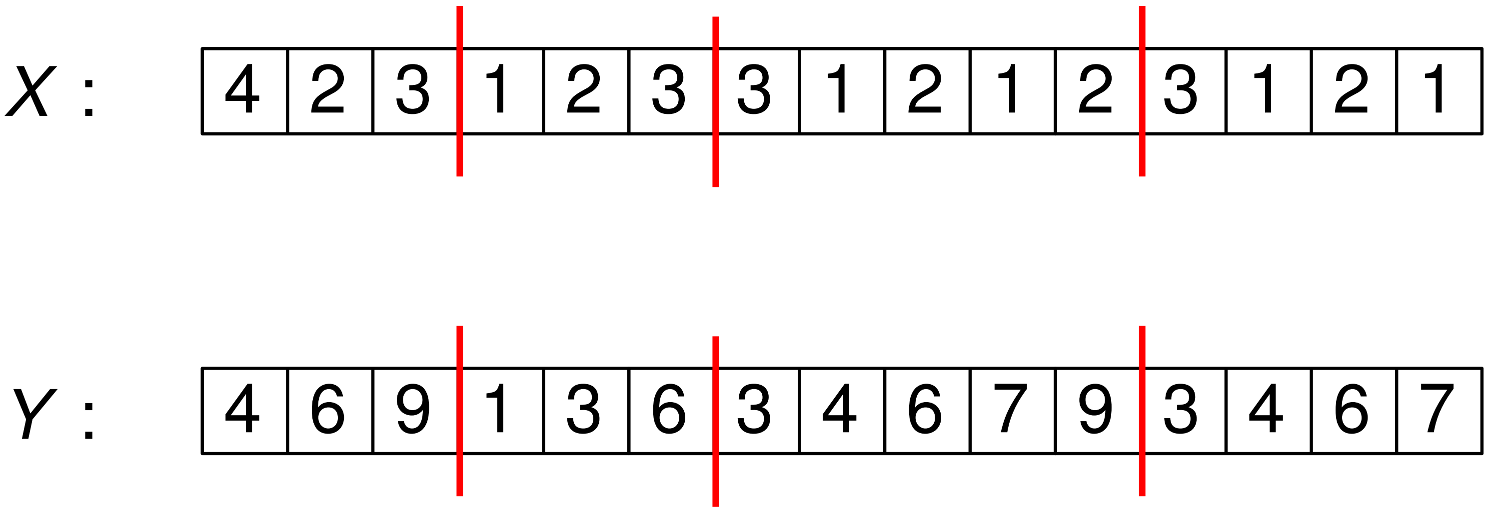
X :

4	2	3	1	2	3	3	1	2	1	2	3	1	2	1

Segmented Prefix Sums



Segmented Prefix Sums



Segmented Prefix Sums

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

Segmented Prefix Sums

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
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Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

Goal: Segmented Prefix Sums via an associative operator $\bullet$

Reminder: Prefix Sums

X :	3	1	5	2	3	4	1	5	7	2	1	5	2	1	6	9
Y :	3	4	9	11	14	18	19	24	31	33	34	39	41	42	48	57

Reminder: Prefix Sums

X : 3 1 5 2 3 4 1 5 7 2 1 5 2 1 6 9

Y : 3 4 9 11 14 18 19 24 31 33 34 39 41 42 48 57

$y_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Reminder: Prefix Sums

X :	3	1	5	2	3	4	1	5	7	2	1	5	2	1	6	9
Y :	3	4	9	11	14	18	19	24	31	33	34	39	41	42	48	57

$y_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

$$y_i = \sum_{k=1}^{i} x_i$$

Reminder: Prefix Sums

X : 3 1 5 2 3 4 1 5 7 2 1 5 2 1 6 9

Y : 3 4 9 11 14 18 19 24 31 33 34 39 41 42 48 57

$$y_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$y_i = \sum_{k=1}^{i} x_i$$

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} + x_i & \text{if } i > 1 \end{cases}$$

Prefix Scan via $\oplus$

Prefix Sums

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} + x_i & \text{if } i > 1 \end{cases}$$

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$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \min(y_{i-1}, x_i) & \text{if } i > 1 \end{cases}$$

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Prefix copy (broadcast)

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \text{COPY}(y_{i-1}, x_i) & \text{if } i > 1 \end{cases}$$

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For any associative operator $\oplus$

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

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double $\oplus$(**double** *a*, **double** *b*)

  return $a + b$ ▷ sum

Prefix Scan via $\oplus$

Prefix Sums

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double $\oplus$(**double** *a*, **double** *b*)

  return min(*a*, *b*) ▷ min

Prefix Scan via $\oplus$

Prefix Sums

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```
double ⊕(double a, double b)
    return a                      ▷ copy
```

Prefix Scan via $\oplus$

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E.g.: $y_4 = ((x_1 \oplus x_2) \oplus x_3) \oplus x_4$

$\qquad = x_1 \oplus x_2 \oplus x_3 \oplus x_4$

Prefix Scan via $\oplus$

Prefix Sums

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$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

E.g.: $y_4 = ((x_1 \oplus x_2) \oplus x_3) \oplus x_4$

$= x_1 \oplus x_2 \oplus x_3 \oplus x_4$

procedure SCAN($\oplus$, X, i)
  if $i = 1$ **then**
    return $X[1]$
  else
    return SCAN($\oplus, X, i - 1$) $\oplus X[i]$

Recursive Segmented Prefix Sums

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

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Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} & \text{if } b_i = 0 \\ & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

Recursive Segmented Prefix Sums

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X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
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$$= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} \times 1 + x_i & \text{if } b_i = 0 \\ y_{i-1} \times 0 + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

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$$= \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \times (1 - b_i) + x_i & \text{if } i > 1 \end{cases}$$

Recursive Segmented Prefix Sums

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
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$$
y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} + x_i & \text{if } b_i = 0 \\ x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}
$$

$$
= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} \times 1 + x_i & \text{if } b_i = 0 \\ \boxed{y_{i-1} \times 0 + x_i} & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}
$$

$0 + x_i = x_i$

$$
= \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \times (1 - b_i) + x_i & \text{if } i > 1 \end{cases}
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Recursive Segmented Prefix Sums

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$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} + x_i & \text{if } b_i = 0 \\ x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

$$= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} \times 1 + x_i & \text{if } b_i = 0 \\ y_{i-1} \times 0 + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

$y_{i-1} \times 1 = y_{i-1}$

$0 + x_i = x_i$

$$= \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \times (1 - b_i) + x_i & \text{if } i > 1 \end{cases}$$

Recursive Segmented Prefix Sums

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
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$$= \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \times (1 - b_i) + x_i & \text{if } i > 1 \end{cases}$$

$I_\times = 1$ is the *identity* of $\times$: $y_{i-1} \times 1 = y_{i-1}$

$0 + x_i = x_i$: $I_+ = 0$ is the *identity* of $+$

$\otimes$ operator

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} (y_{i-1} \times 1) + x_i & \text{if } b_i = 0 \\ (y_{i-1} \times 0) + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

$$= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} + x_i & \text{if } b_i = 0 \\ 0 + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

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$\otimes$ operator

$$
y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} (y_{i-1} \times 1) \oplus x_i & \text{if } b_i = 0 \\ (y_{i-1} \times 0) \oplus x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}
$$

$$
= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} + x_i & \text{if } b_i = 0 \\ 0 + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}
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$$= \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} + x_i & \text{if } b_i = 0 \\ I_+ + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

$\otimes$ operator

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \boxed{\begin{cases} (y_{i-1} \otimes b_i) \oplus x_i & \text{if } b_i = 0 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } b_i = 1 \end{cases}} & \text{if } i > 1 \end{cases}$$

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$$\otimes : \mathbb{R} \times \{0, 1\} \rightarrow \mathbb{R}$$

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$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} (y_{i-1} \otimes b_i) \oplus x_i & \text{if } b_i = 0 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

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Examples:

$\oplus$	$I_\oplus$	$\otimes$
$+$		
MIN		
COPY		

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$0 + z = z$

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Examples:

$\oplus$	$I_\oplus$	$\otimes$
$+$	0	$z \cdot (1 - b)$
MIN		
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Examples:

$\oplus$	$I_\oplus$	$\otimes$	
$+$	0	$z \cdot (1 - b)$	$0 + z = z$
MIN	??		$\min(??, z) = z$
COPY			

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Examples:

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MIN	∞	
COPY		

$0 + z = z$

$\min(\infty, z) = z$

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Examples:

$\oplus$	$I_\oplus$	$\otimes$	
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MIN	∞	$z \cdot (1 - b) + \infty \cdot b$	$\min(\infty, z) = z$
COPY			

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$\oplus$	$I_\oplus$	$\otimes$	
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MIN	∞	$z \cdot (1 - b) + \infty \cdot b$	$\min(\infty, z) = z$
COPY	??	*undefined*	$copy(??, z) = z$

Segmented Scan via $\oplus$ and $\otimes$

Segmented Prefix Sums

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ \begin{cases} y_{i-1} \times 1 + x_i & \text{if } b_i = 0 \\ y_{i-1} \times 0 + x_i & \text{if } b_i = 1 \end{cases} & \text{if } i > 1 \end{cases}$$

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b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

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Algebra with $\oplus$ and $\otimes$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
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Distributive Property of $\otimes$

Property 1. $\otimes$ *distributes over* $\oplus$*:*
$(x \oplus y) \otimes b = (x \otimes b) \oplus (y \otimes b)$

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Simplifying Expression

$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \Big(\big((x_1 \otimes b_2) \oplus x_2 \big) \otimes b_3 \Big) \oplus x_3$$

Simplifying Expression

$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \left(\boxed{\big((x_1 \otimes b_2) \oplus x_2\big)} \otimes \boxed{b_3}\right) \oplus x_3$$

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$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \Big(\big((x_1 \otimes b_2) \oplus x_2\big) \otimes b_3\Big) \oplus x_3$$

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Is $\otimes$ associative?

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$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_{\oplus} & \text{if } b = 1 \end{cases}$$

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Interface: $\left(\mathbb{R} \otimes \{0, 1\}\right) \otimes \{0, 1\}$

$$\mathbb{R} \otimes \{0, 1\}$$

$$\downarrow$$

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Truth Table

b_2	b_3	$(x_1 \otimes b_2) \otimes b_3$	$b_2 \vee b_3$	$x_1 \otimes (b_2 \vee b_3)$
0	0		0	
0	1		1	
1	0		1	
1	1		1	

Simplifying Expression

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Property 2. $\otimes$ *is NOT associative*

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Property 2. *$\otimes$ is semi-associative, with $\vee$ as the companion operator:* $(x_1 \otimes b_1) \otimes b_2 = x_1 \otimes (b_1 \vee b_2)$

Simplifying Expression

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Simplifying Expression

$$
\begin{aligned}
y_3 = (y_2 \otimes b_3) \oplus x_3 &= \Big(\big((x_1 \otimes b_2) \oplus x_2 \big) \otimes b_3 \Big) \oplus x_3 \\
&= \Big(\big((x_1 \otimes b_2) \otimes b_3 \big) \oplus \big(x_2 \otimes b_3 \big) \Big) \oplus x_3 \\
&= \Big(\big(x_1 \otimes (b_2 \vee b_3) \big) \oplus \big(x_2 \otimes b_3 \big) \Big) \oplus x_3 \\
&= \big(x_1 \otimes (b_2 \vee b_3) \big) \oplus \big(x_2 \otimes b_3 \big) \oplus x_3
\end{aligned}
$$

Simplifying Expression

$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \Big(\big((x_1 \otimes b_2) \oplus x_2\big) \otimes b_3\Big) \oplus x_3$$

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$$y_4 = (y_3 \otimes b_4) \oplus x_4 = \Big[\Big(\big(x_1 \otimes (b_2 \vee b_3)\big) \oplus \big(x_2 \otimes b_3\big) \oplus x_3\Big) \otimes b_4\Big] \oplus x_4$$

Simplifying Expression

$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \Big(\big((x_1 \otimes b_2) \oplus x_2\big) \otimes b_3\Big) \oplus x_3$$

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$$= \Big[\big([x_1 \otimes (b_2 \vee b_3)] \otimes b_4\big) \oplus \big((x_2 \otimes b_3) \otimes b_4\big) \oplus \big(x_3 \otimes b_4\big)\Big] \oplus x_4$$

Simplifying Expression

$$y_3 = (y_2 \otimes b_3) \oplus x_3 = \Big(\big((x_1 \otimes b_2) \oplus x_2\big) \otimes b_3\Big) \oplus x_3$$

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Property 2. $\otimes$ *is semi-associative, with* $\vee$ *as the companion operator:*
$(x_1 \otimes b_1) \otimes b_2 = x_1 \otimes (b_1 \vee b_2)$

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$$y_1 = x_1$$

$$y_2 = (x_1 \otimes b_2) \oplus x_2$$

$$y_3 = \big(x_1 \otimes (b_2 \vee b_3)\big) \oplus \big(x_2 \otimes b_3\big) \oplus x_3$$

$$y_4 = \big(x_1 \otimes (b_2 \vee b_3 \vee b_4)\big) \oplus \big(x_2 \otimes (b_3 \vee b_4)\big) \oplus \big(x_3 \otimes b_4\big) \oplus x_4$$

$$y_1 = x_1$$

$$y_2 = (x_1 \otimes b_2) \oplus x_2$$

$$y_3 = \big(x_1 \otimes (b_2 \vee b_3)\big) \oplus \big(x_2 \otimes b_3\big) \oplus x_3$$

$$y_4 = \big(x_1 \otimes (b_2 \vee b_3 \vee b_4)\big) \oplus \big(x_2 \otimes (b_3 \vee b_4)\big) \oplus \big(x_3 \otimes b_4\big) \oplus x_4$$

$$y_5 = \big(x_1 \otimes (b_2 \vee b_3 \vee b_4 \vee b_5)\big) \oplus \big(x_2 \otimes (b_3 \vee b_4 \vee b_5)\big)$$

$$\oplus\big(x_3 \otimes (b_4 \vee b_5)\big) \oplus (x_4 \vee b_5) \oplus x_5$$

Operator ●

Operator $\bullet$

$$\bullet : \big(\mathbb{R} \times \{0, 1\}\big) \times \big(\mathbb{R} \times \{0, 1\}\big) \rightarrow \big(\mathbb{R} \times \{0, 1\}\big)$$

Operator $\bullet$

$$\bullet : \left(\mathbb{R} \times \{0, 1\}\right) \times \left(\mathbb{R} \times \{0, 1\}\right) \to \left(\mathbb{R} \times \{0, 1\}\right)$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Segmented Scan via $\bullet$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Segmented Scan via $\bullet$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

Segmented Scan via $\bullet$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$$

Segmented Scan via $\bullet$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} y_i \\ b'_i \end{pmatrix} = \begin{cases} \begin{pmatrix} x_1 \\ b_1 \end{pmatrix} & \text{if } i = 1 \\ \begin{pmatrix} y_{i-1} \\ b'_{i-1} \end{pmatrix} \bullet \begin{pmatrix} x_i \\ b_i \end{pmatrix} = \begin{pmatrix} (y_{i-1} \otimes b_i) \oplus x_i \\ b'_{i-1} \vee b_i \end{pmatrix} & \text{if } i > 1 \end{cases}$$

Segmented Scan via $\bullet$

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} y_i \\ b'_i \end{pmatrix} = \begin{cases} \begin{pmatrix} x_1 \\ b_1 \end{pmatrix} & \text{if } i = 1 \\ \begin{pmatrix} y_{i-1} \\ b'_{i-1} \end{pmatrix} \bullet \begin{pmatrix} x_i \\ b_i \end{pmatrix} = \begin{pmatrix} (y_{i-1} \otimes b_i) \oplus x_i \\ b'_{i-1} \vee b_i \end{pmatrix} & \text{if } i > 1 \end{cases}$$

Segmented Scan via •

b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ (y_{i-1} \otimes b_i) \oplus x_i & \text{if } i > 1 \end{cases}$$

$$\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} = \begin{pmatrix} (x \otimes b_2) \oplus y \\ b_1 \vee b_2 \end{pmatrix}$$

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

Segmented scan

$$\begin{pmatrix} y_i \\ b'_i \end{pmatrix} = \begin{cases} \begin{pmatrix} x_1 \\ b_1 \end{pmatrix} & \text{if } i = 1 \\ \begin{pmatrix} y_{i-1} \\ b'_{i-1} \end{pmatrix} \bullet \begin{pmatrix} x_i \\ b_i \end{pmatrix} = \begin{pmatrix} (y_{i-1} \otimes b_i) \oplus x_i \\ b'_{i-1} \vee b_i \end{pmatrix} & \text{if } i > 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
Y :															
b' :															

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
Y :	4														
b' :	1														

$$\binom{y_1}{b'_1} = \binom{x_1}{b_1} = \binom{4}{1}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \rightarrow \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via •

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
Y :	4	6													
b' :	1	1													

$$\binom{y_2}{b'_2} = \binom{(y_1 \otimes b_2) \oplus x_2}{b'_1 \vee b_2} = \binom{(4 \otimes 0) \oplus 2}{1 \vee 0}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_{\oplus} & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0

Y :	4	6	9												
b' :	1	1	1												

$$\binom{y_3}{b'_3} = \binom{(y_2 \otimes b_3) \oplus x_3}{b'_2 \vee b_3} = \binom{(6 \otimes 0) \oplus 3}{1 \vee 0}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0

Y :	4	6	9	1											
b' :	1	1	1	1											

$$\binom{y_4}{b'_4} = \binom{(y_3 \otimes b_4) \oplus x_4}{b'_3 \vee b_4} = \binom{(9 \otimes 1) \oplus 1}{1 \vee 1}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via •

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
Y :	4	6	9	1	3										
b' :	1	1	1	1	1										

$$\binom{y_5}{b'_5} = \binom{(y_4 \otimes b_5) \oplus x_5}{b'_4 \vee b_5} = \binom{(1 \otimes 0) \oplus 2}{1 \vee 0}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_{\oplus} & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via •

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0
Y :	4	6	9	1	3	6									
b' :	1	1	1	1	1	1									

$$\binom{y_6}{b'_6} = \binom{(y_5 \otimes b_6) \oplus x_6}{b'_5 \vee b_6} = \binom{(3 \otimes 0) \oplus 3}{1 \vee 0}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0

Y :	4	6	9	1	3	6	3								
b' :	1	1	1	1	1	1	1								

$$\binom{y_7}{b'_7} = \binom{(y_6 \otimes b_7) \oplus x_7}{b'_6 \vee b_7} = \binom{(6 \otimes 1) \oplus 3}{1 \vee 1}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_\oplus & \text{if } b = 1 \end{cases}$$

Segmented Prefix Sums via $\bullet$

X :	4	2	3	1	2	3	3	1	2	1	2	3	1	2	1
b :	1	0	0	1	0	0	1	0	0	0	0	1	0	0	0

Y :	4	6	9	1	3	6	3	4	6	7	9	3	4	6	7
b' :	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} = \binom{(y_{i-1} \otimes b_i) \oplus x_i}{b'_{i-1} \vee b_i} & \text{if } i > 1 \end{cases}$$

$$\otimes : \mathbb{R} \times \{0,1\} \to \mathbb{R}$$

$$z \otimes b = \begin{cases} z & \text{if } b = 0 \\ I_{\oplus} & \text{if } b = 1 \end{cases}$$

$\bullet$ is associative

$$\bullet : \big(\mathbb{R} \times \{0,1\}\big) \times \big(\mathbb{R} \times \{0,1\}\big) \to \big(\mathbb{R} \times \{0,1\}\big)$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix}\right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix}\right)$$

$\bullet$ is associative

$$\bullet : \left(\mathbb{R} \times \{0,1\}\right) \times \left(\mathbb{R} \times \{0,1\}\right) \to \left(\mathbb{R} \times \{0,1\}\right)$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix}\right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3}$$

$\bullet$ is associative

$$\bullet : \big(\mathbb{R} \times \{0,1\}\big) \times \big(\mathbb{R} \times \{0,1\}\big) \to \big(\mathbb{R} \times \{0,1\}\big)$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix}\right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\Big(\big((x \otimes b_2) \oplus y\big) \otimes b_3\Big) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\boxed{\left((x \otimes b_2) \oplus y\right) \otimes b_3}\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. *$\bullet$ is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix} \right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} \right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2} \right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\otimes$ distributes over $\oplus$:

$(x \oplus y) \otimes b = (x \otimes b) \oplus (y \otimes b)$

• is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix}\right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$$= \binom{\left(\left((x \otimes b_2) \otimes b_3\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\otimes$ *distributes over* $\oplus$*:*

$(x \oplus y) \otimes b = (x \otimes b) \oplus (y \otimes b)$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2} \right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3} \right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2} \right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y \right) \otimes b_3 \right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$$= \binom{\left(\boxed{\left((x \otimes b_2) \otimes b_3 \right)} \oplus \left(y \otimes b_3 \right) \right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$$= \binom{\left(\left((x \otimes b_2) \otimes b_3\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\otimes$ *is semi-associative:*
$(x \otimes b_1) \otimes b_2) = x \otimes (b_1 \vee b_2)$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

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$$= \binom{\left(\left(x \otimes (b_2 \vee b_3)\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

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$$\bullet : \big(\mathbb{R} \times \{0,1\}\big) \times \big(\mathbb{R} \times \{0,1\}\big) \to \big(\mathbb{R} \times \{0,1\}\big)$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

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$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

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$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

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$$= \binom{\left(\left(x \otimes (b_2 \vee b_3)\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$\vee$ *is associative:*

$(b_1 \vee b2) \vee b_3 = b_1 \vee (b_2 \vee b_3)$

$\bullet$ is associative

$$\bullet : \left(\mathbb{R} \times \{0,1\}\right) \times \left(\mathbb{R} \times \{0,1\}\right) \rightarrow \left(\mathbb{R} \times \{0,1\}\right)$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \begin{pmatrix} y \\ b_2 \end{pmatrix}\right) \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix} = \begin{pmatrix} x \\ b_1 \end{pmatrix} \bullet \left(\begin{pmatrix} y \\ b_2 \end{pmatrix} \bullet \begin{pmatrix} z \\ b_3 \end{pmatrix}\right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

$$= \binom{\left(\left((x \otimes b_2) \otimes b_3\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{\boxed{(b_1 \vee b_2) \vee b_3}}$$

$$= \binom{\left(\left(x \otimes (b_2 \vee b_3)\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{b_1 \vee (b_2 \vee b_3)}$$

$\vee$ *is associative:*

$(b_1 \vee b2) \vee b_3 = b_1 \vee (b_2 \vee b_3)$

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$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2} \right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3} \right)$$

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2} \right) \bullet \binom{z}{b_3} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2} \bullet \binom{z}{b_3} = \binom{\left(\left((x \otimes b_2) \oplus y\right) \otimes b_3\right) \oplus z}{(b_1 \vee b_2) \vee b_3}$$

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$$= \binom{\left(\left(x \otimes (b_2 \vee b_3)\right) \oplus \left(y \otimes b_3\right)\right) \oplus z}{b_1 \vee (b_2 \vee b_3)}$$

$\bullet$ is associative

$$\bullet : \big(\mathbb{R}\times\{0,1\}\big)\times\big(\mathbb{R}\times\{0,1\}\big)\rightarrow\big(\mathbb{R}\times\{0,1\}\big)$$

$$\binom{x}{b_1}\bullet\binom{y}{b_2}=\binom{(x\otimes b_2)\oplus y}{b_1\vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1}\bullet\binom{y}{b_2}\right)\bullet\binom{z}{b_3}=\binom{x}{b_1}\bullet\left(\binom{y}{b_2}\bullet\binom{z}{b_3}\right)$$

$$\left(\binom{x}{b_1}\bullet\binom{y}{b_2}\right)\bullet\binom{z}{b_3}=\binom{(x\otimes b_2)\oplus y}{b_1\vee b_2}\bullet\binom{z}{b_3}=\binom{\Big(\big((x\otimes b_2)\oplus y\big)\otimes b_3\Big)\oplus z}{(b_1\vee b_2)\vee b_3}$$

$$=\binom{\Big(\big((x\otimes b_2)\otimes b_3\big)\oplus\big(y\otimes b_3\big)\Big)\oplus z}{(b_1\vee b_2)\vee b_3}$$

$$=\binom{\Big(\big(x\otimes(b_2\vee b_3)\big)\oplus\big(y\otimes b_3\big)\Big)\oplus z}{b_1\vee(b_2\vee b_3)}$$

$\oplus$ *is associative:*

$$(p\oplus q)\oplus r)=p\oplus(q\oplus r)$$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

Claim. $\bullet$ *is associative:*

$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

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$$= \binom{\left(x \otimes (b_2 \vee b_3)\right) \oplus \left((y \otimes b_3) \oplus z\right)}{b_1 \vee (b_2 \vee b_3)}$$

$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

$$\binom{x}{b_1} \bullet \binom{y}{b_2} = \binom{(x \otimes b_2) \oplus y}{b_1 \vee b_2}$$

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$\bullet$ is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

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$$\left(\binom{x}{b_1} \bullet \binom{y}{b_2}\right) \bullet \binom{z}{b_3} = \binom{x}{b_1} \bullet \left(\binom{y}{b_2} \bullet \binom{z}{b_3}\right)$$

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$$b' = b_2 \vee b_3$$
$$y' = (y \otimes b_3) \oplus z$$

• is associative

$$\bullet : (\mathbb{R} \times \{0,1\}) \times (\mathbb{R} \times \{0,1\}) \to (\mathbb{R} \times \{0,1\})$$

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Claim. • *is associative:*

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$$= \binom{\left(x \otimes b'\right) \oplus y'}{b_1 \vee b'}$$

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$$= \binom{\big(x \otimes b'\big) \oplus y'}{b_1 \vee b'} = \binom{x}{b_1} \bullet \binom{y'}{b'}$$

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$$\bullet : \big(\mathbb{R} \times \{0,1\}\big) \times \big(\mathbb{R} \times \{0,1\}\big) \to \big(\mathbb{R} \times \{0,1\}\big)$$

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$$= \binom{\big(x \otimes (b_2 \vee b_3)\big) \oplus \big((y \otimes b_3) \oplus z\big)}{b_1 \vee (b_2 \vee b_3)}$$

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$$b' = b_2 \vee b_3$$
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$\bullet$ is associative

$$\bullet : \big(\mathbb{R} \times \{0,1\}\big) \times \big(\mathbb{R} \times \{0,1\}\big) \to \big(\mathbb{R} \times \{0,1\}\big)$$

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$$= \binom{\left(x \otimes (b_2 \vee b_3)\right) \oplus \left((y \otimes b_3) \oplus z\right)}{b_1 \vee (b_2 \vee b_3)}$$

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$$b' = b_2 \vee b_3$$
$$y' = (y \otimes b_3) \oplus z$$

Segmented scan via prefix sums

Scan	Segmented scan
$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$	$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$

Segmented scan via prefix sums

Scan	Segmented scan
$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$	$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$

procedure PREFIX-SUMS($x[1..n]$)
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x'[i] = x[2i-1] + x[2i]$
  PREFIX-SUMS($x'[1..\frac{n}{2}]$)
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x[2i] = x'[i]$
    if $i \neq \frac{n}{2}$ **then**
      $x[2i+1] = x[2i+1] + x'[i]$

Segmented scan via prefix sums

Scan	Segmented scan
$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$	$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$

procedure $\text{SCAN}(x[1..n], \oplus)$
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x'[i] = x[2i-1] \oplus x[2i]$
  $\text{SCAN}(x'[1..\frac{n}{2}])$
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x[2i] = x'[i]$
    if $i \neq \frac{n}{2}$ **then**
      $x[2i+1] = x[2i+1] \oplus x'[i]$

Segmented scan via prefix sums

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$$

procedure SCAN($x[1..n], \oplus$)
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x'[i] = x[2i-1] \oplus x[2i]$
  SCAN($x'[1..\frac{n}{2}]$)
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x[2i] = x'[i]$
    if $i \neq \frac{n}{2}$ **then**
      $x[2i+1] = x[2i+1] \oplus x'[i]$

procedure SEG-SCAN($x[1..n], b[1..n], \bullet$)
  $X[1..n]$ = new array of PAIRS

Segmented scan via prefix sums

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$$

class PAIR
    double first
    bit second

procedure SCAN($x[1..n], \oplus$)
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x'[i] = x[2i-1] \oplus x[2i]$
  SCAN($x'[1..\frac{n}{2}]$)
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x[2i] = x'[i]$
    if $i \neq \frac{n}{2}$ **then**
      $x[2i+1] = x[2i+1] \oplus x'[i]$

procedure SEG-SCAN($x[1..n], b[1..n], \bullet$)
  $X[1..n]$ = new array of PAIRs

Segmented scan via prefix sums

Scan

$$y_i = \begin{cases} x_1 & \text{if } i = 1 \\ y_{i-1} \oplus x_i & \text{if } i > 1 \end{cases}$$

Segmented scan

$$\binom{y_i}{b'_i} = \begin{cases} \binom{x_1}{b_1} & \text{if } i = 1 \\ \binom{y_{i-1}}{b'_{i-1}} \bullet \binom{x_i}{b_i} & \text{if } i > 1 \end{cases}$$

class PAIR
  double first
  bit second

procedure SCAN($x[1..n], \oplus$)
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x'[i] = x[2i-1] \oplus x[2i]$
  SCAN($x'[1..\frac{n}{2}]$)
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
    $x[2i] = x'[i]$
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      $x[2i+1] = x[2i+1] \oplus x'[i]$

procedure SEG-SCAN($x[1..n], b[1..n], \bullet$)
  $X[1..n]$ = new array of PAIRs
  for $i = 1$ to n **in parallel do**
    $X[i].first = x[i]$
    $X[i].second = b[i]$

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  SCAN($X[1..n], \bullet$)
  for $i = 1$ to n **in parallel do**
    $x[i] = X[i].first$