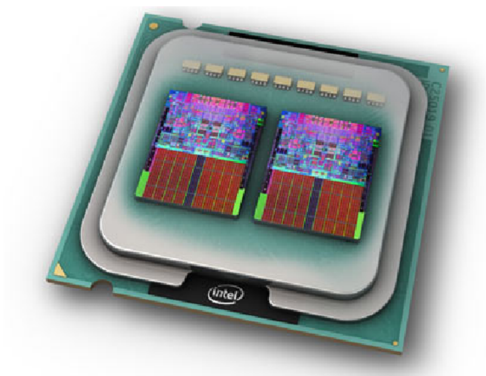
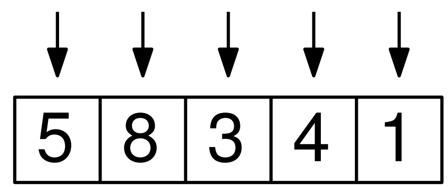
INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

AlgoPARC

ICS 443: Parallel Algorithms

Prof. Nodari Sitchinava





Lecture 4: Prefix Sums

Prefix Sums

for $i = 2$ to n **do**

$\quad a[i] = a[i] + a[i-1]$

return $a[n]$

5	8	3	4	1

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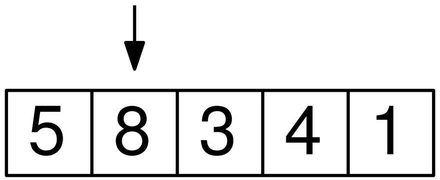

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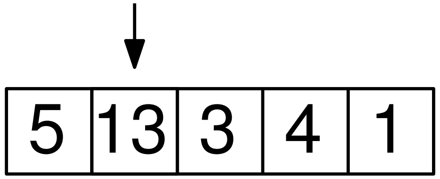

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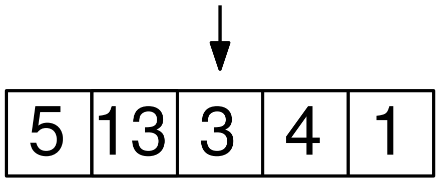

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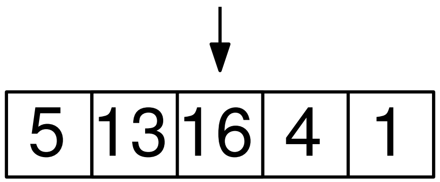

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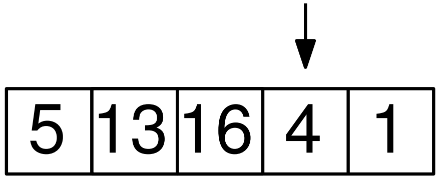

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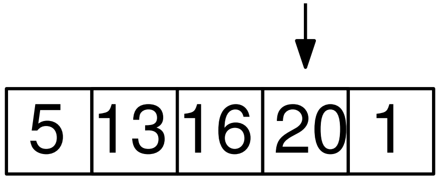

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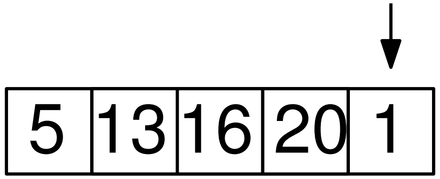

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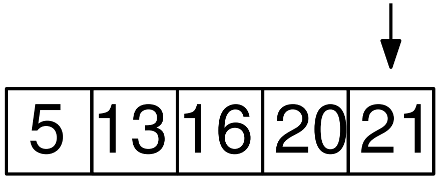

Prefix Sums

Time

for i = 2 to n **do**

$a[i] = a[i] + a[i-1]$

return $a[n]$

5	13	16	20	21

$O(n)$

Prefix Sums

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Prefix Sums

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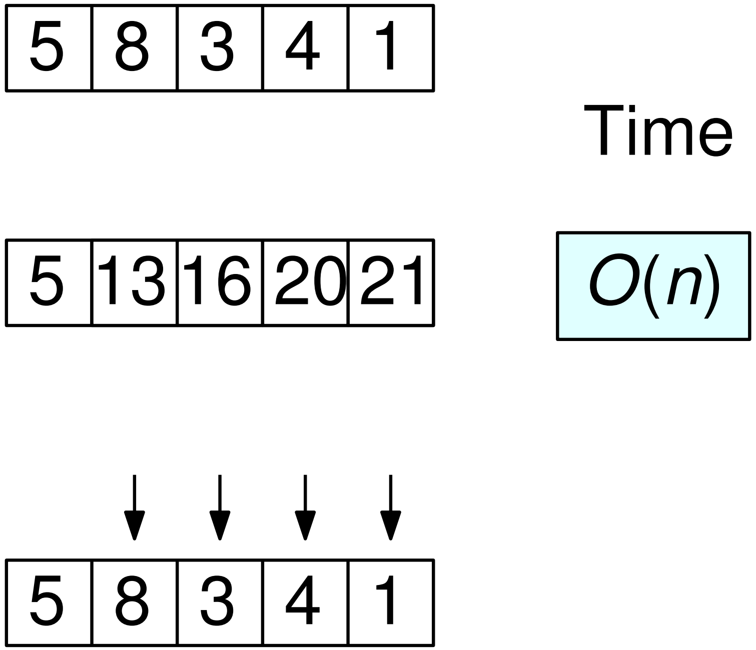

for $i = 2$ to n **do**
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for $i = 2$ to n **in parallel do**
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Start $n-1$ threads $t_2, \ldots t_n$
Each thread t_i (where $i = 2, \ldots, n$) **do**:
    READ $r0_i \leftarrow a[i]$
    EXECUTE
    WRITE
    READ $r1_i \leftarrow a[i-1]$
    EXECUTE $r0_i \leftarrow r0_i + r1_i$
    WRITE $a[i] \leftarrow r0_i$

Prefix Sums

5	8	3	4	1

Time

for $i = 2$ to n **do**
  $a[i] = a[i] + a[i-1]$
return $a[n]$

5	13	16	20	21

$O(n)$

for $i = 2$ to n **in parallel do**
  $a[i] = a[i] + a[i-1]$
return $a[n]$

5	13	11	7	5

Start $n-1$ threads $t_2, \ldots t_n$
Each thread t_i (where $i = 2, \ldots, n$) **do**:
  READ $r0_i \leftarrow a[i]$
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Prefix Sums

5	8	3	4	1	**Time**
5	13	16	20	21	$O(n)$
5	13	11	7	5	$O(1)$

for $i = 2$ to n **do**
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for $i = 2$ to n **in parallel do**
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Parallel Prefix Sums

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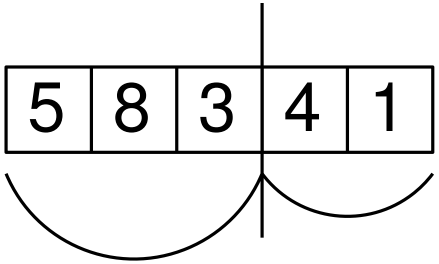

Parallel Prefix Sums

5	8	3	4	1

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4	1

Parallel Prefix Sums

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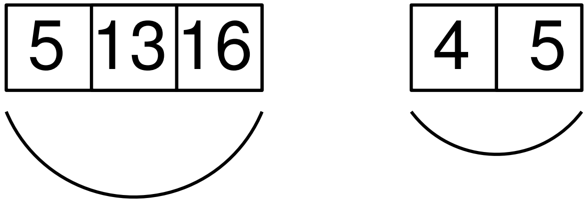

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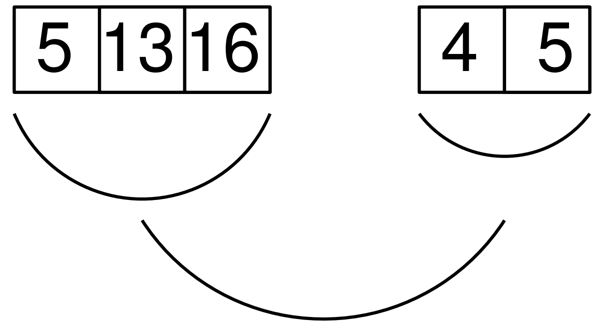

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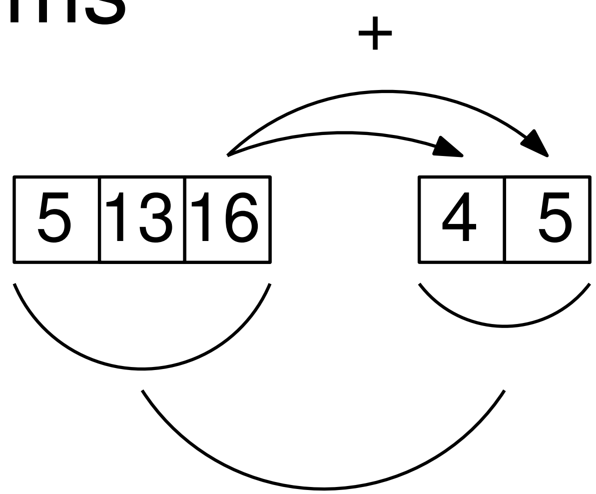

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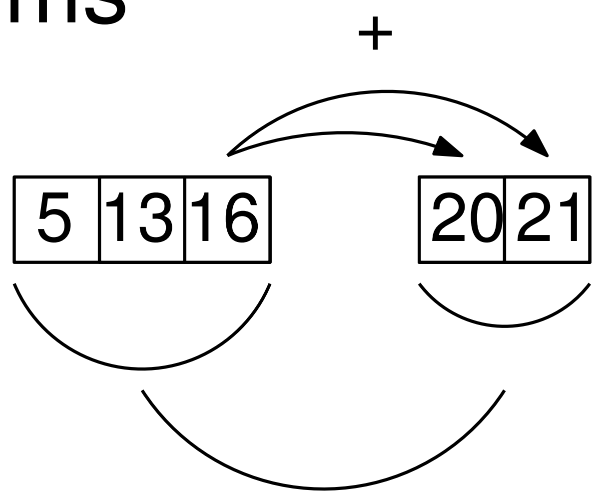

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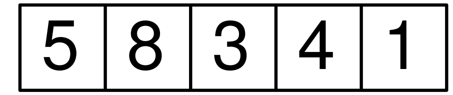
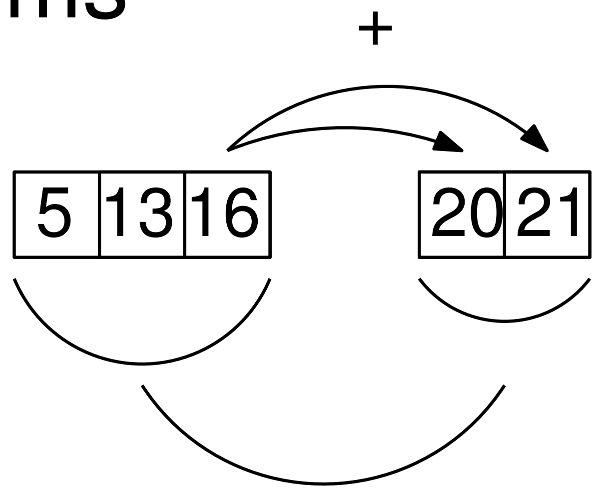




function PREFIX-SUMS(A, i, j)

  if $i \geq j$ **then return** ▷ Base case

  $mid = \left\lfloor \frac{i+j}{2} \right\rfloor$

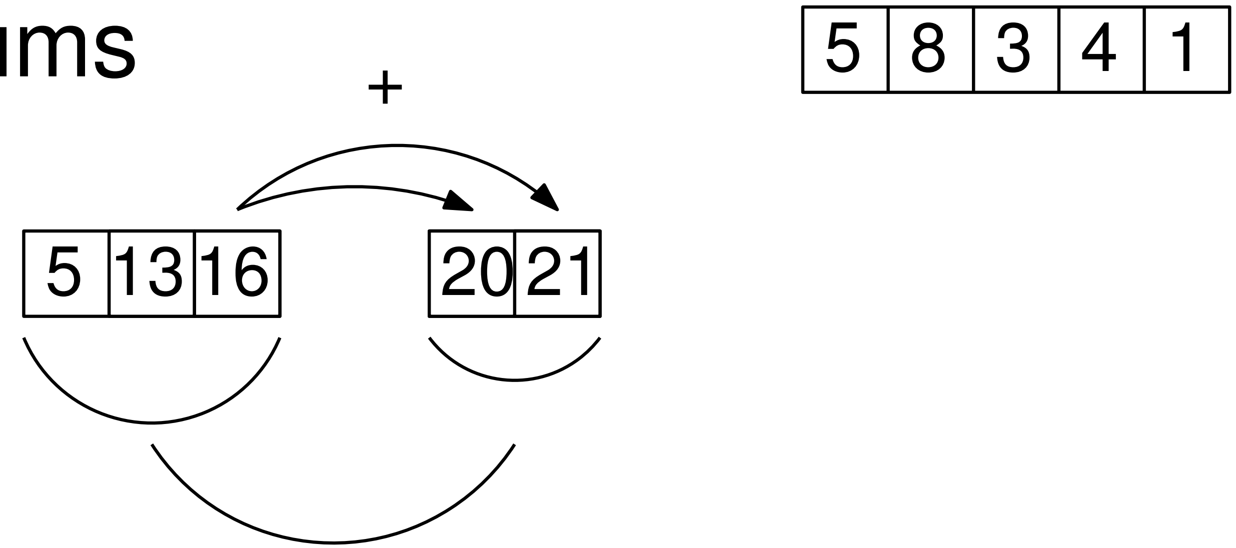
  PREFIX-SUMS(A, i, mid)

  PREFIX-SUMS($A, mid + 1, j$)

  for $k = mid + 1$ to j **do**

    $A[k] = A[k] + A[mid]$

Parallel Prefix Sums



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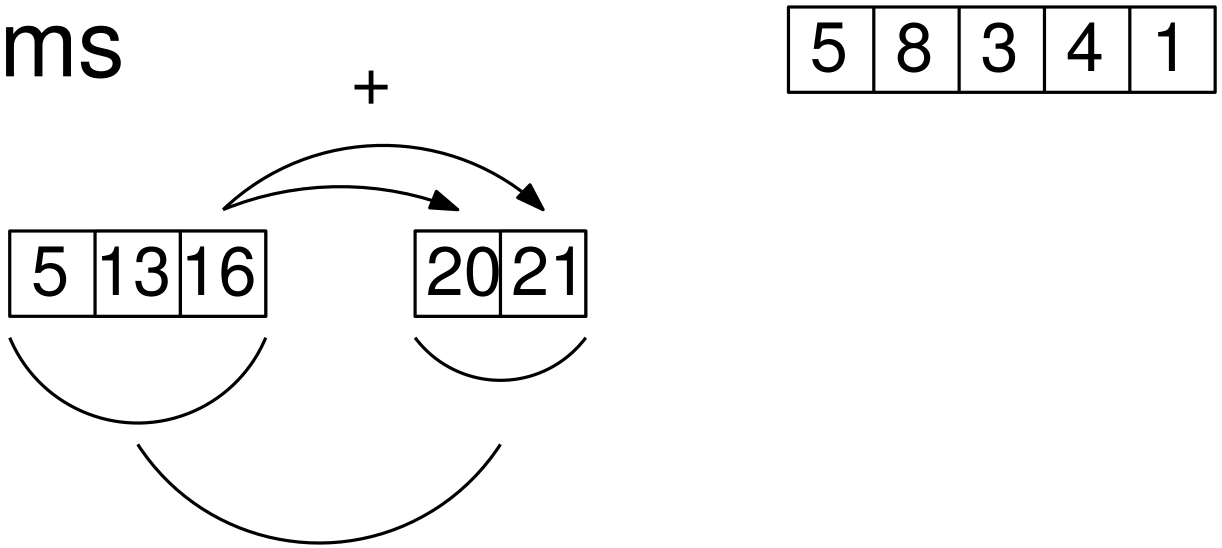
 for $k = mid + 1$ to j **do**
  $A[k] = A[k] + A[mid]$

$$\begin{aligned} W(n) &= 2W(n/2) + O(n) \\ &= O(n \log n) \end{aligned}$$

Parallel Prefix Sums



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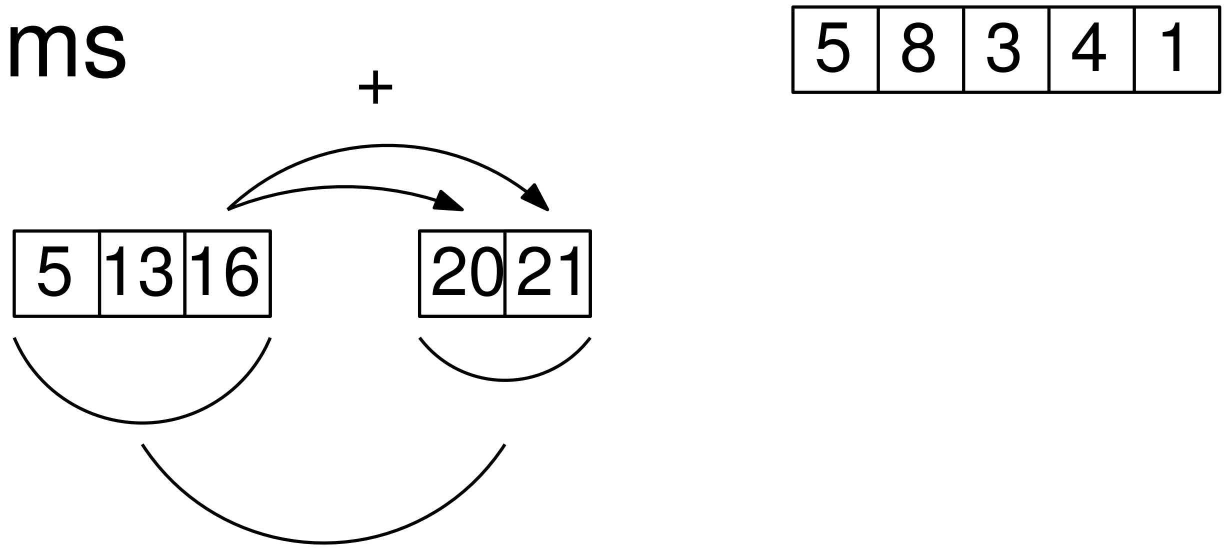
Parallel Prefix Sums



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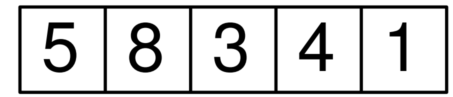

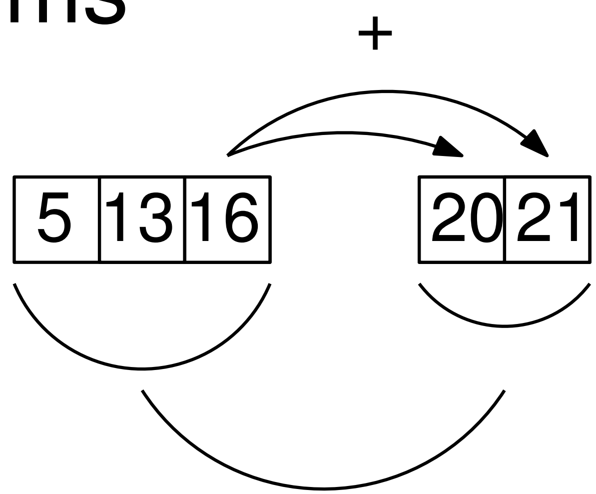

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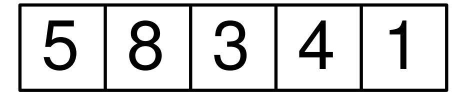

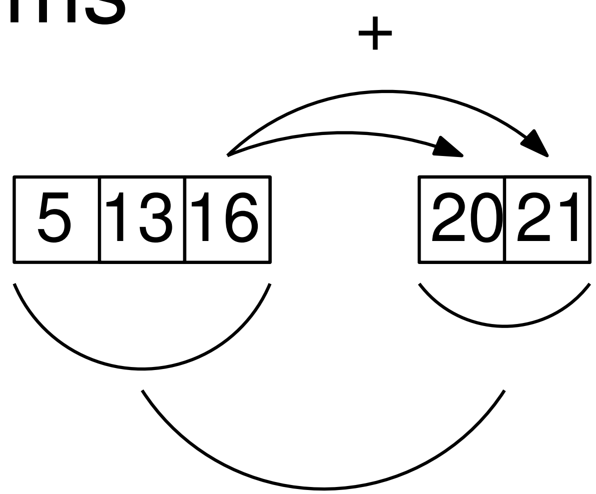

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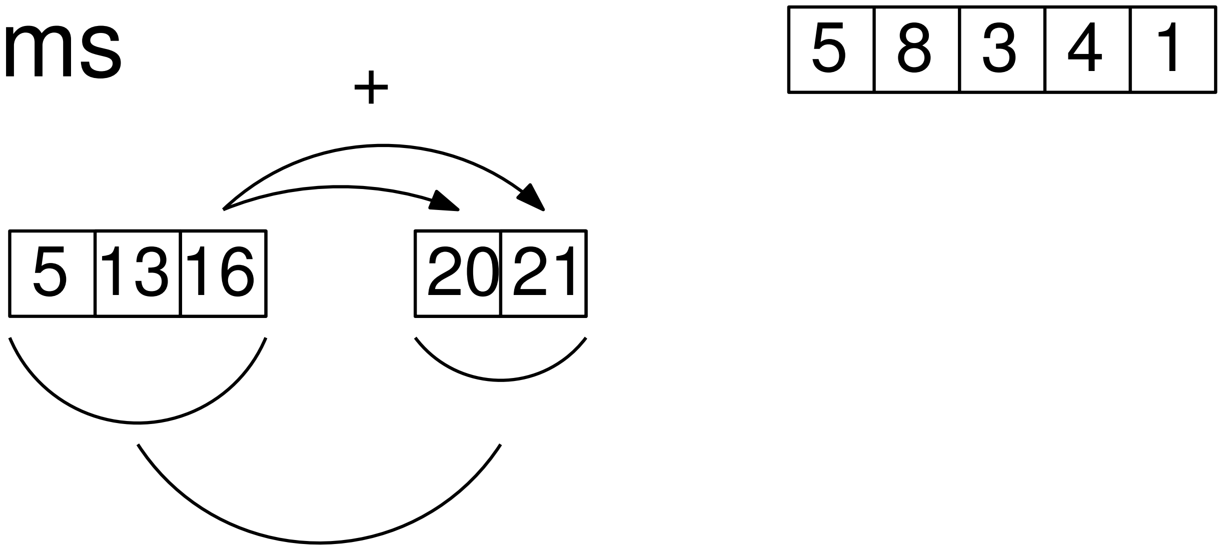
(t_1, t_2) = STARTTWOTHREADS()
t_1 **do**: PREFIX-SUMS(A, i, mid)
t_2 **do**: PREFIX-SUMS($A, mid+1, j$)
WAITUNTILFINISHED(t_1, t_2)

$$T(n) = 2T(n/2) + O(1) = O(n)$$

Parallel Prefix Sums



function $\text{PREFIX-SUMS}(A, i, j)$

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$\text{PREFIX-SUMS}(A, i, mid)$

$\text{PREFIX-SUMS}(A, mid + 1, j)$

for $k = mid + 1$ to j **in parallel**

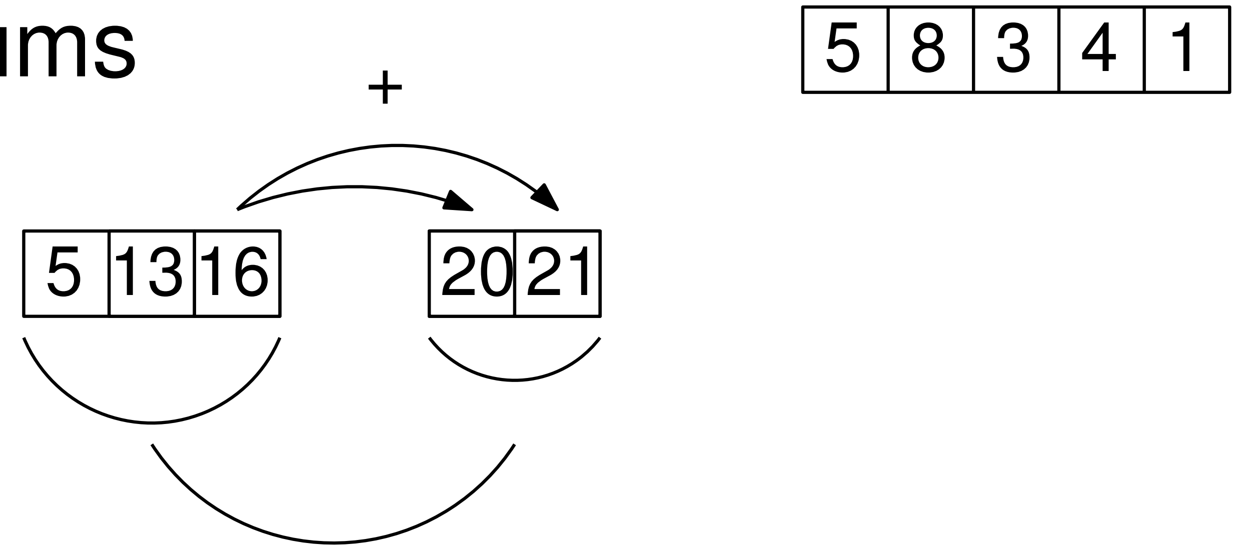
$A[k] = A[k] + A[mid]$

$(t_1, t_2) = \text{STARTTWOTHREADS}()$
t_1 **do**: $\text{PREFIX-SUMS}(A, i, mid)$
t_2 **do**: $\text{PREFIX-SUMS}(A, mid + 1, j)$
$\text{WAITUNTILFINISHED}(t_1, t_2)$

$$T(n) = \max\left\{T\left(\left\lceil \frac{n}{2} \right\rceil\right), T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right\} + O(1)$$

$$\leq T(n/2) + O(1)$$

Parallel Prefix Sums



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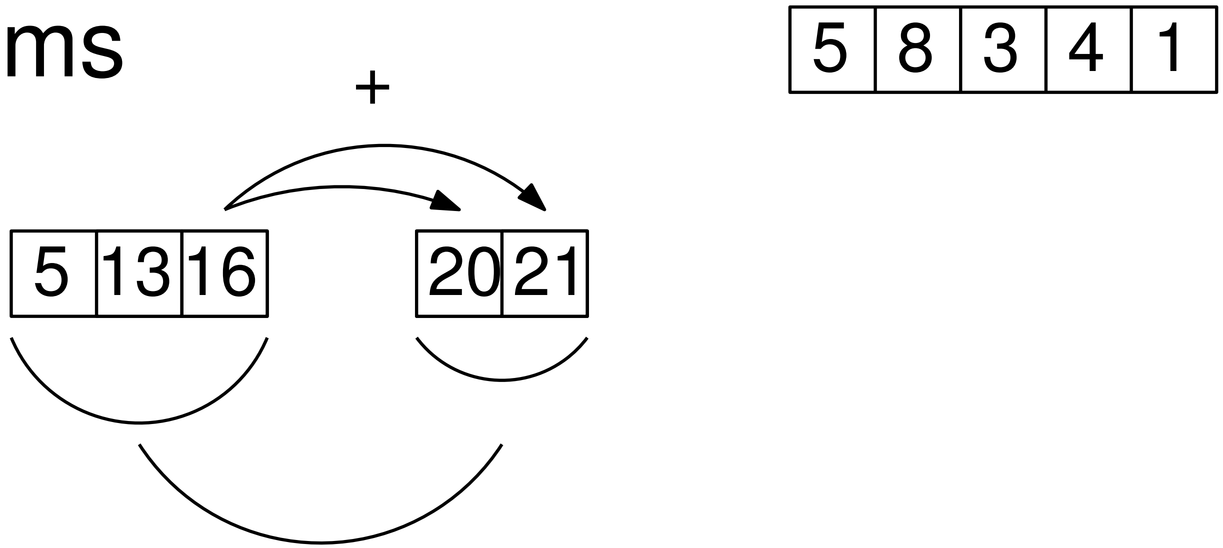
$$T(n) = \max\left\{T\left(\left\lceil \frac{n}{2} \right\rceil\right), T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)\right\} + O(1)$$
$$\leq T(n/2) + O(1)$$
$$= O(\log n)$$

Parallel Prefix Sums



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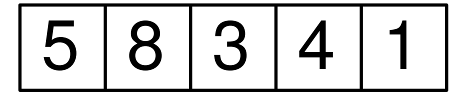
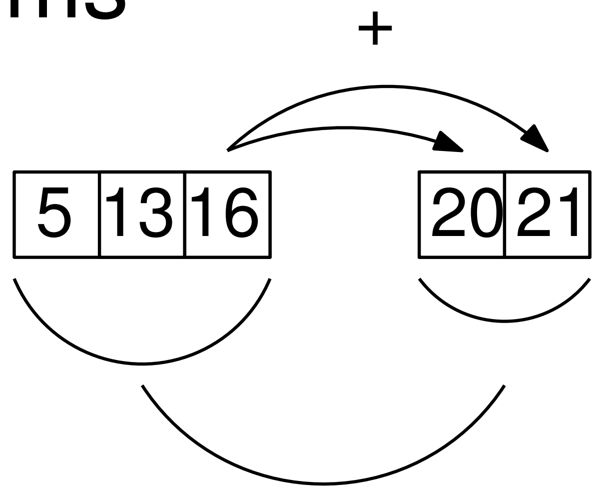
Work: $W(n) = O(n \log n)$

Time: $T(n) = O(\log n)$

Parallel Prefix Sums





Best sequential time

function PREFIX-SUMS(A, i, j)
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Parallel Prefix Sums

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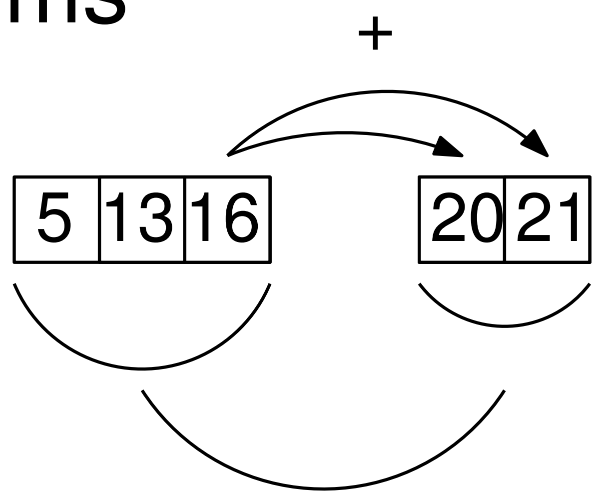

Best sequential time

$T(n) = O(n)$

function PREFIX-SUMS(A, i, j)

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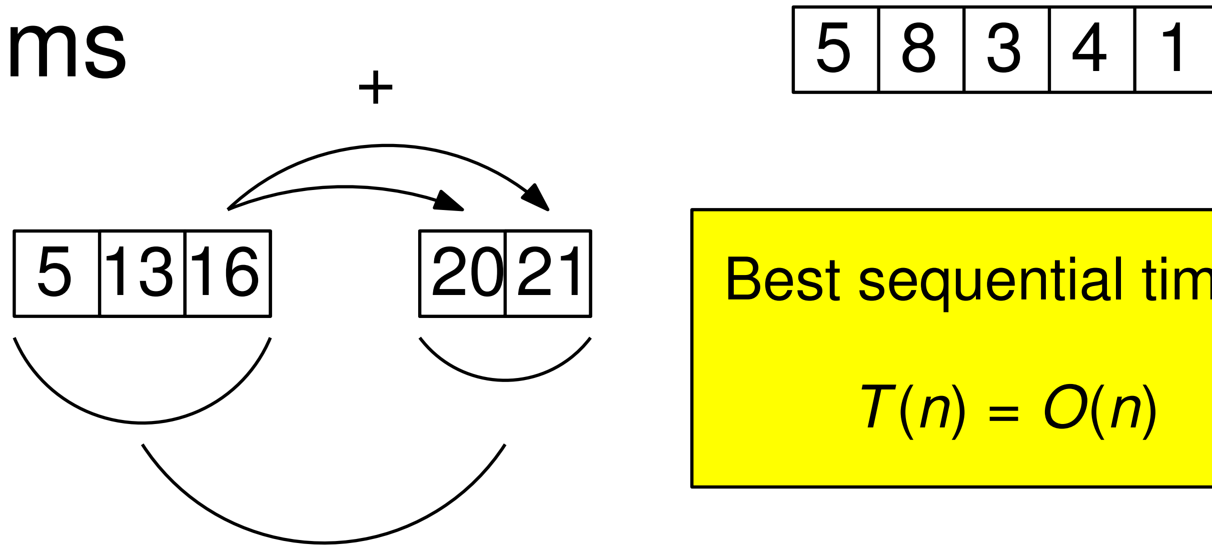

Best sequential time

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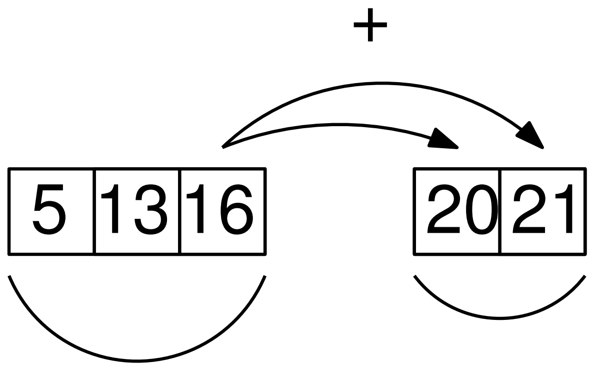
Not work-efficient!

Work: $W(n) = O(n \log n)$

Time: $T(n) = O(\log n)$

Work-efficient Prefix Sums



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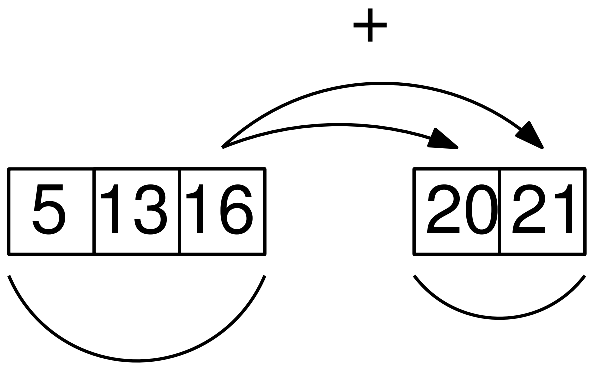
Work-efficient Prefix Sums



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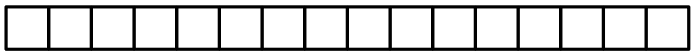
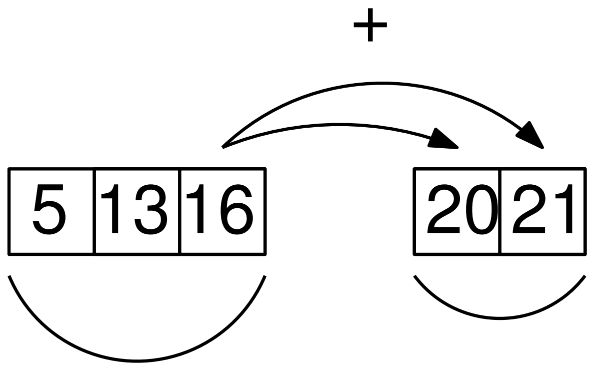
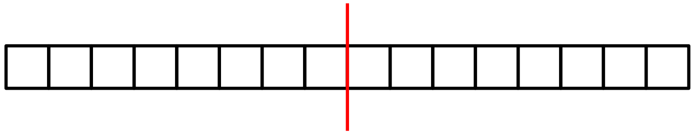


Work-efficient Prefix Sums



$$W(n) = 2W(n/2) + O(n)$$
$$= O(n \log n)$$



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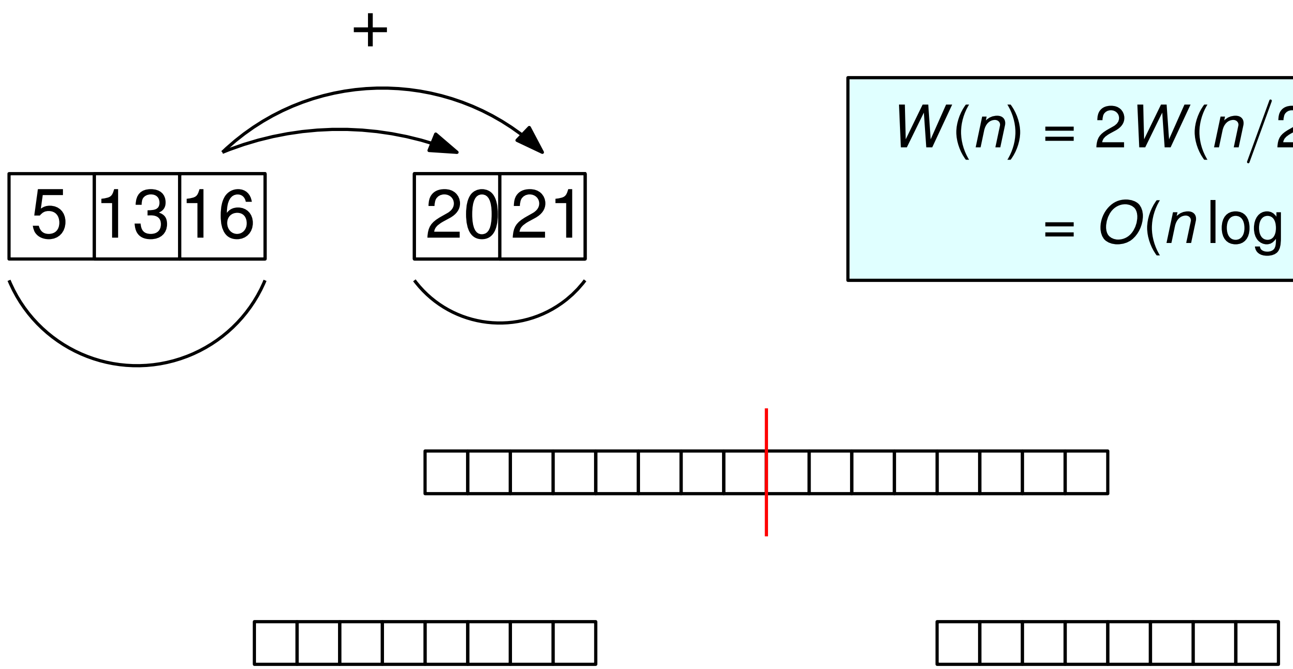

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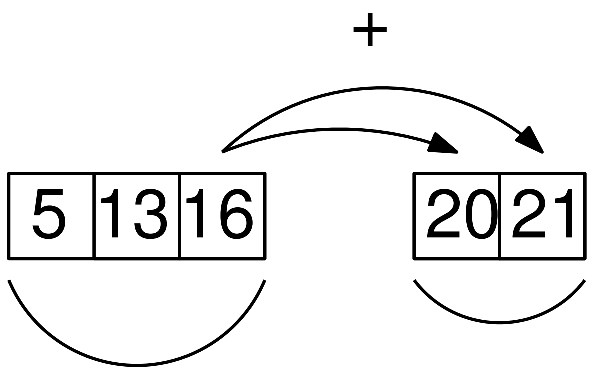

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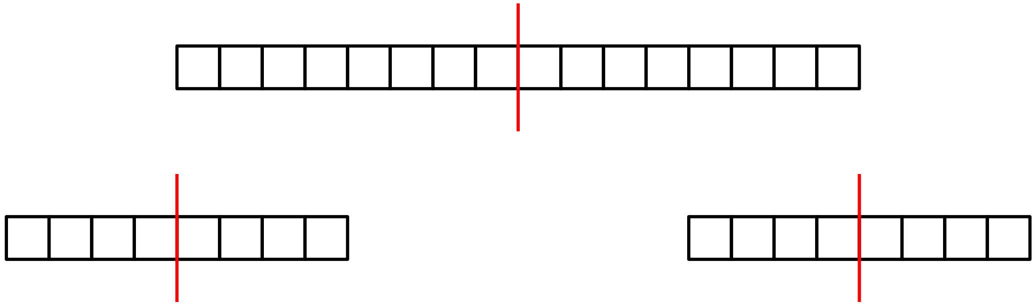

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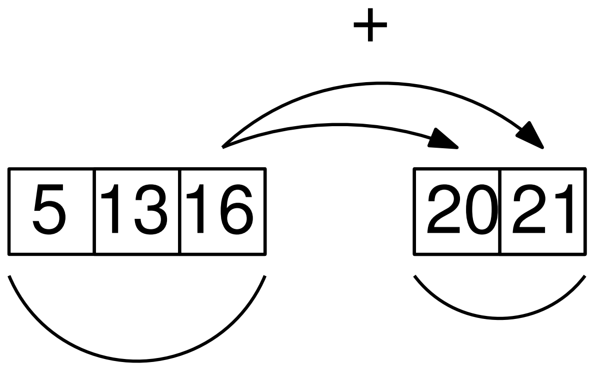

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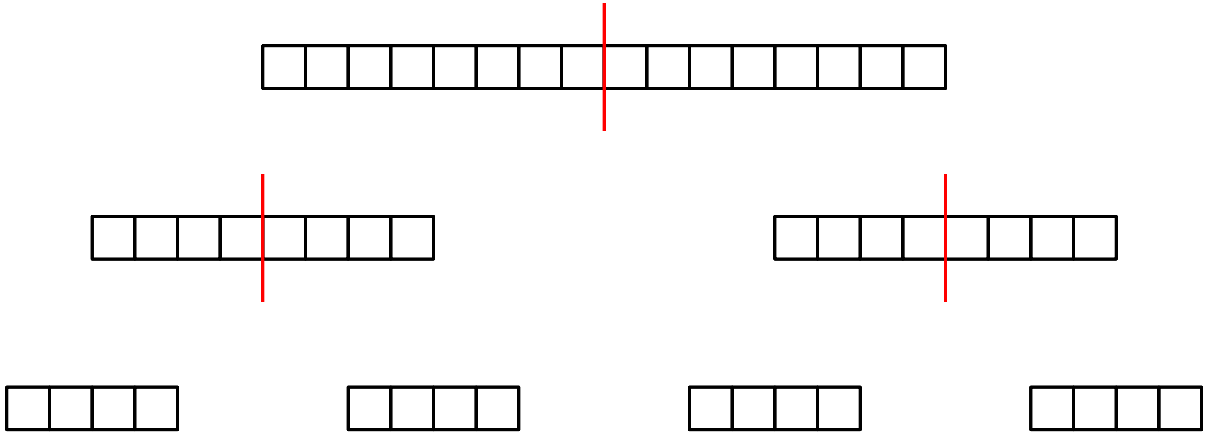

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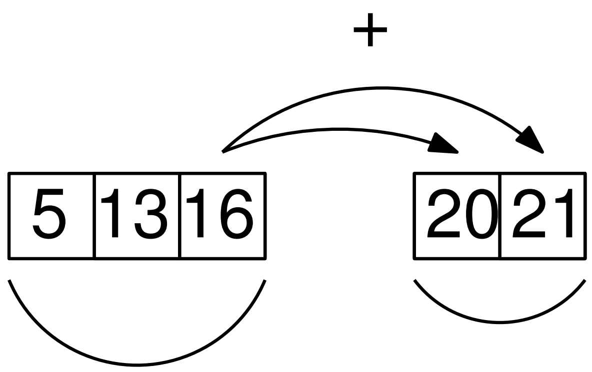

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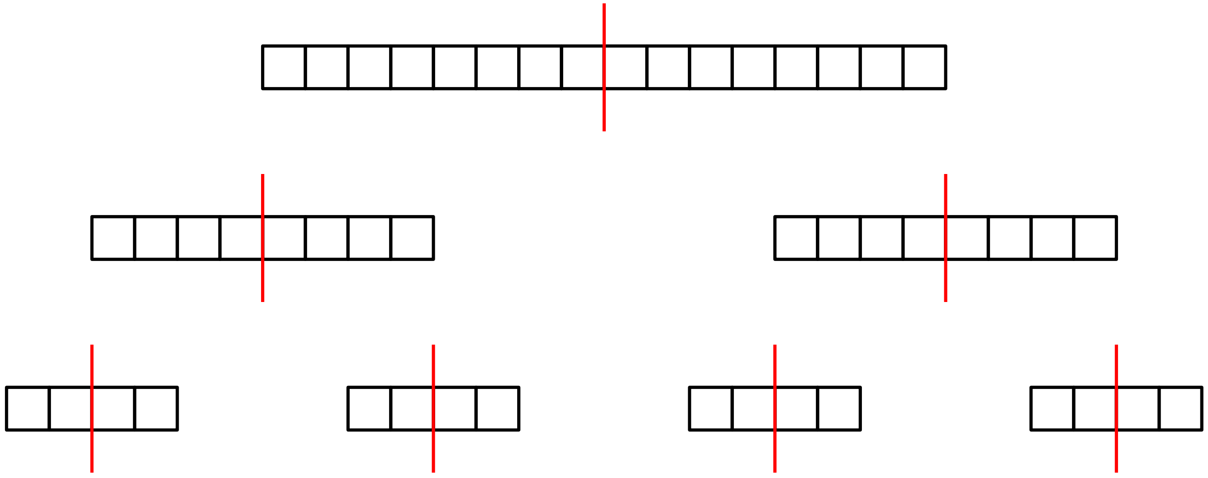

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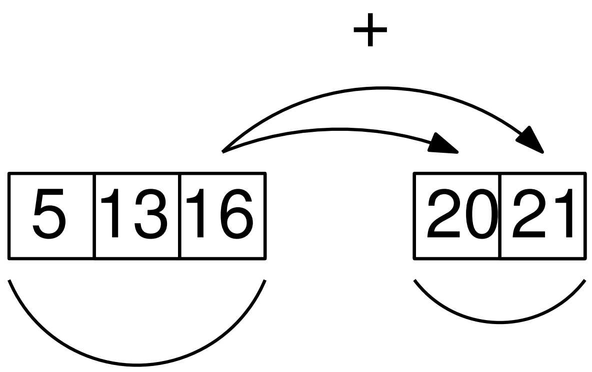

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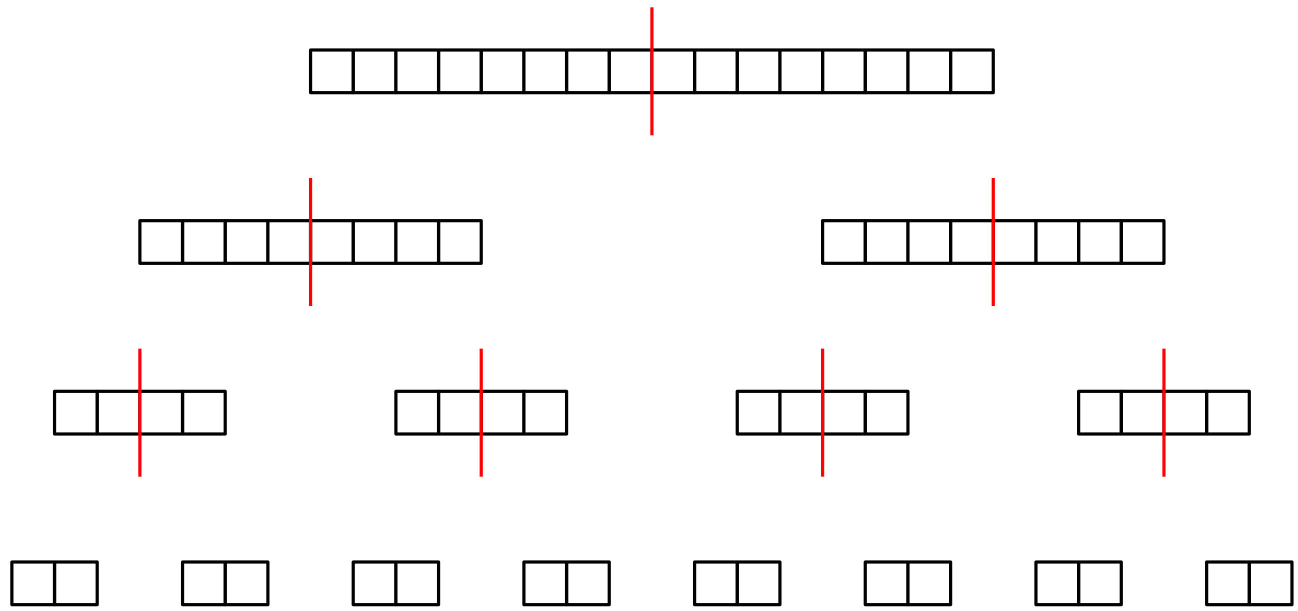

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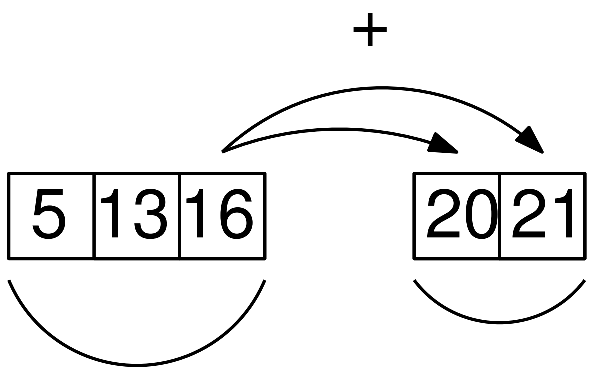

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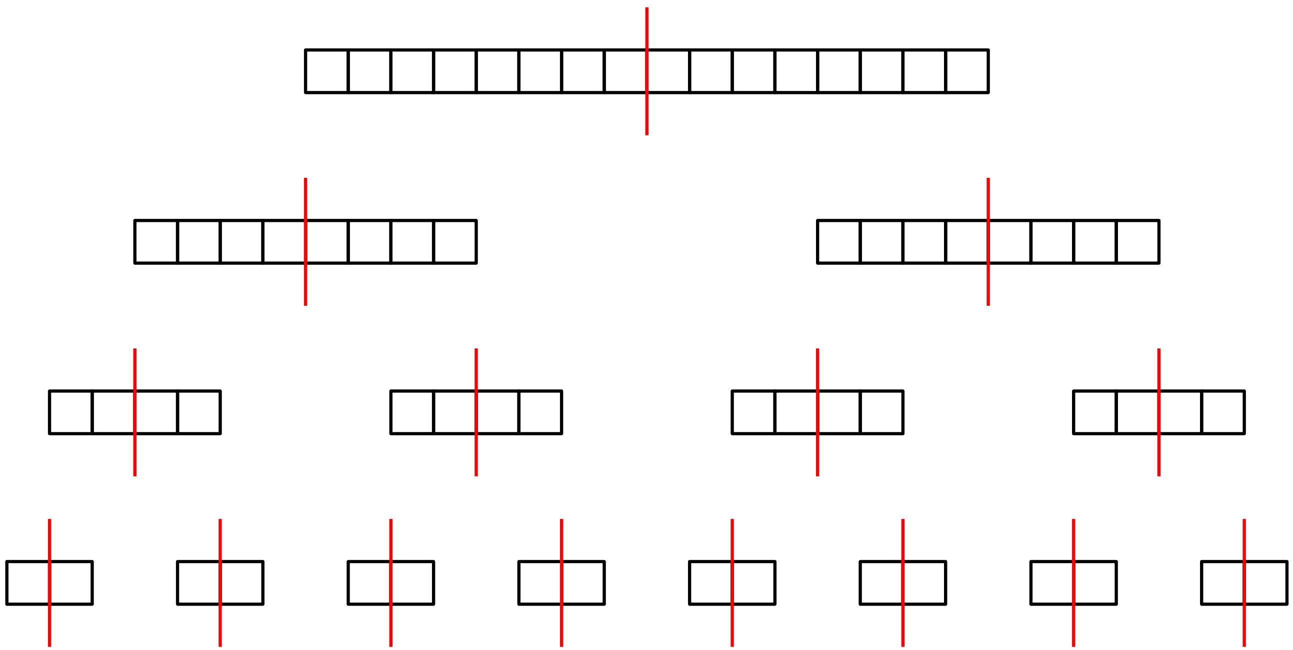

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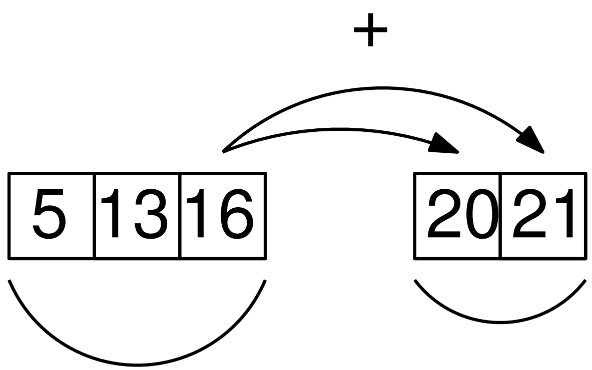

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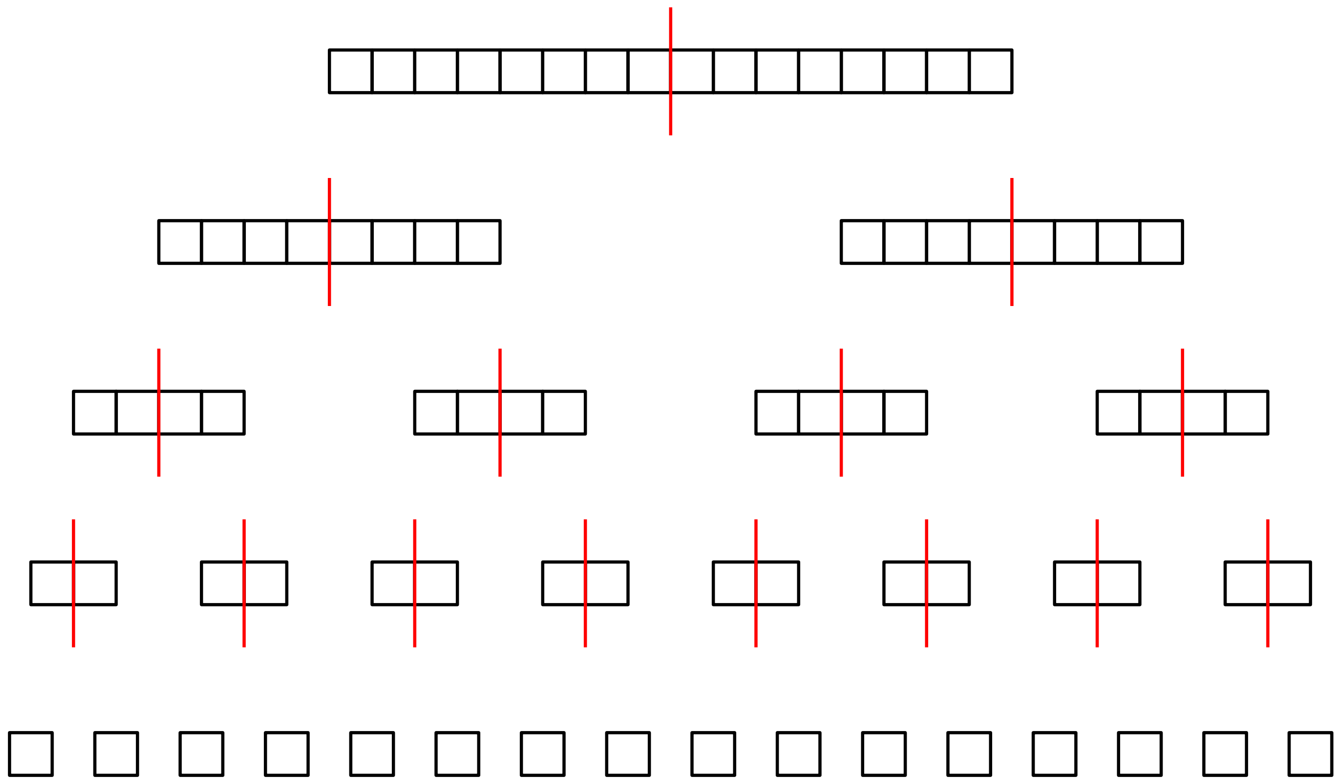

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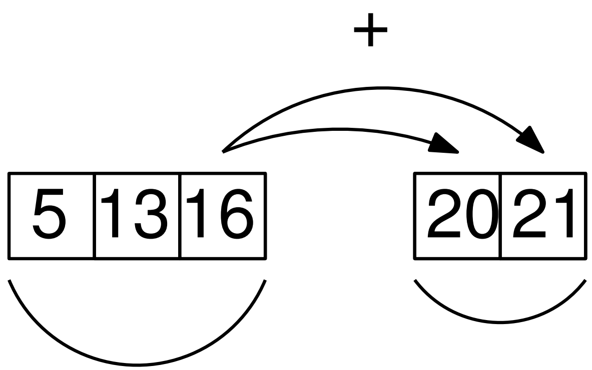

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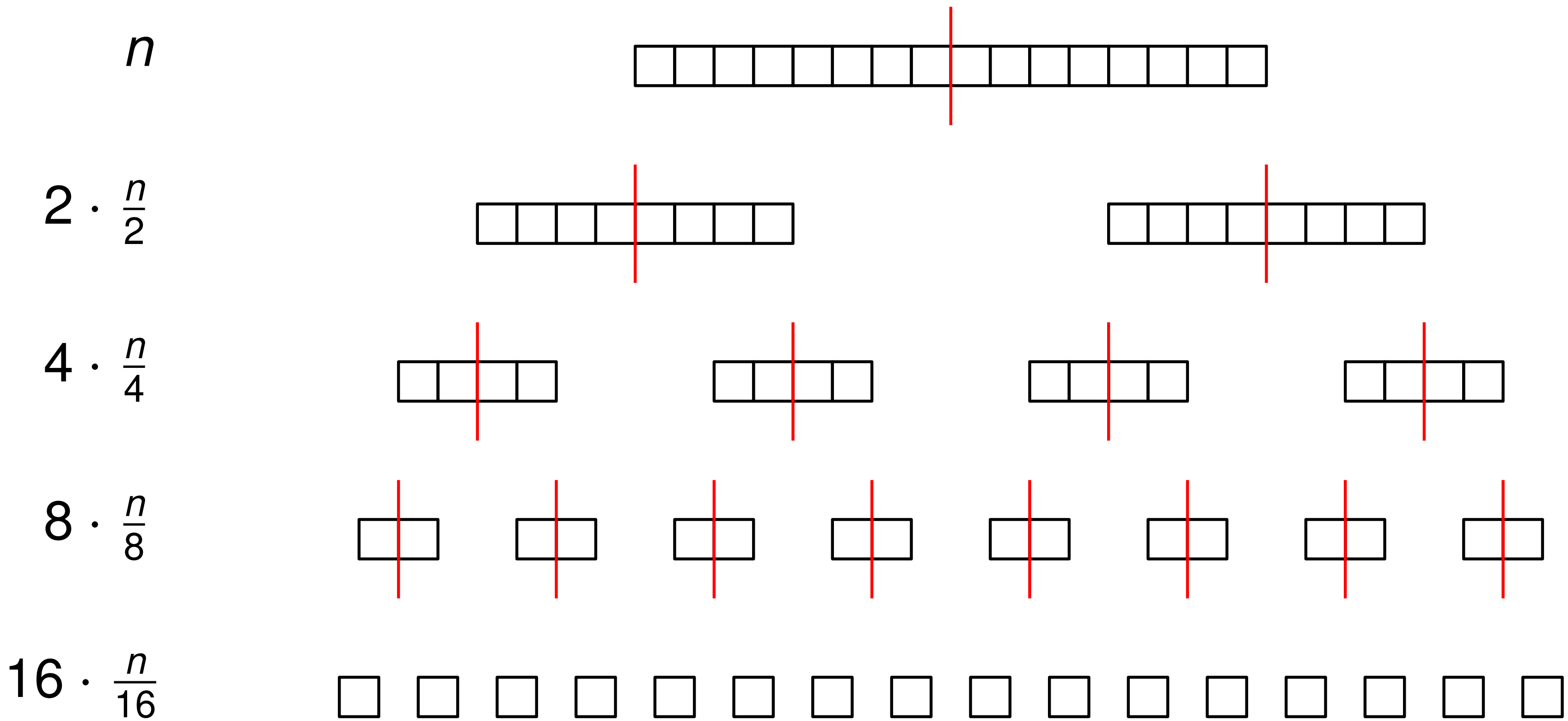

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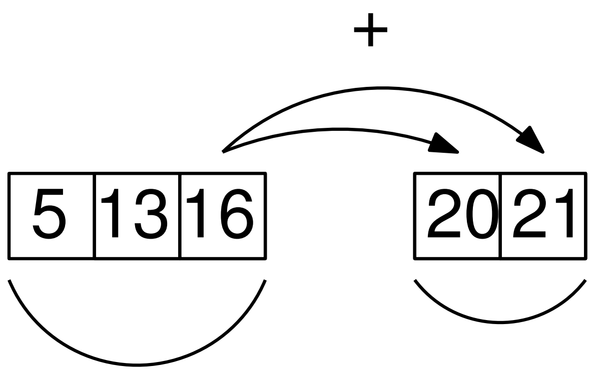

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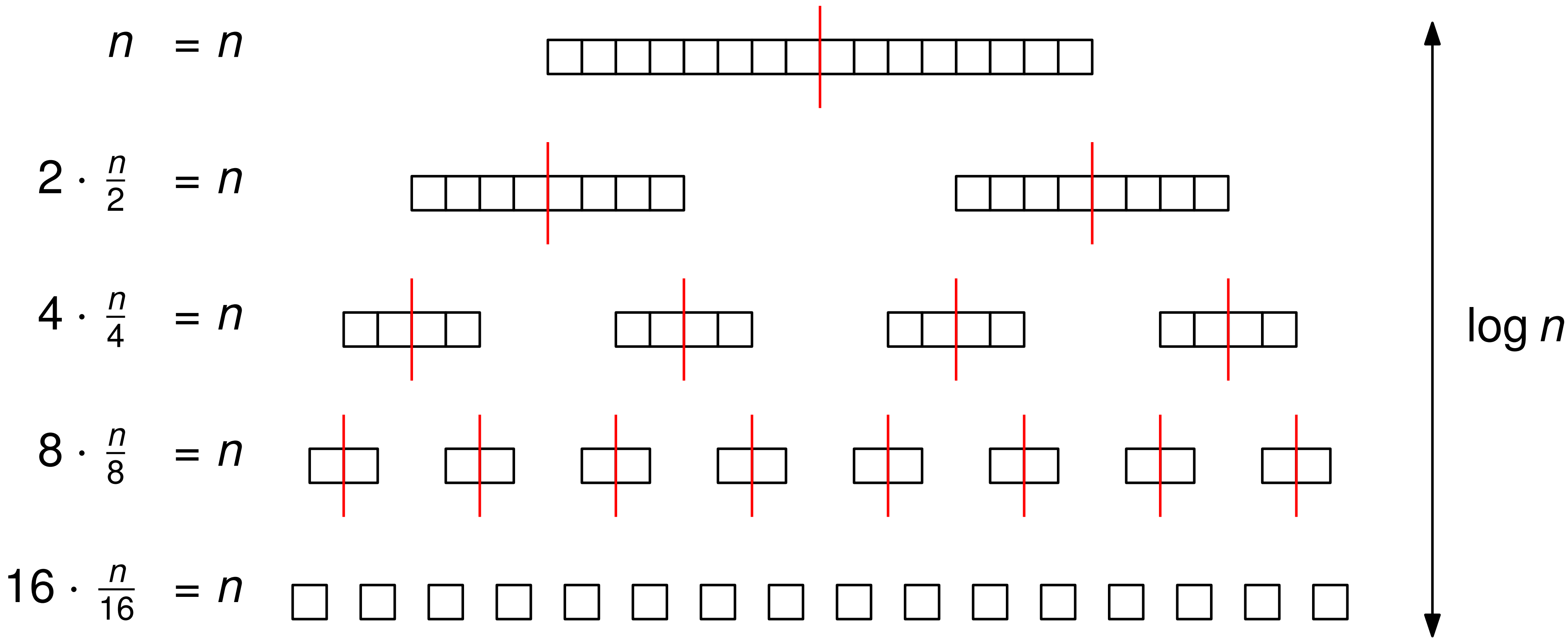

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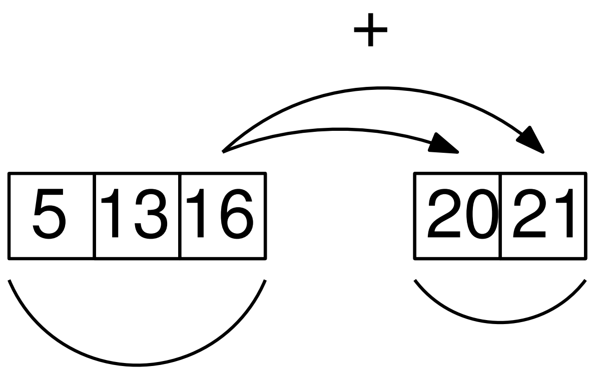

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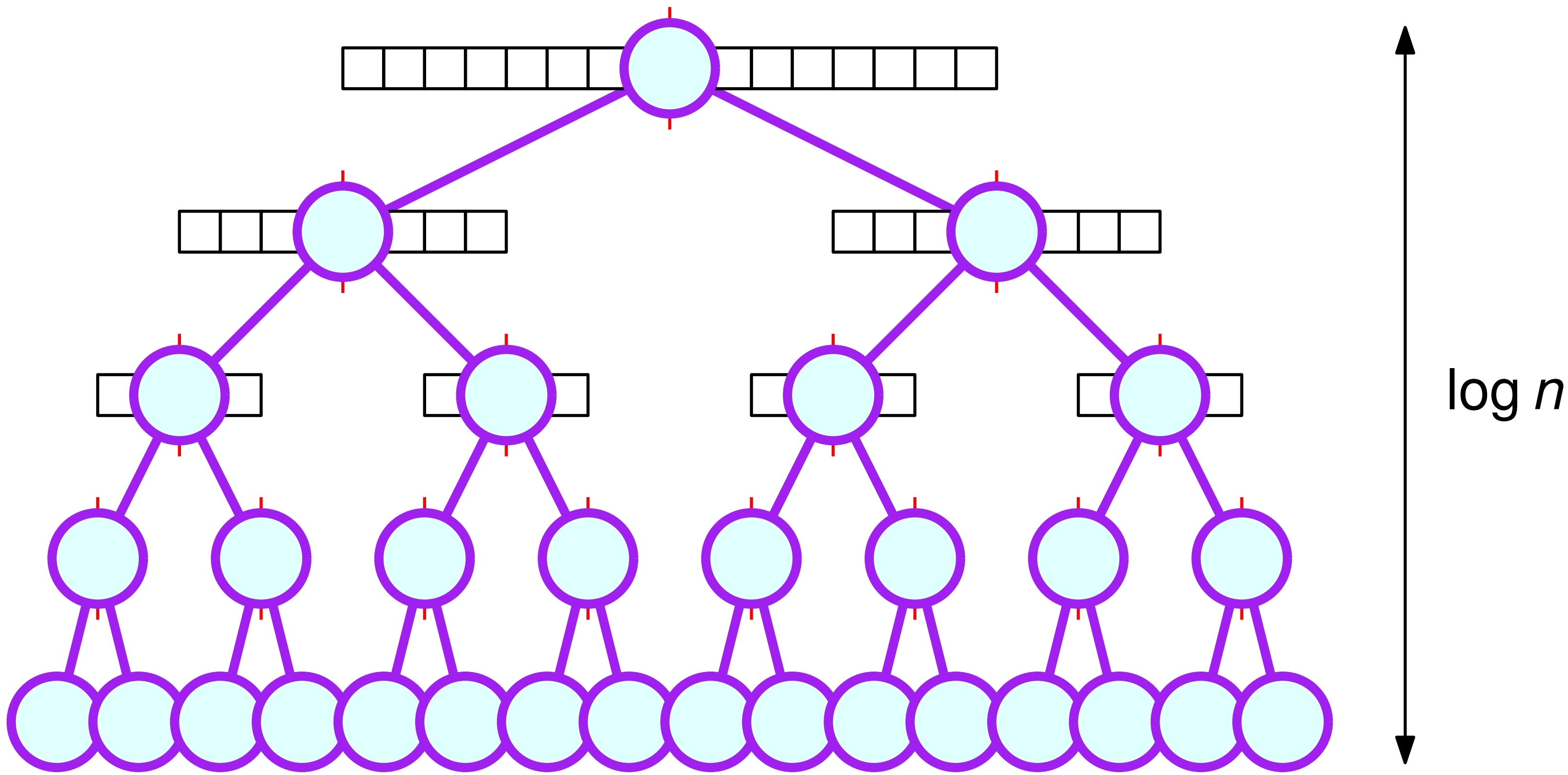
$n = n$

$2 \cdot \frac{n}{2} = n$

$4 \cdot \frac{n}{4} = n$

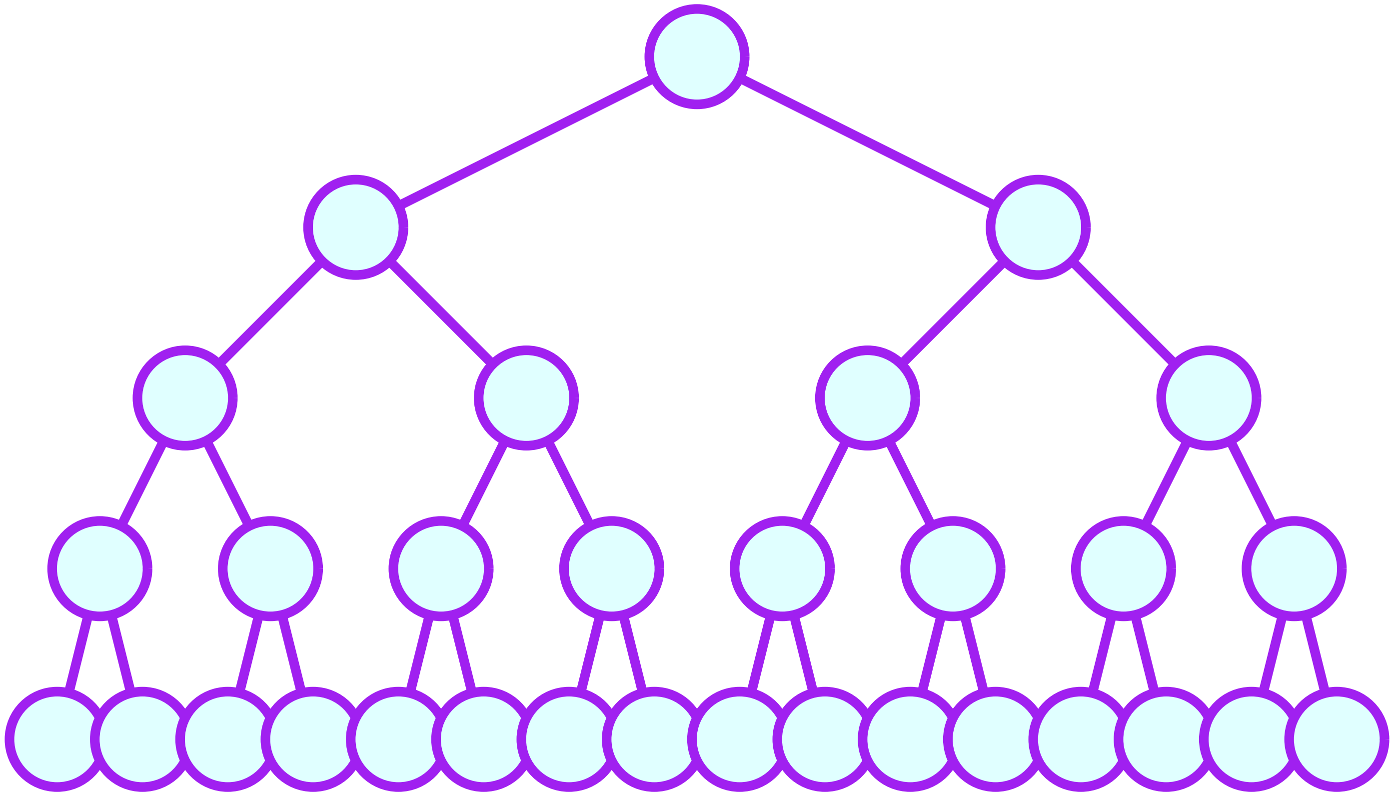
$8 \cdot \frac{n}{8} = n$

$16 \cdot \frac{n}{16} = n$



Number of Nodes in Perfect Binary Trees

Problem. *Using induction, prove that a perfect binary tree with n leaves has* $2n - 1$ *total nodes.*



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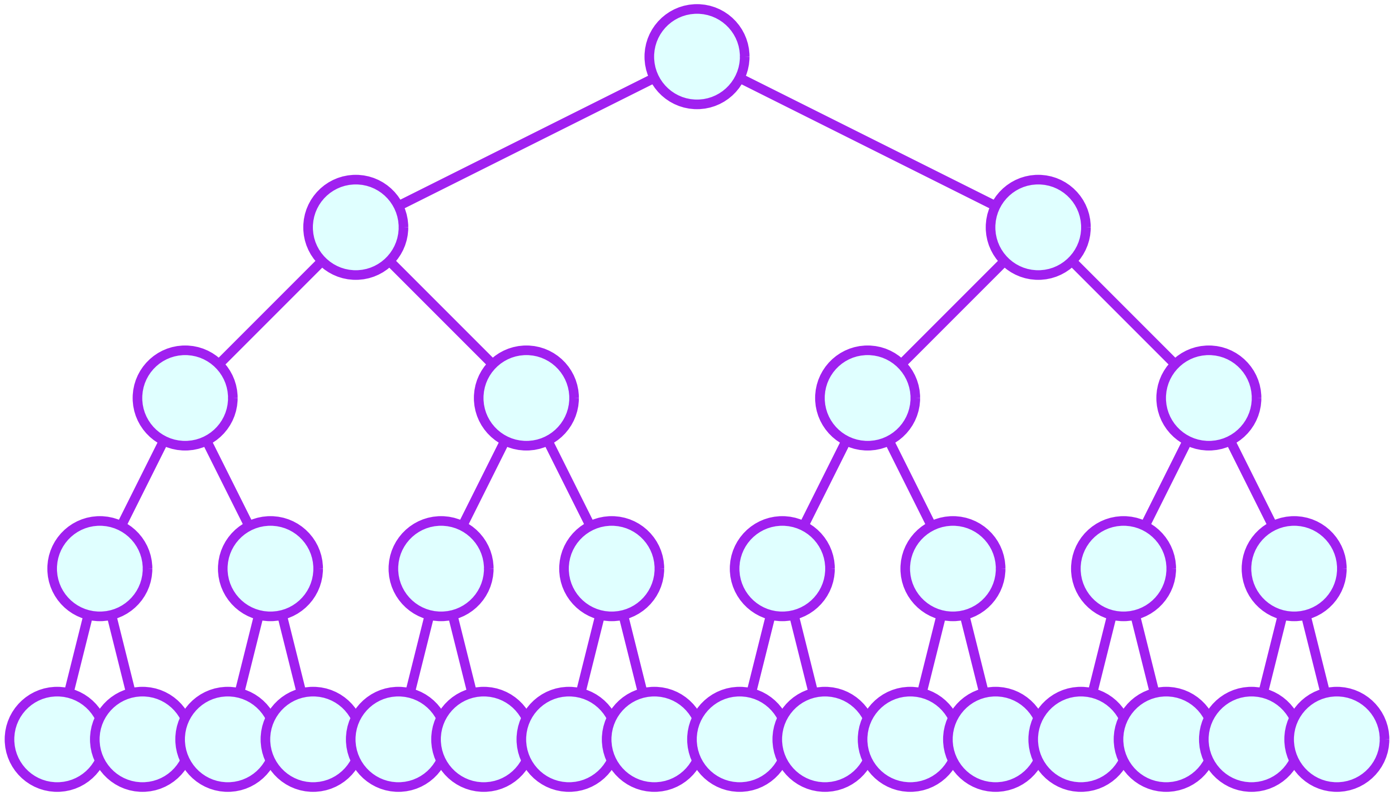
Inductive Hypothesis: For all $k < n$, a perfect binary tree with k leaves has $2k - 1$ total nodes.



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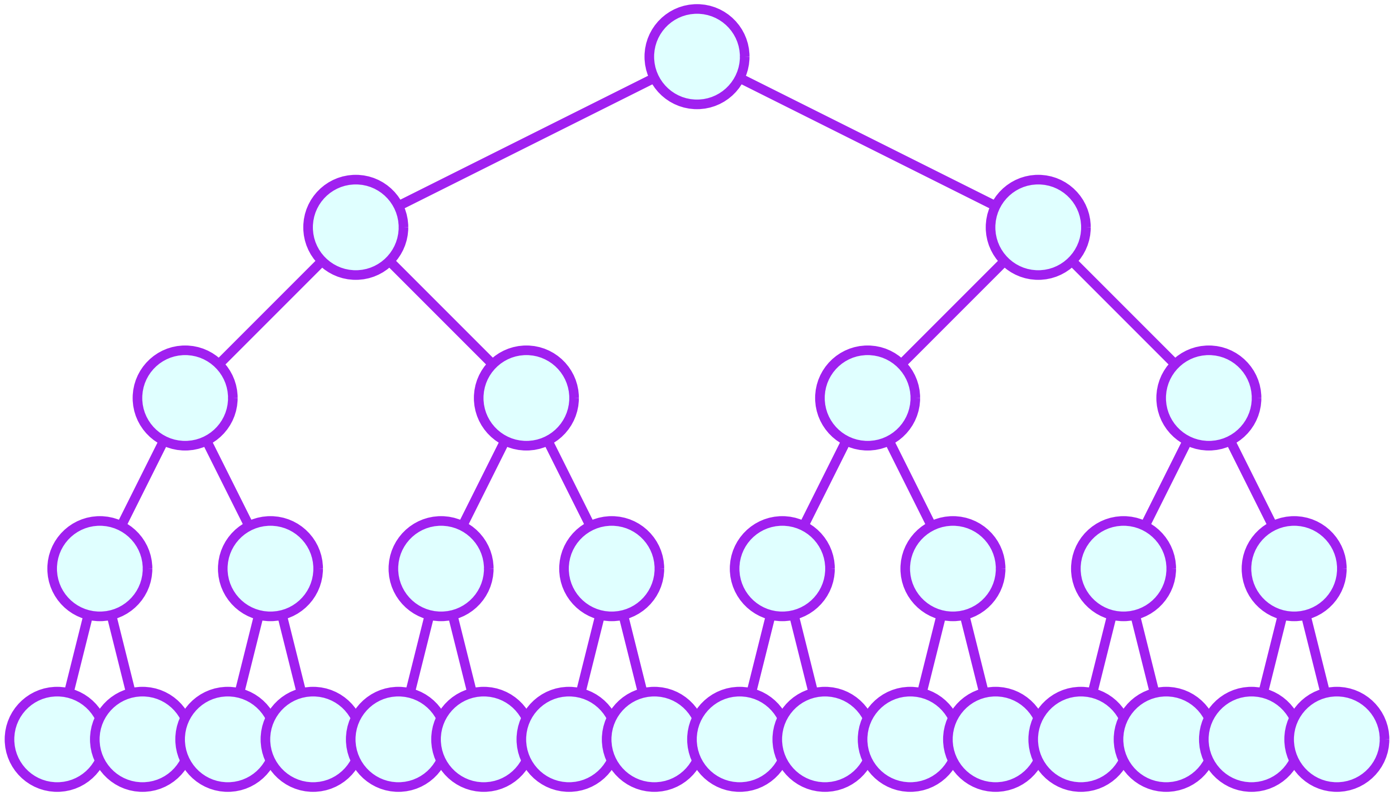
Let $f(n)$ = total number of nodes in a perfect tree with n leaves



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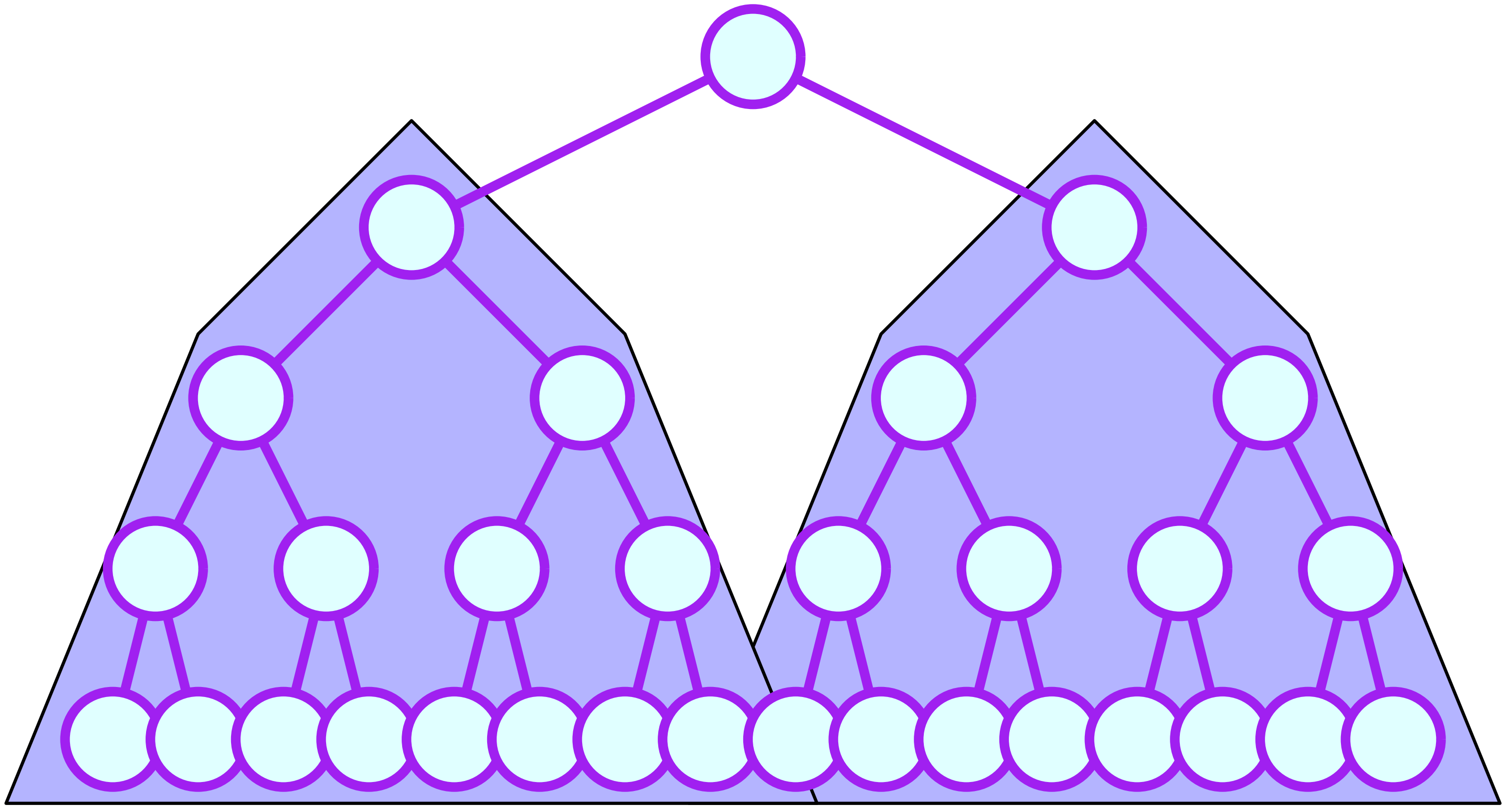

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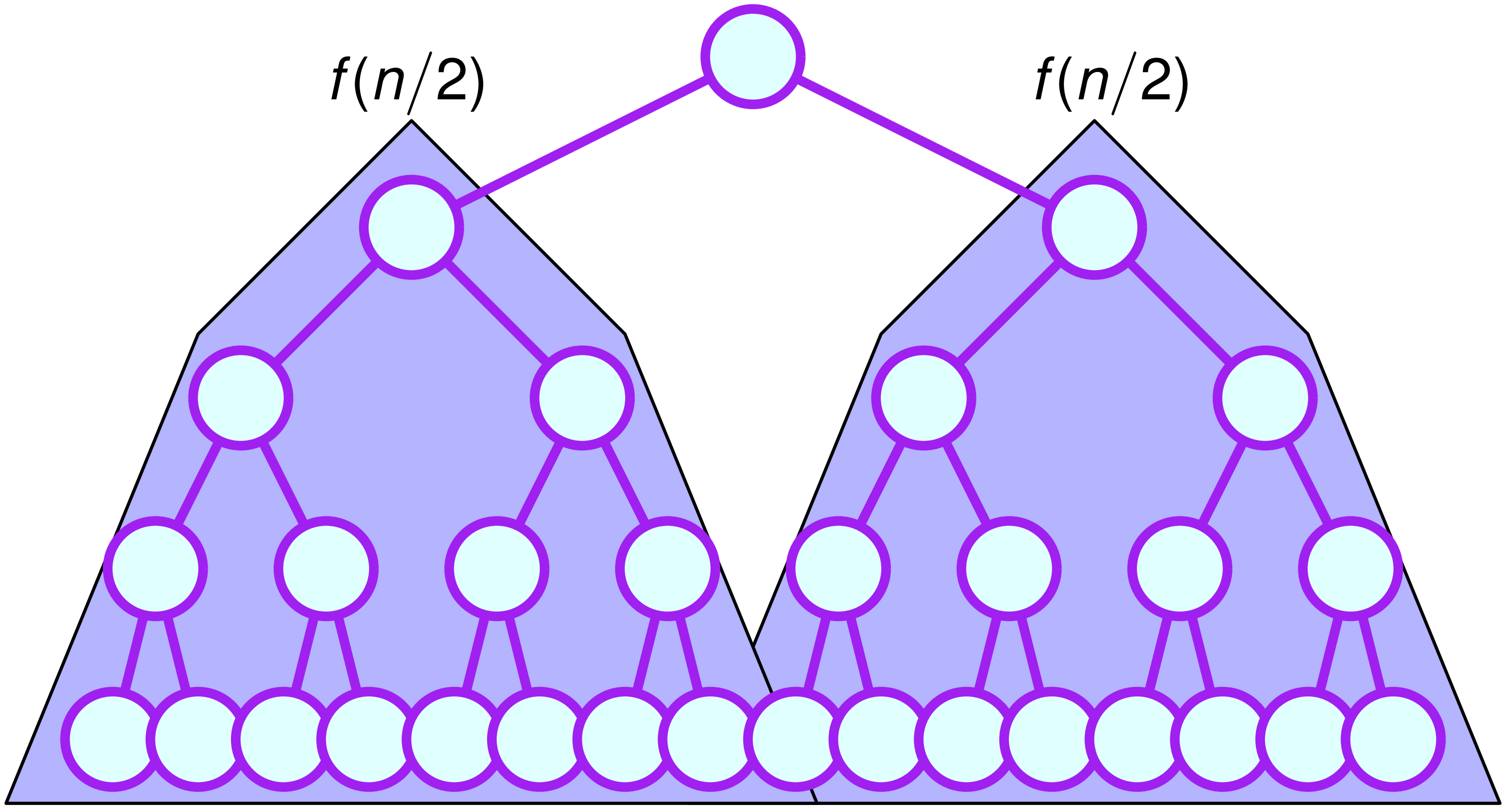

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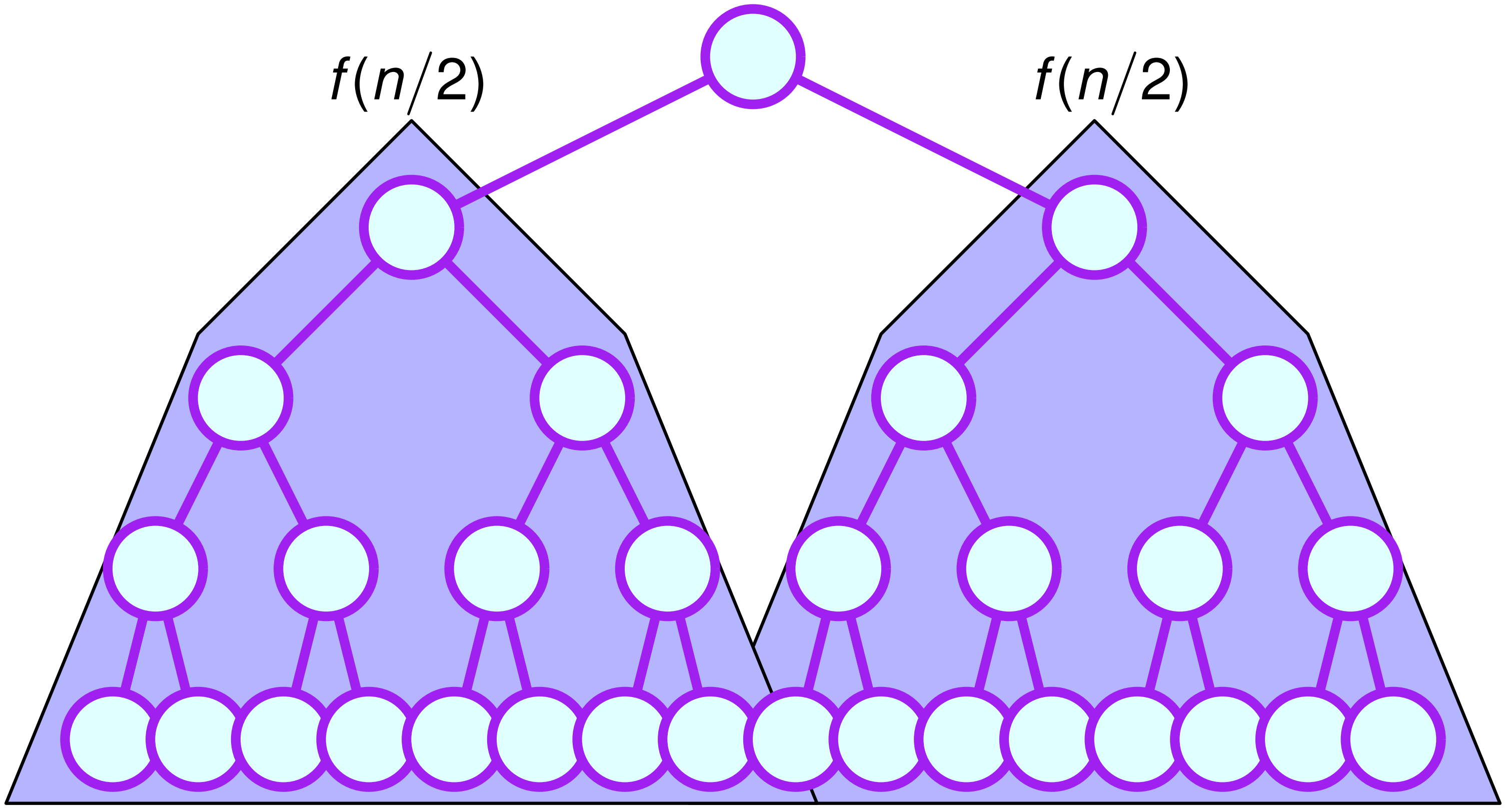
Inductive Hypothesis: For all $k < n$, a perfect binary tree with k leaves has $2k - 1$ total nodes.

$f(n) = 1 + 2 \cdot f(n/2)$



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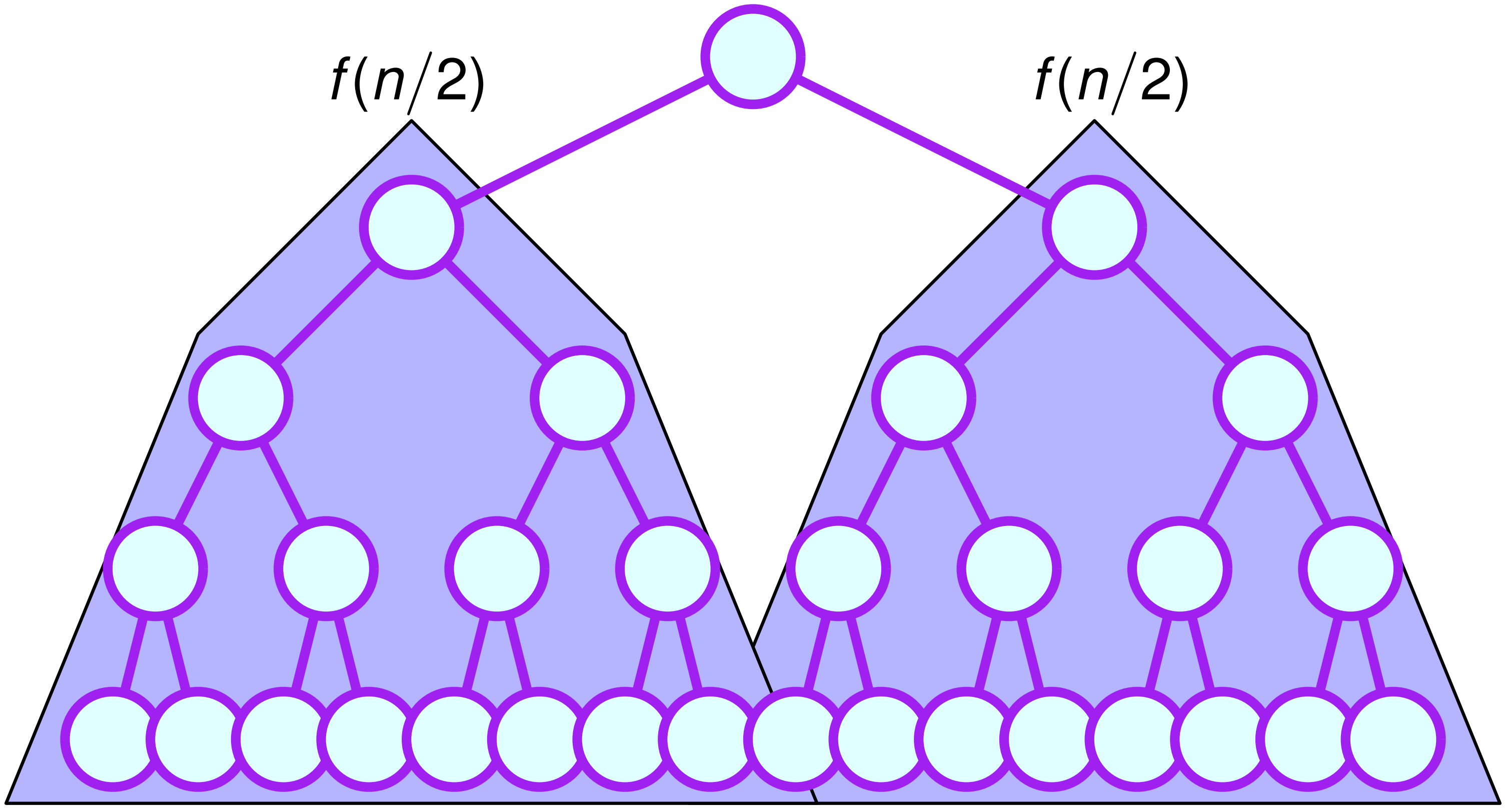
$$f(n) = 1 + 2 \cdot f(n/2) = 1 + 2 \cdot \left(2 \cdot \frac{n}{2} - 1\right)$$



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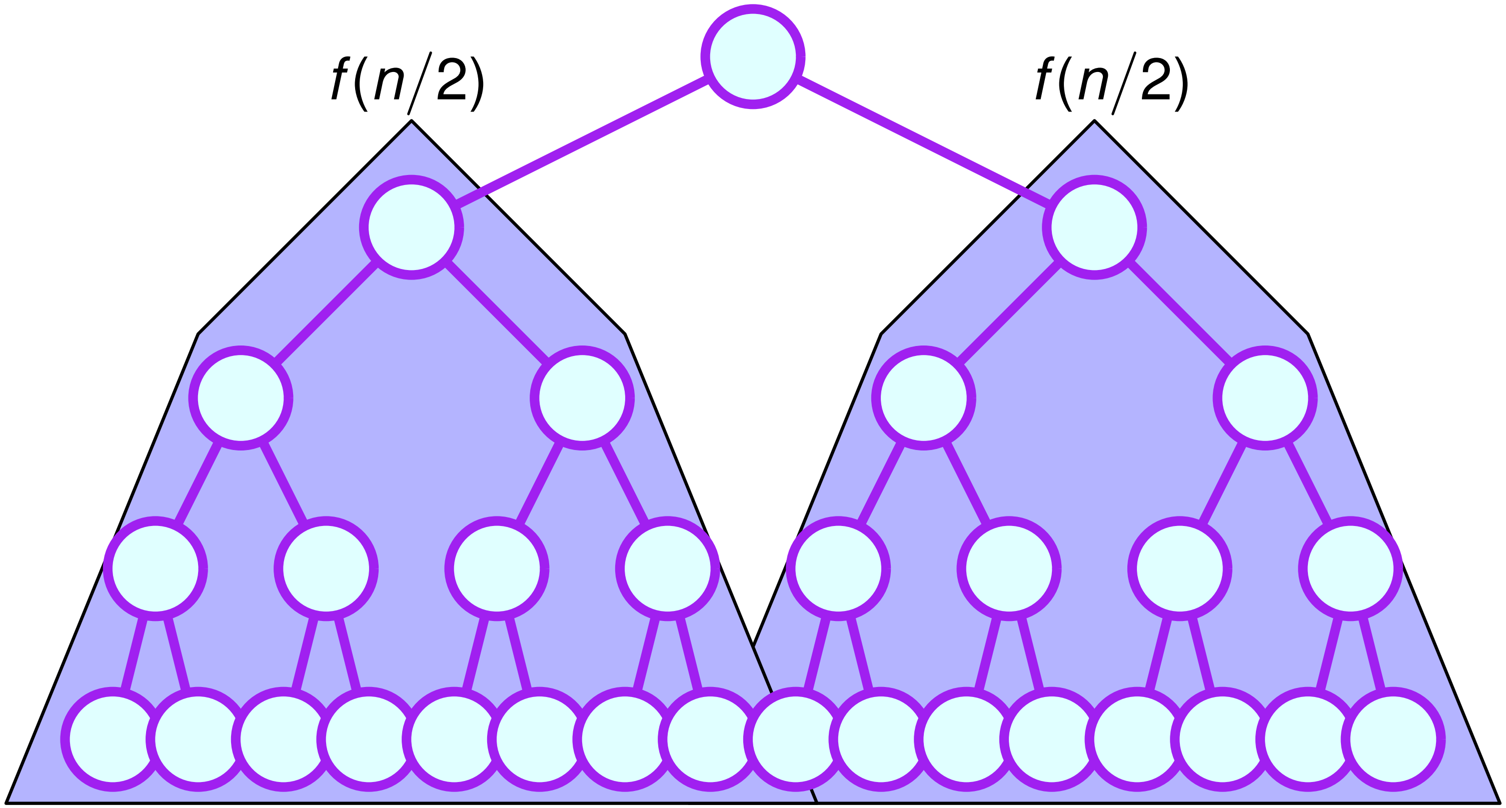

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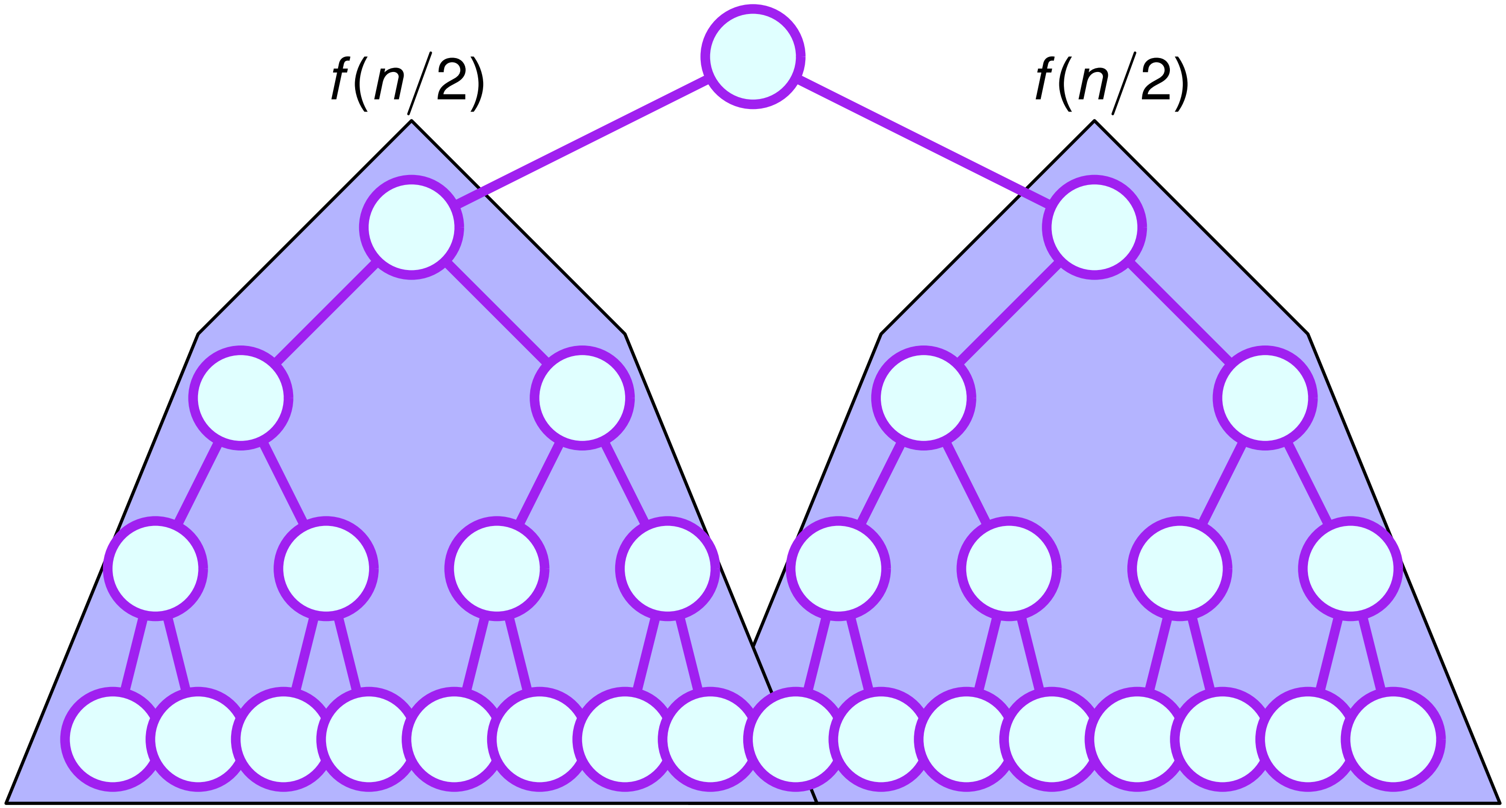
$$f(n) = 1 + 2 \cdot f(n/2) = 1 + 2 \cdot \left(2 \cdot \frac{n}{2} - 1\right) = 1 + 2 \cdot (n - 1) = 1 + (2n - 2) = 2n - 1$$



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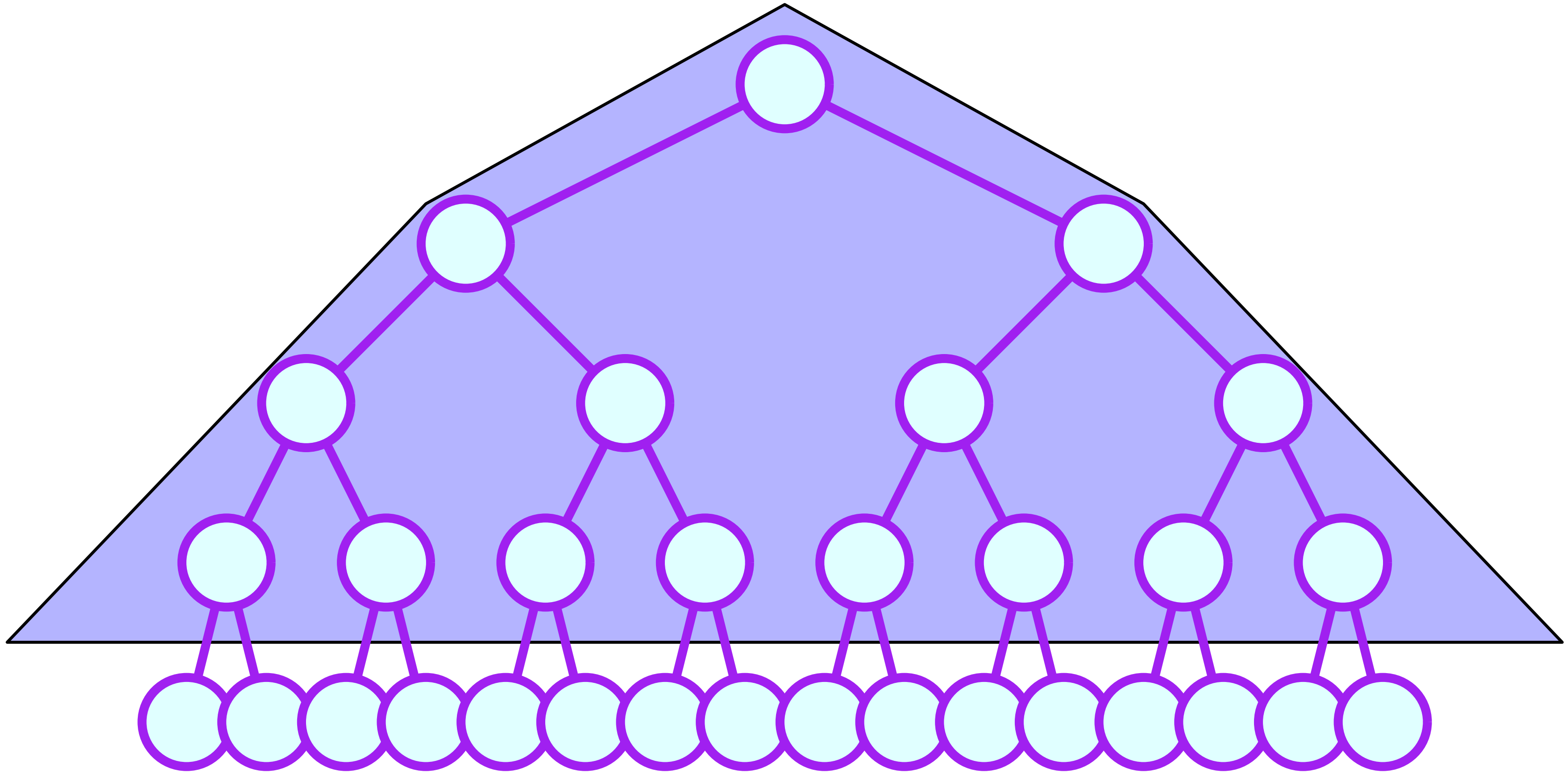

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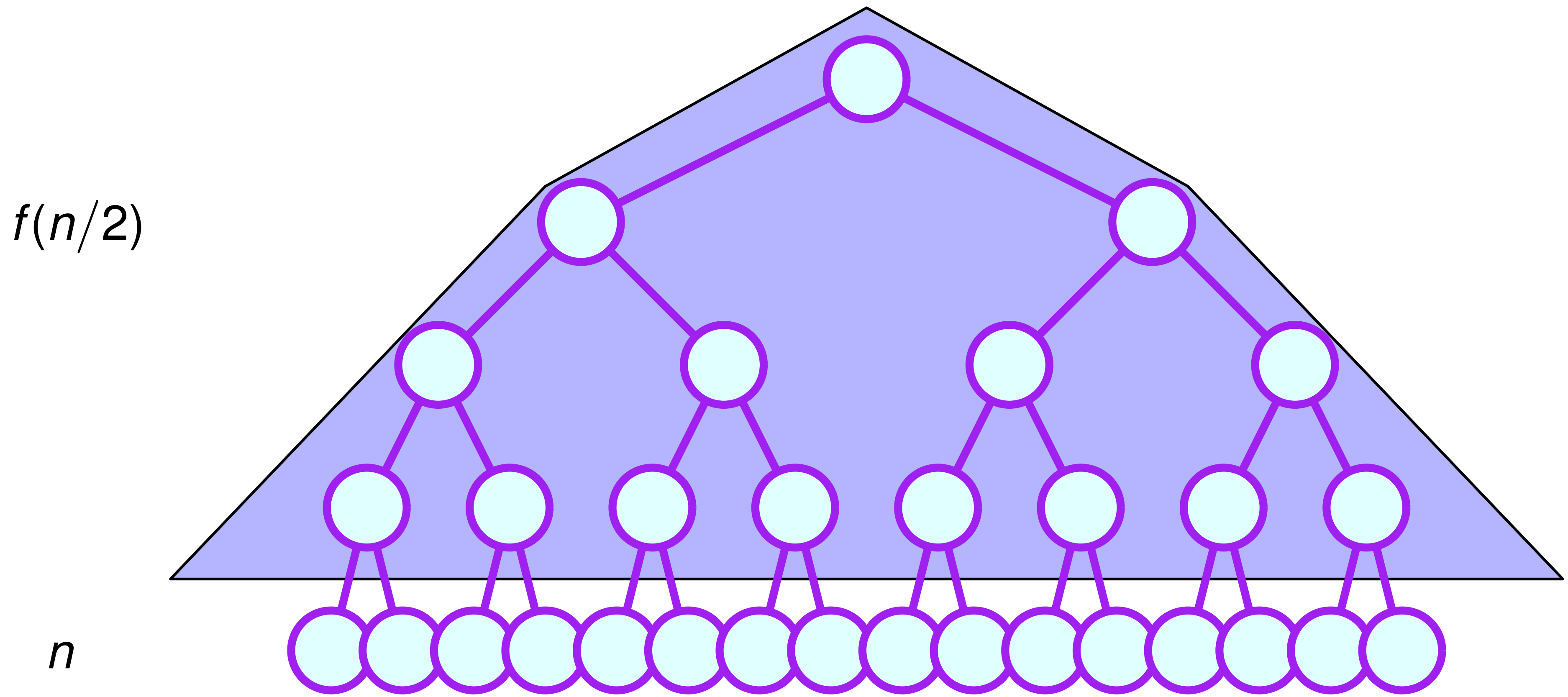

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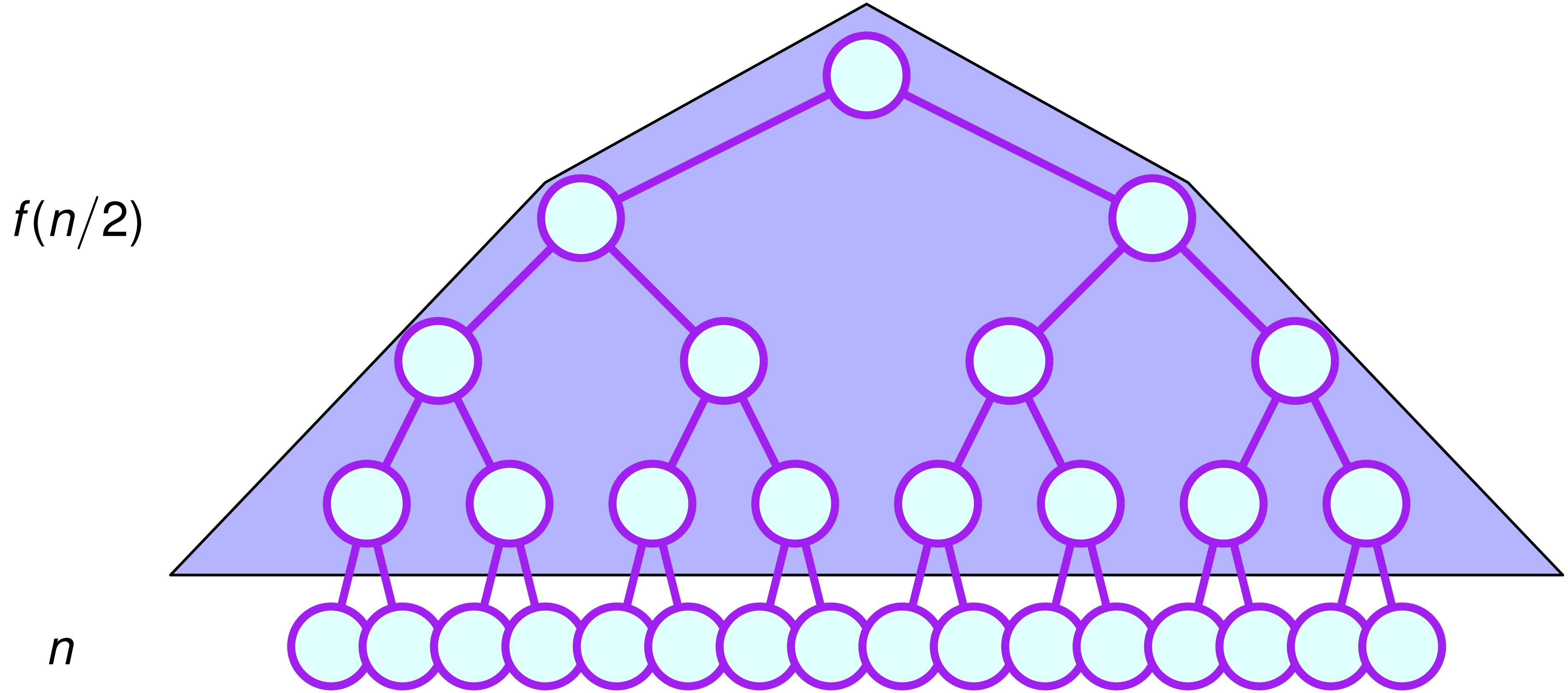
$f(n) = f(n/2) + n$



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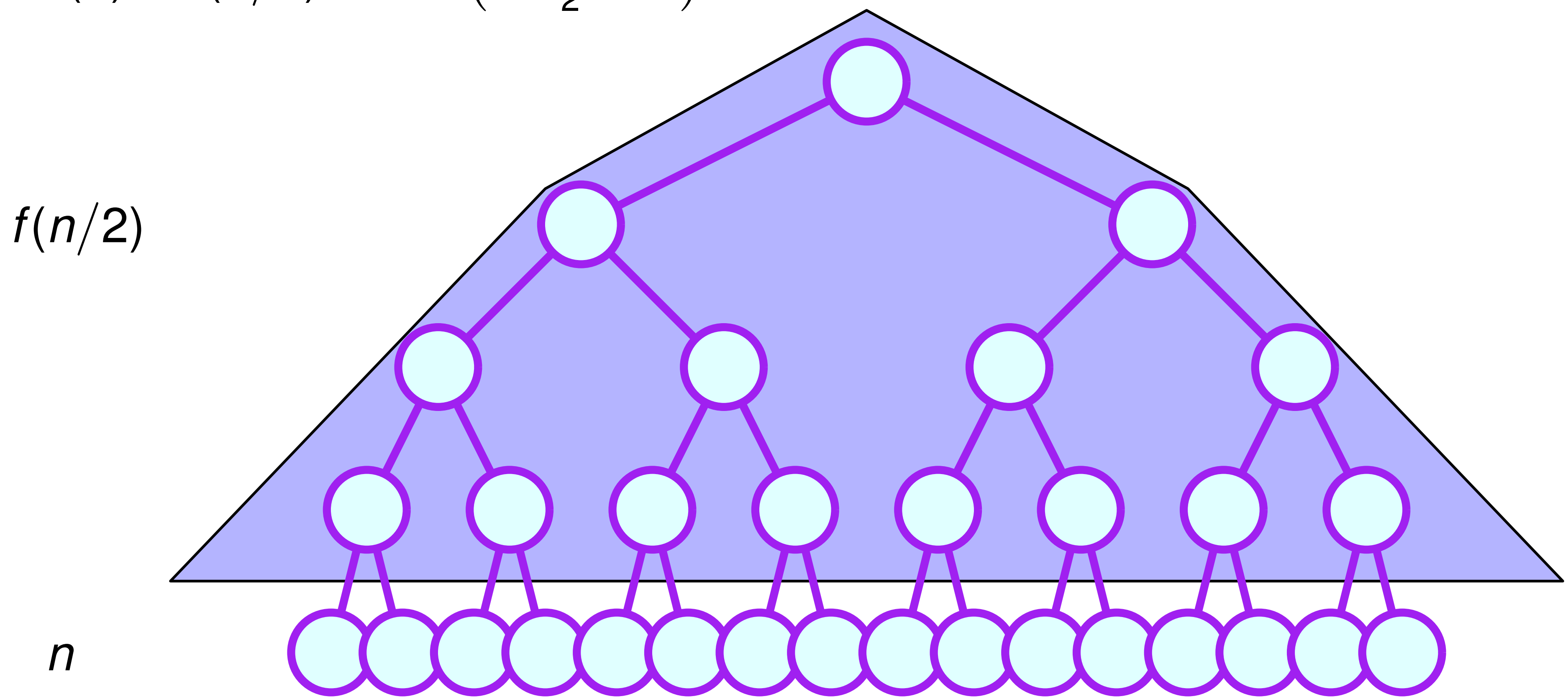

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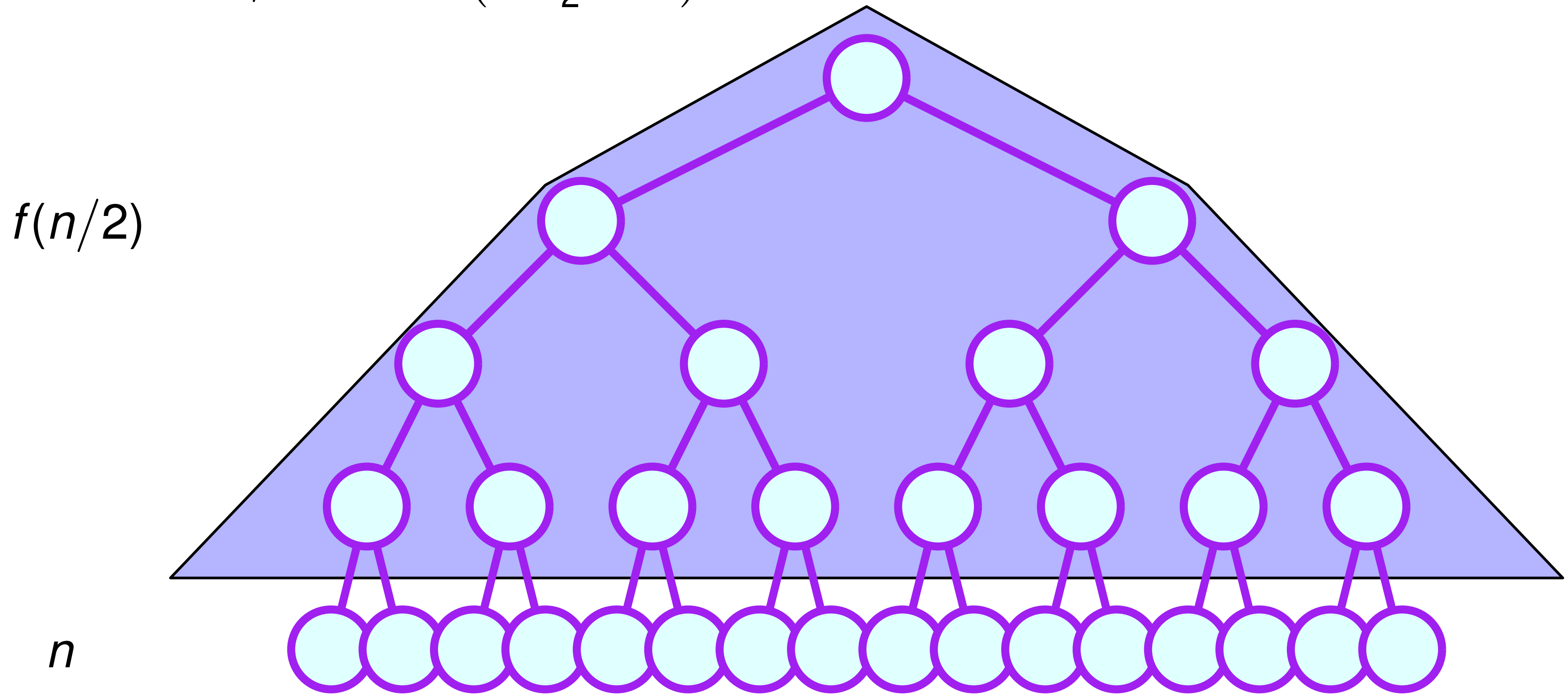
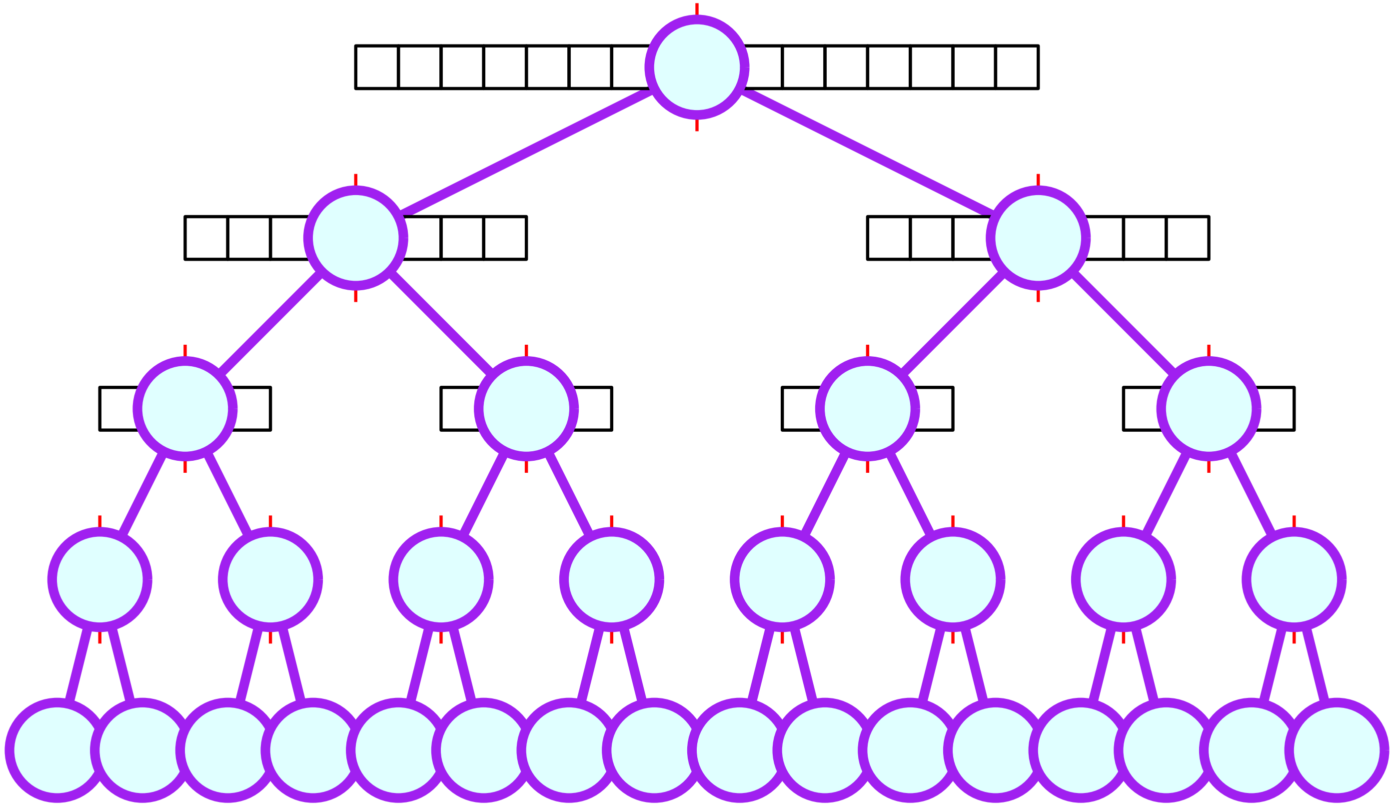


Work-efficient Prefix Sums



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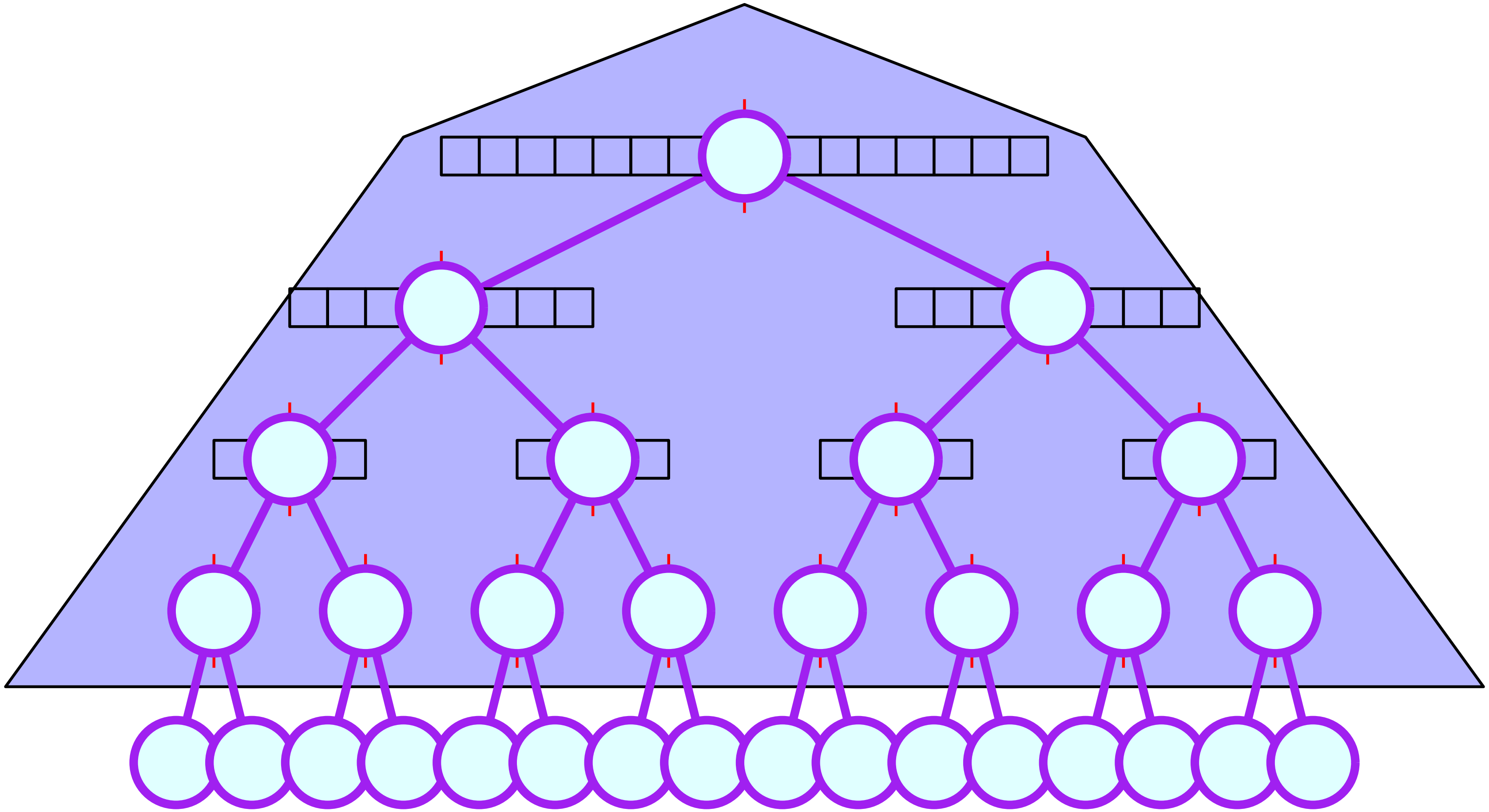

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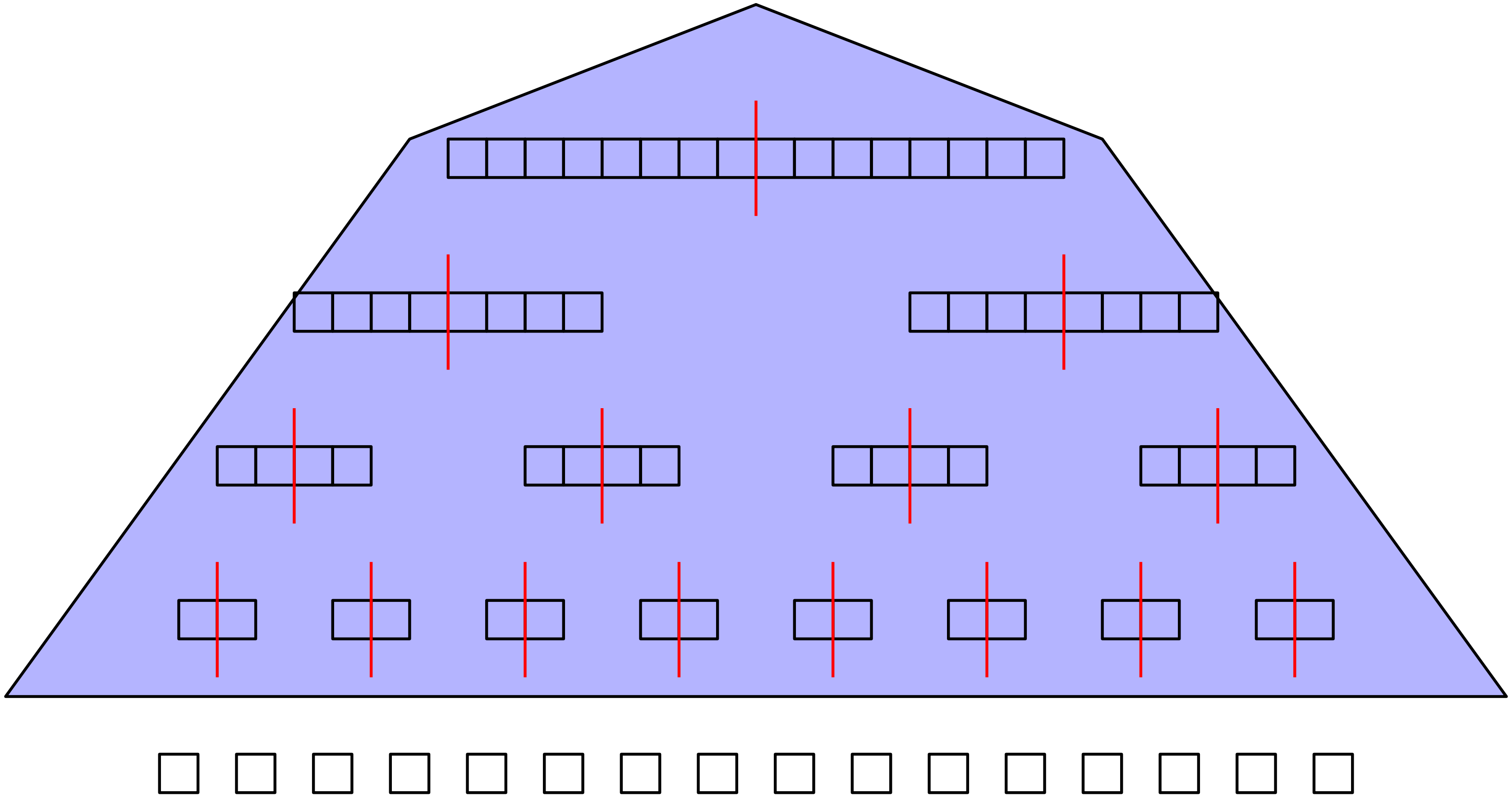

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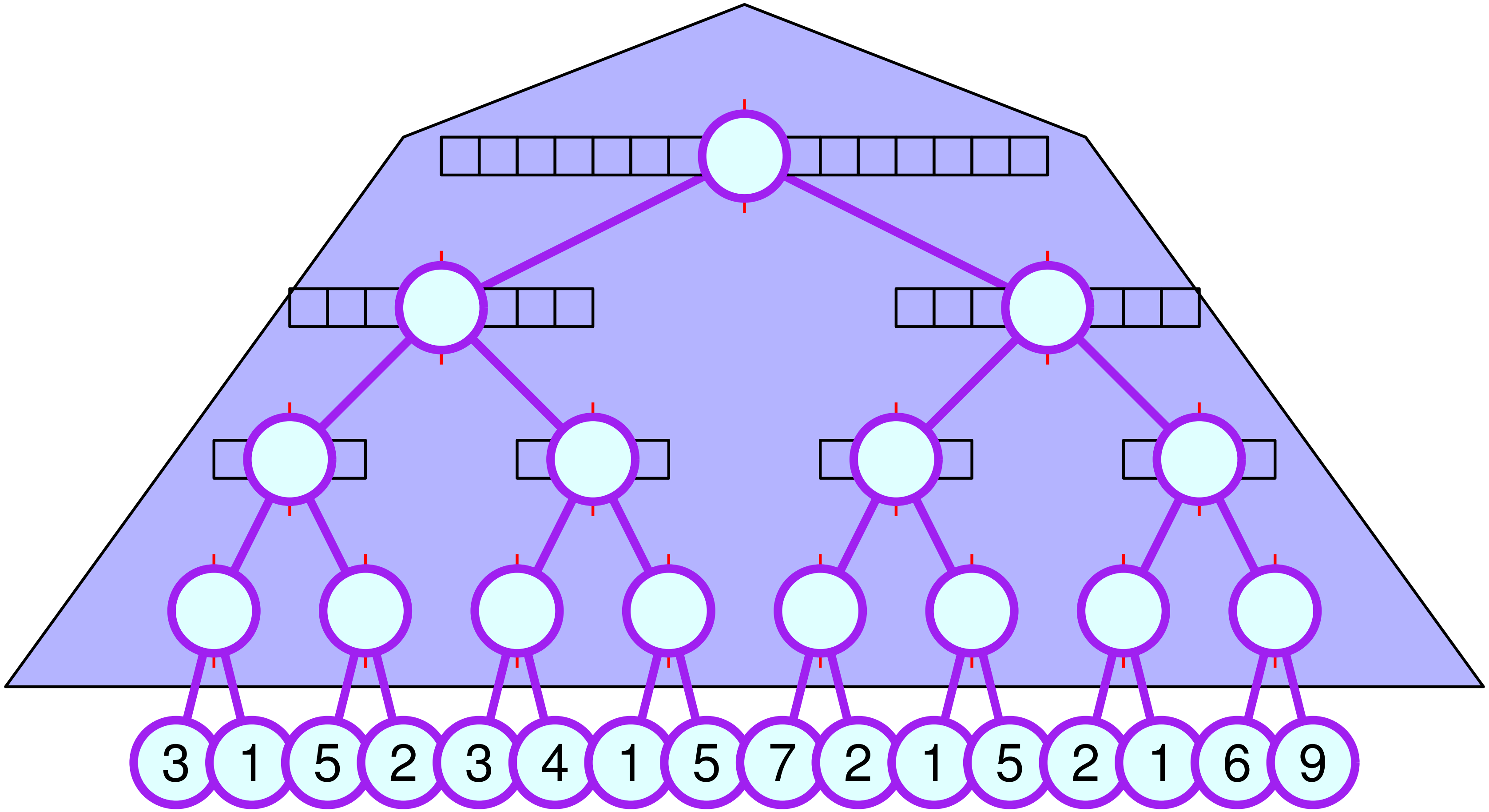

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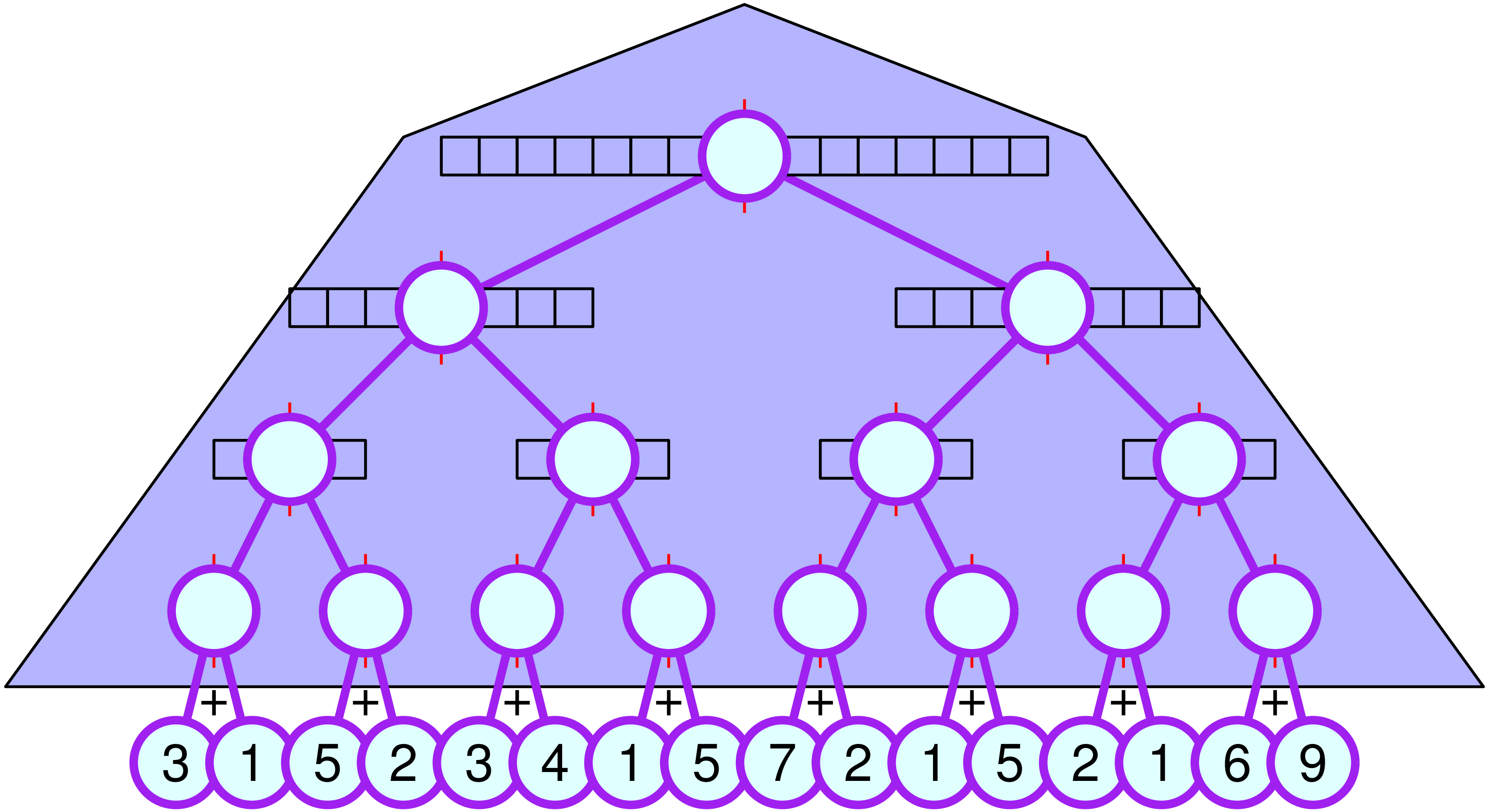

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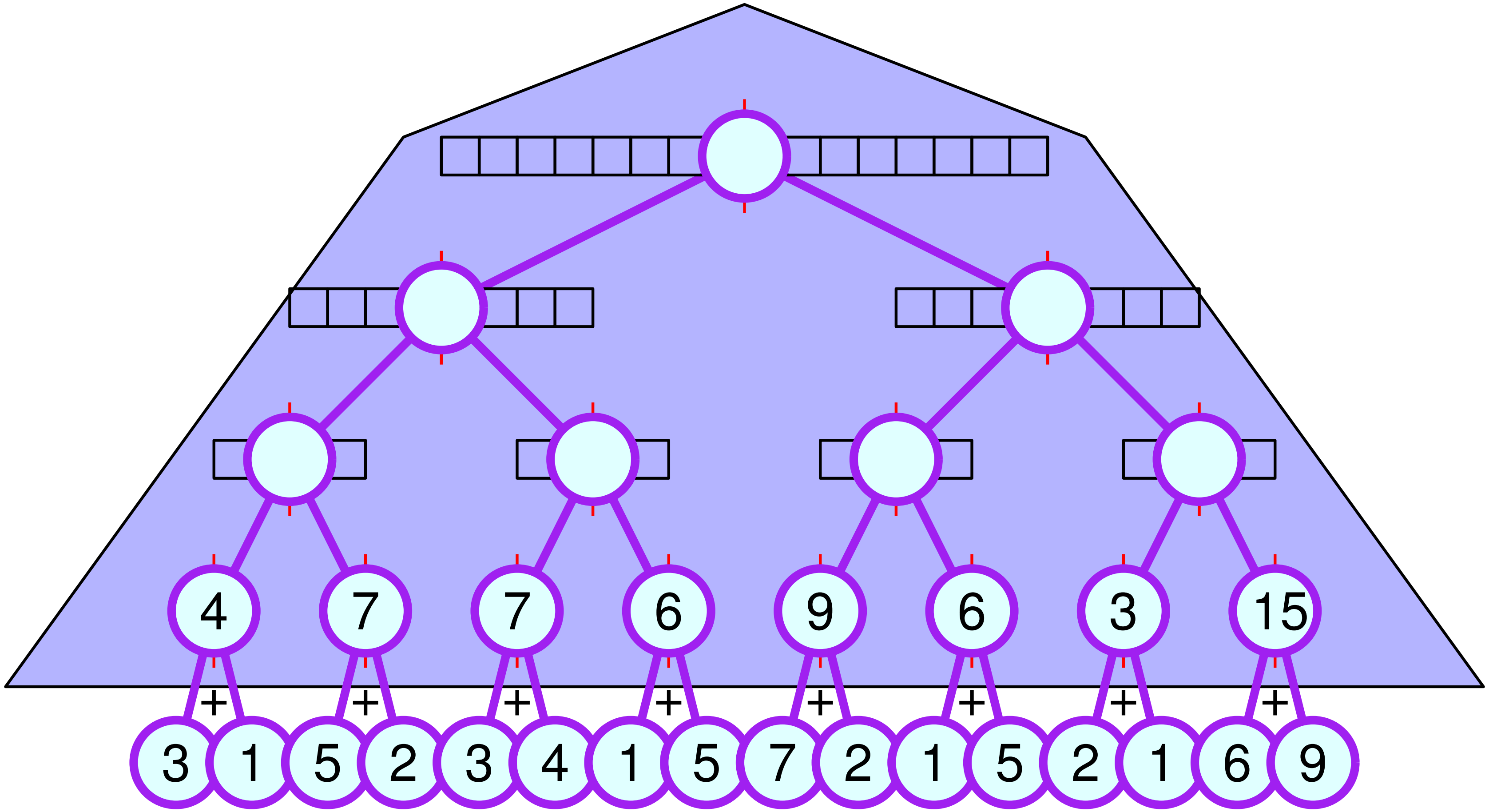

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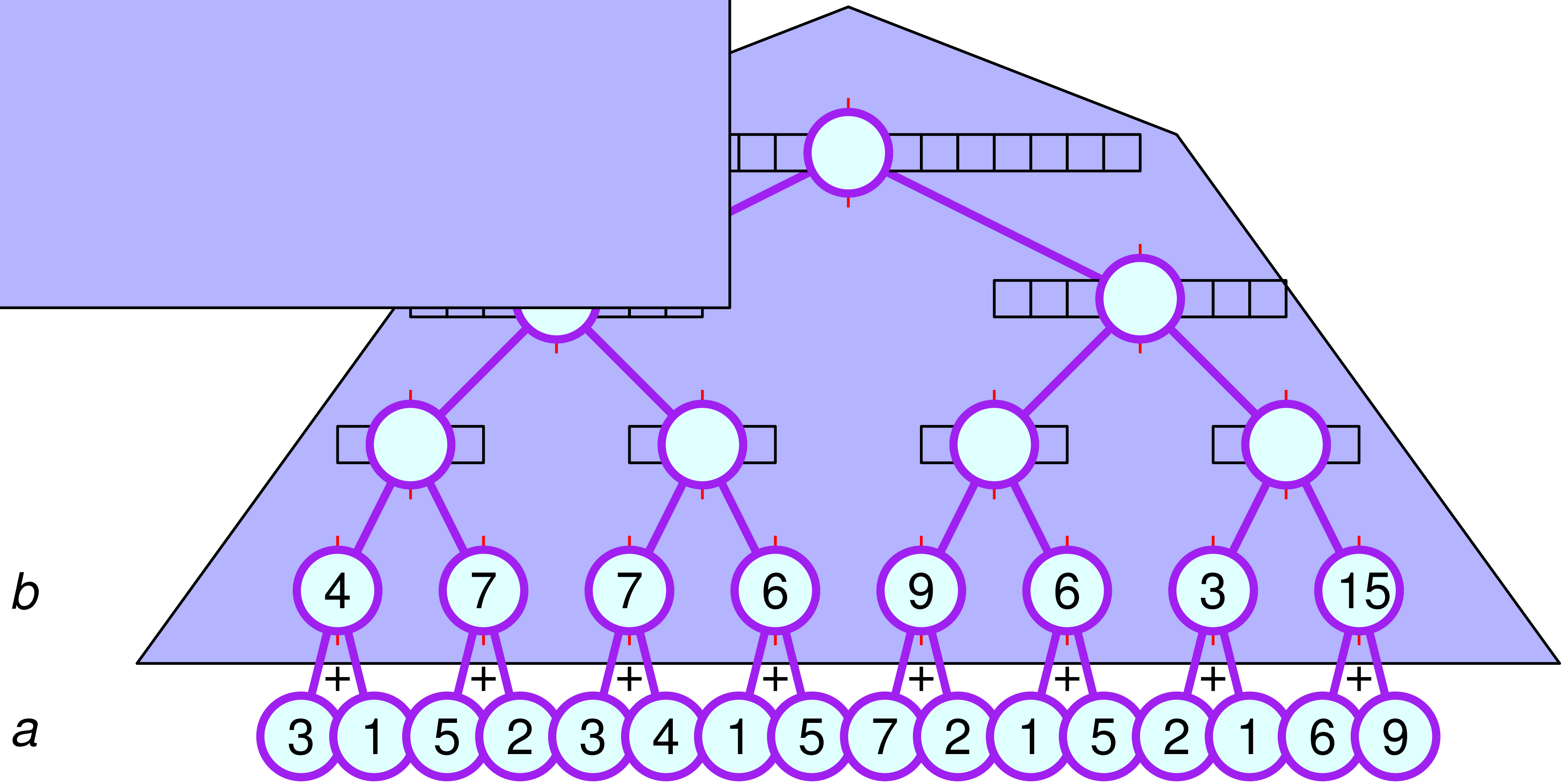
procedure PREFIX-SUMS($a[1..n]$)

for $i = 1$ to $\frac{n}{2}$ **in parallel do**
$b[i] = a[2i-1] + a[2i]$



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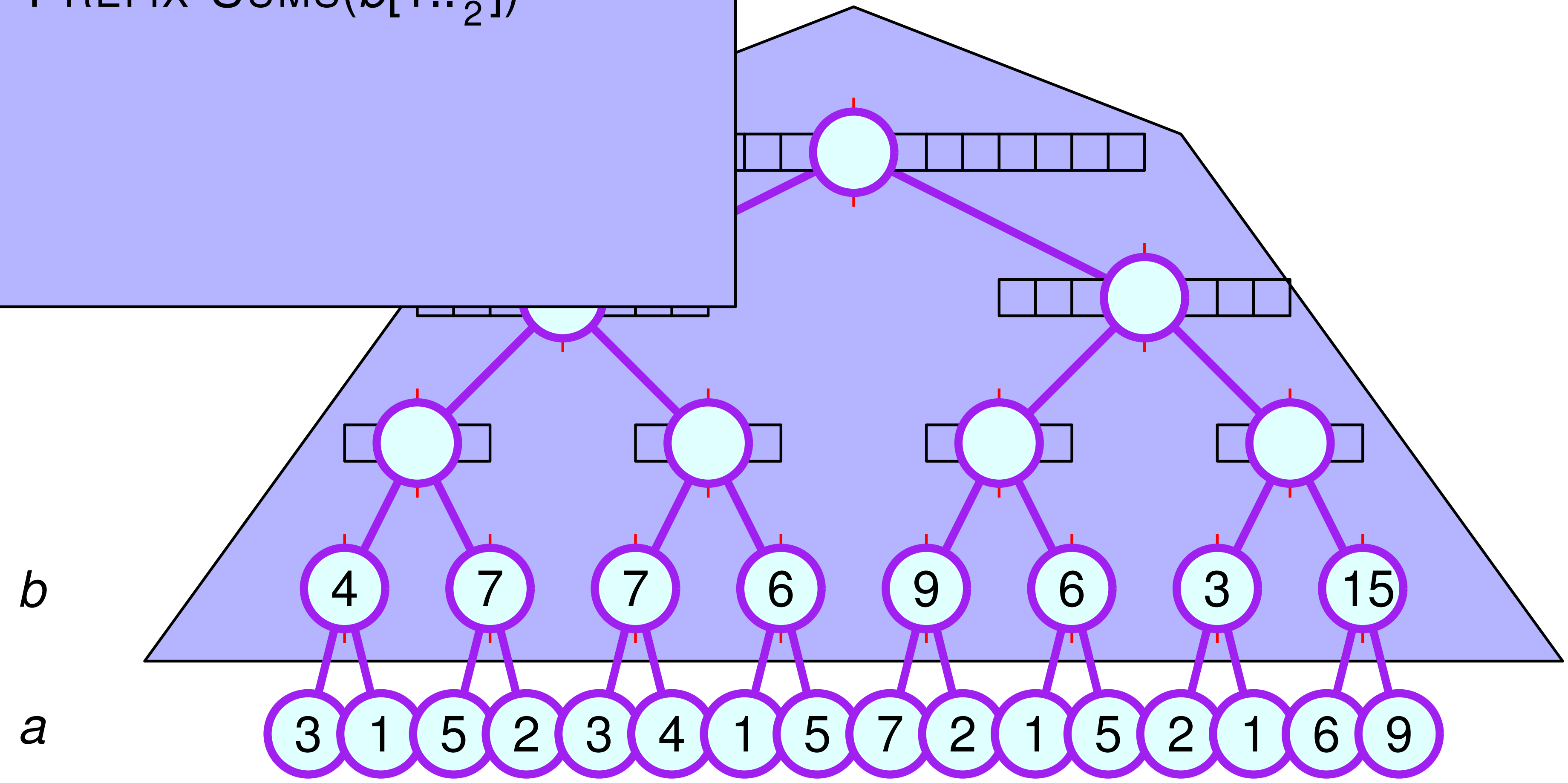

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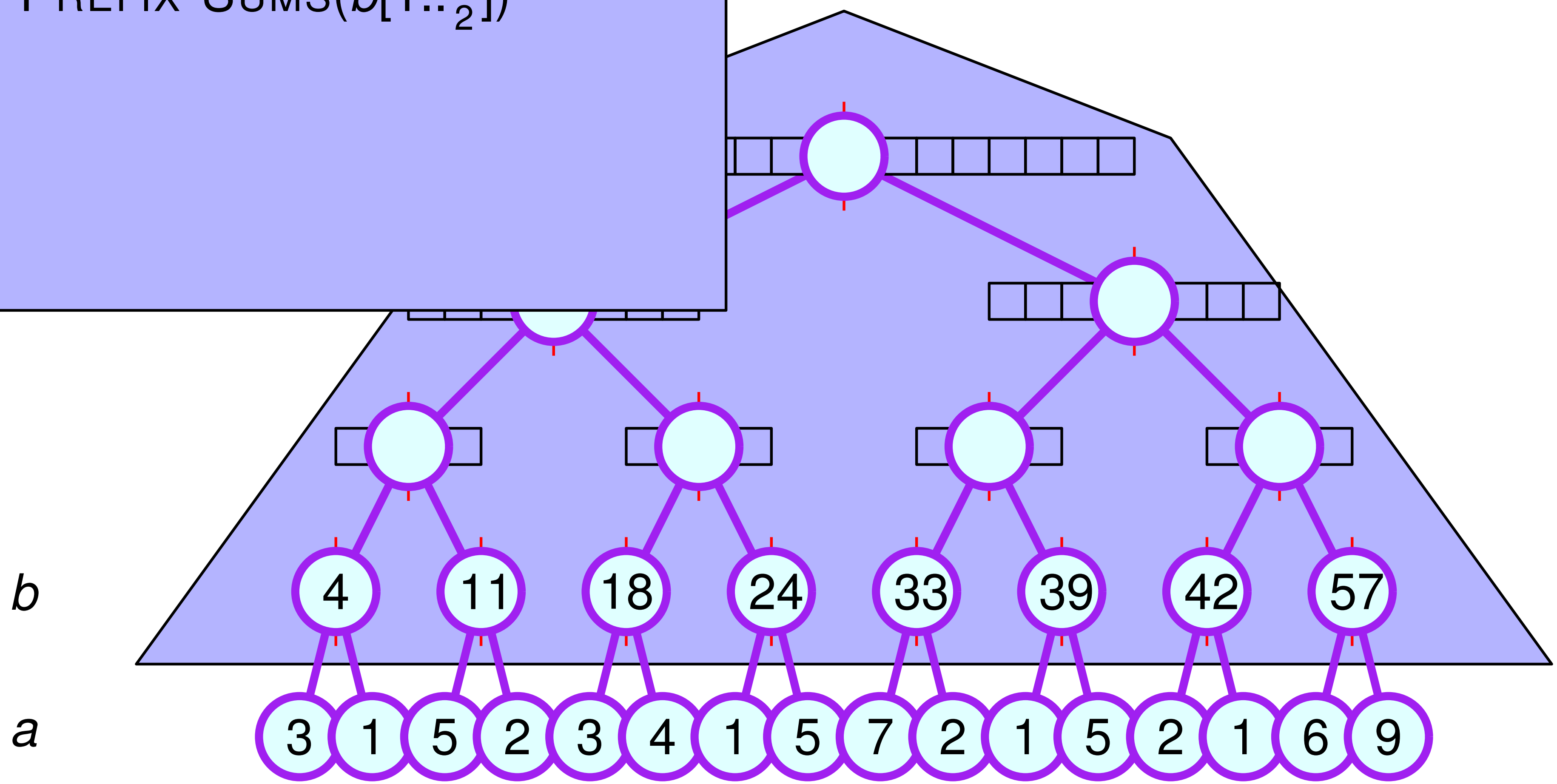

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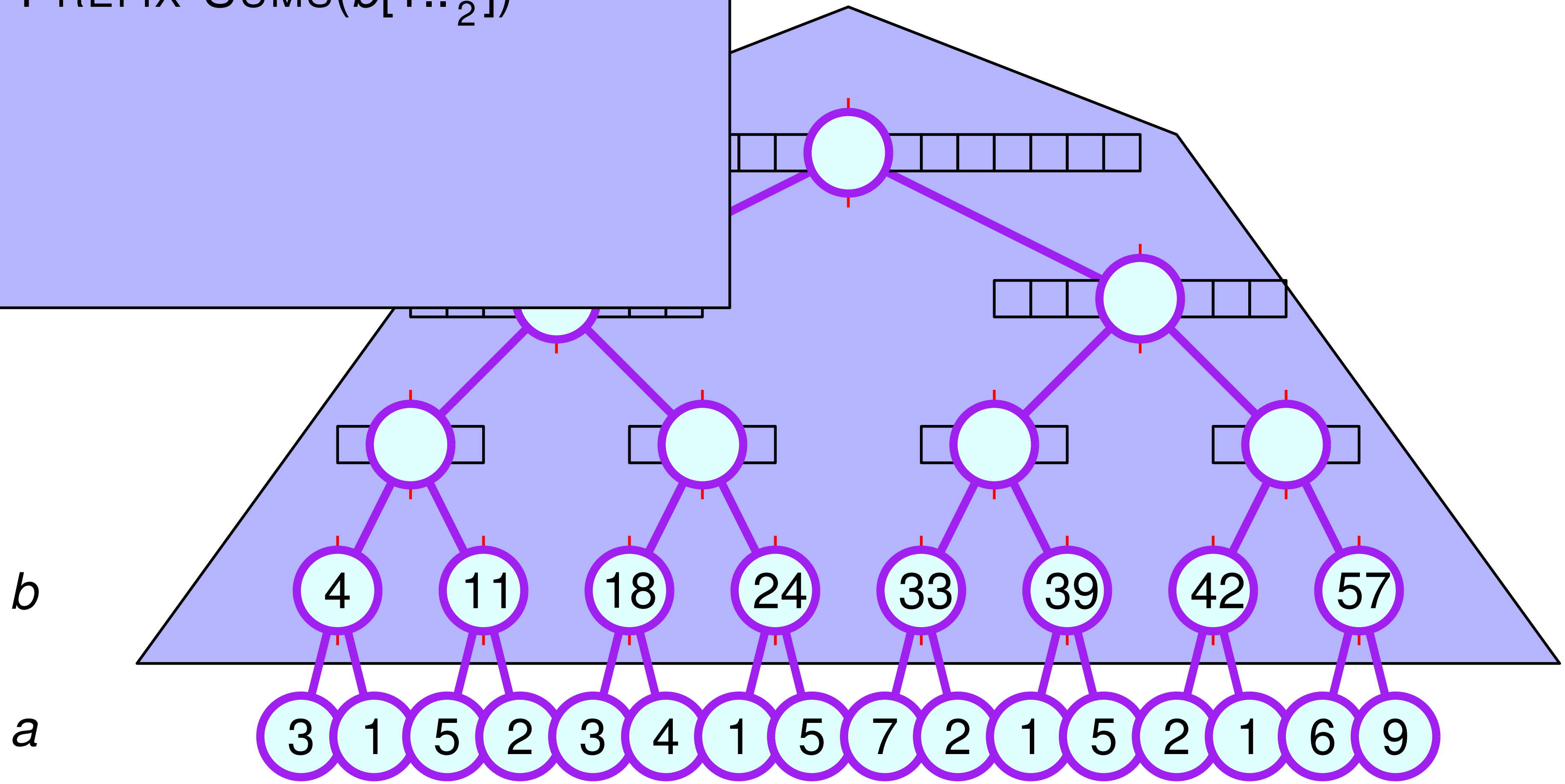
Claim. $b[i] = \sum_{k=1}^{2i} a[k]$



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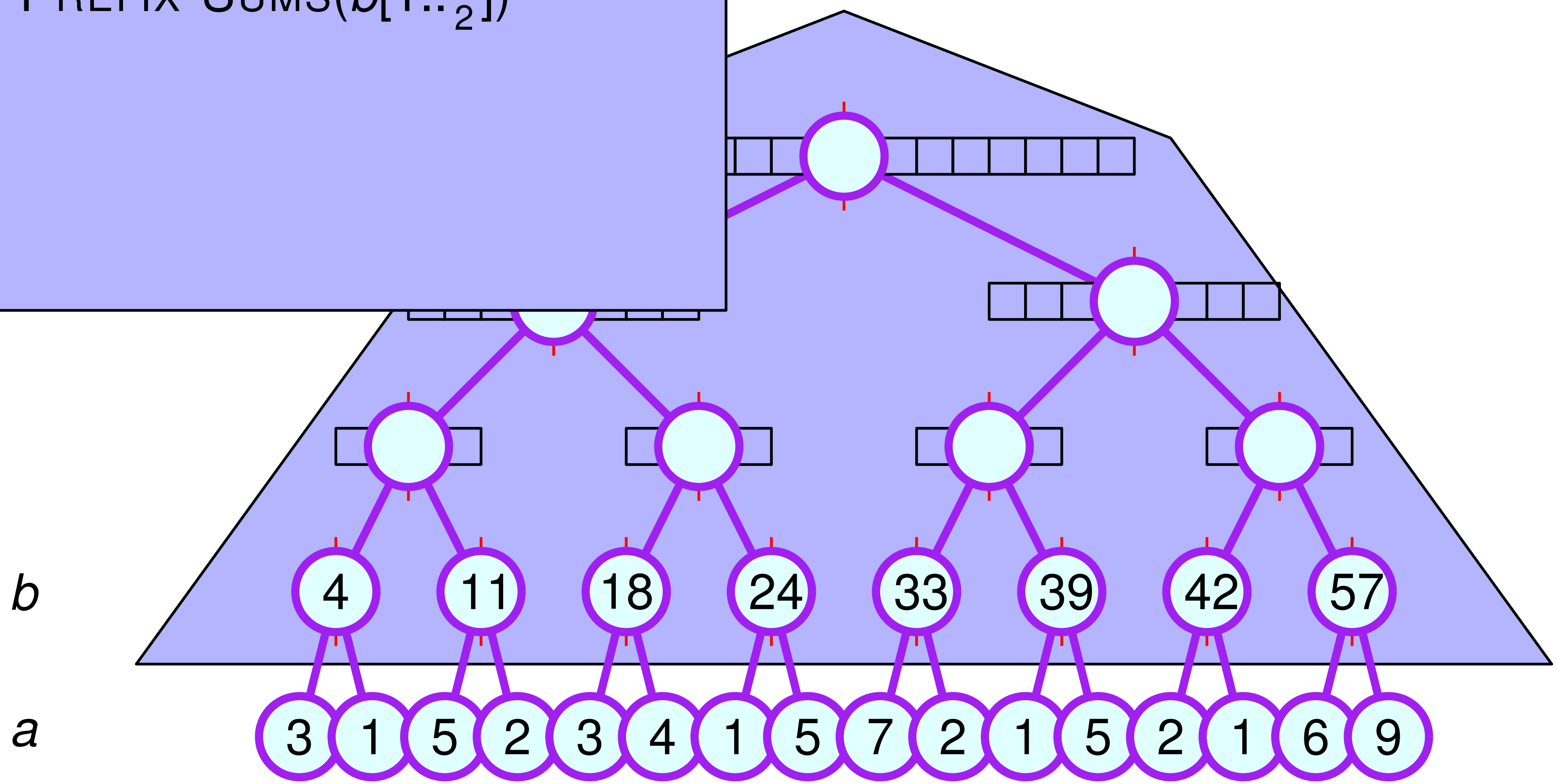
Proof. By I.H. $b[i] = \sum_{k=1}^{i} b[k]$



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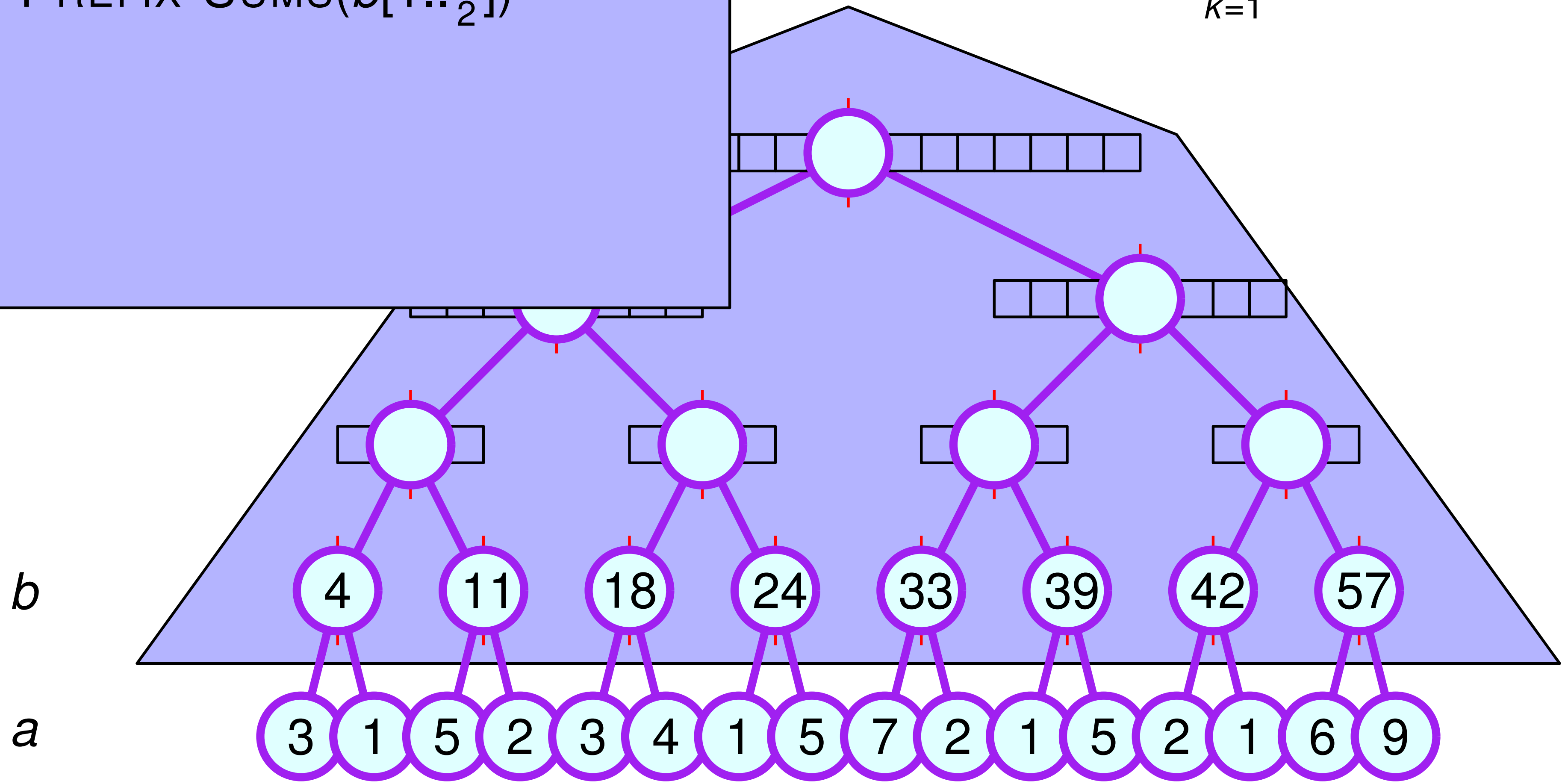
$$= \sum_{k=1}^{i} (a[2k - 1] + a[2k])$$



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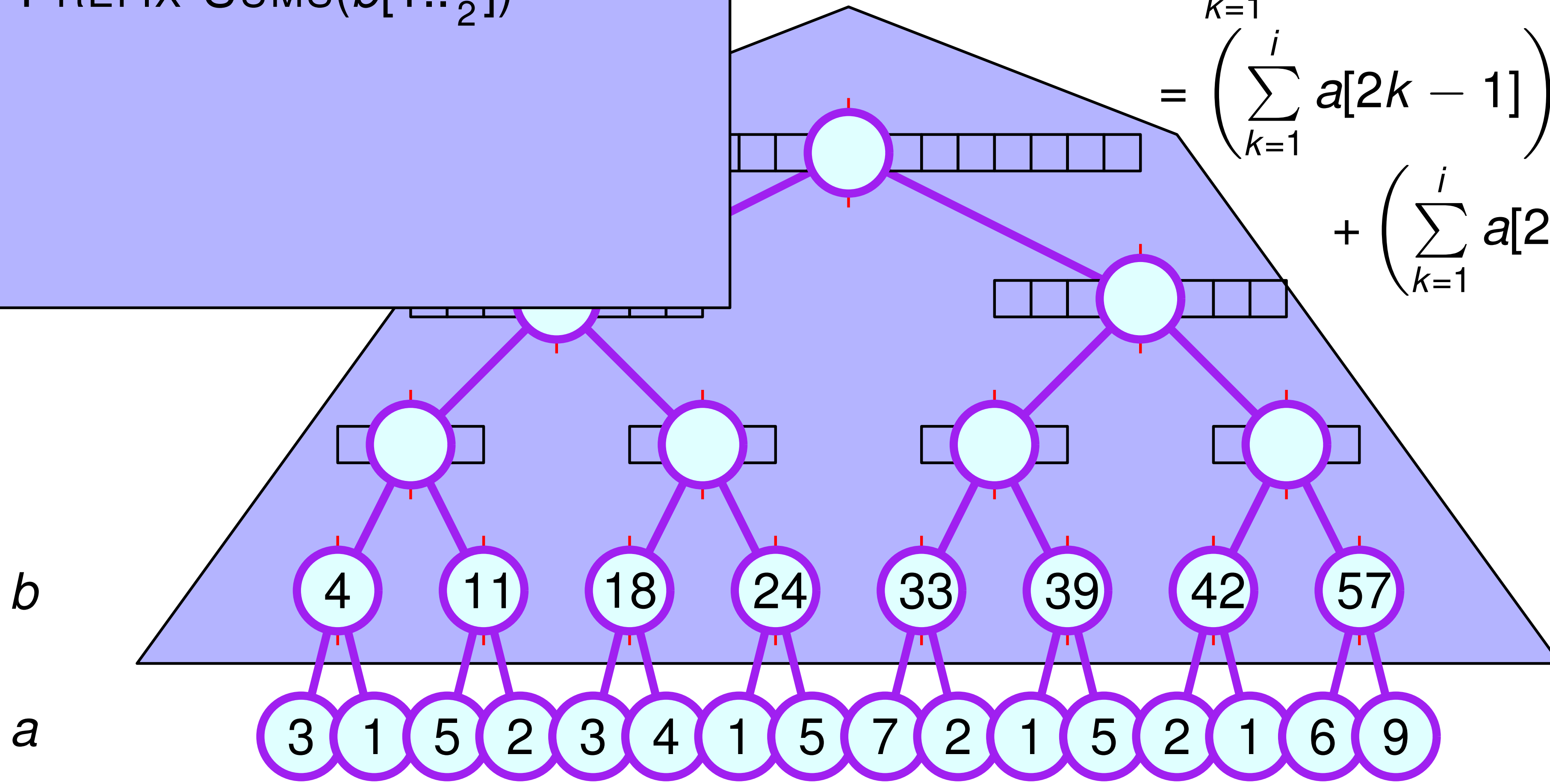

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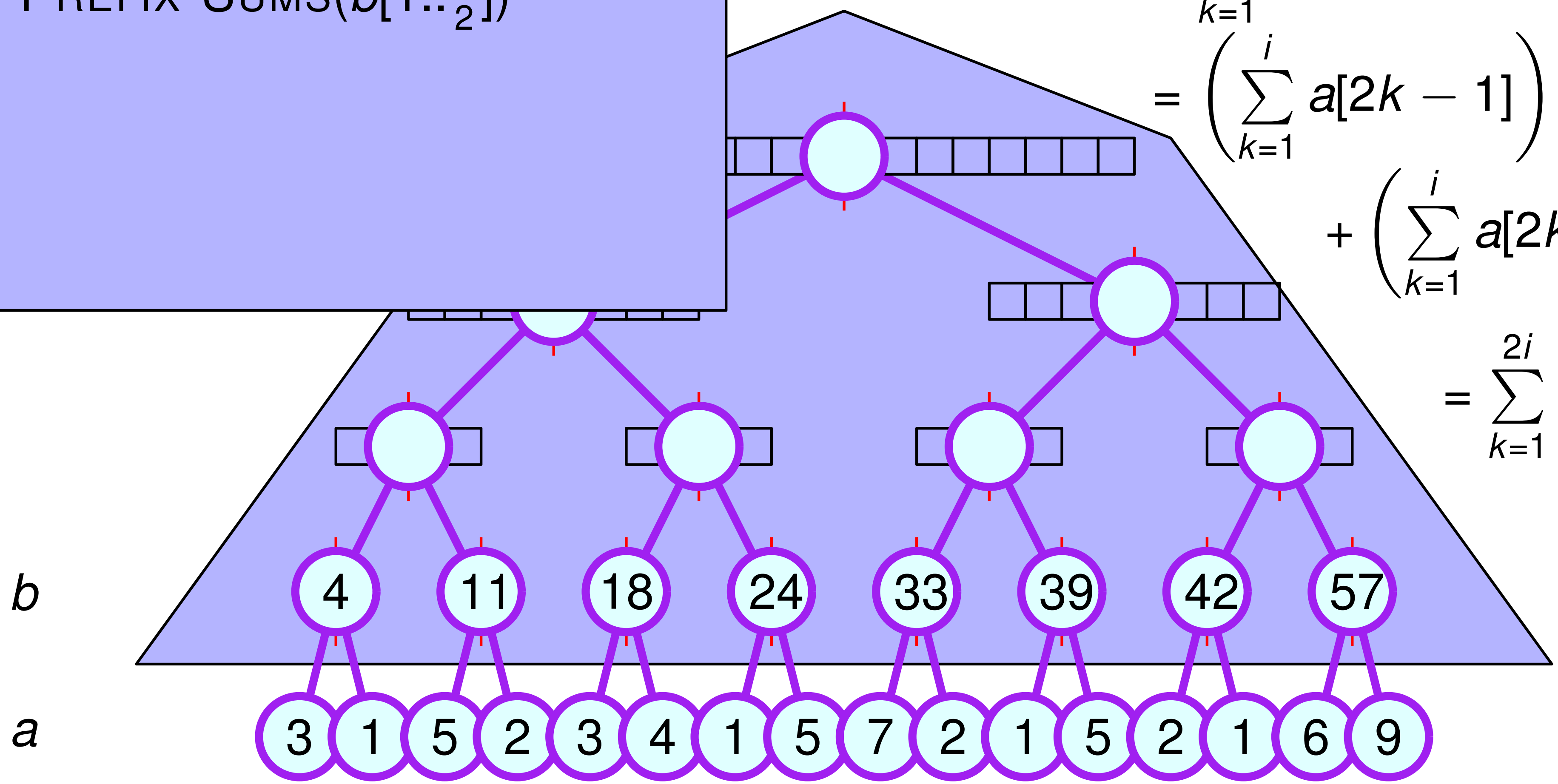

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$b[i] = a[2i-1] + a[2i]$
PREFIX-SUMS($b[1..\frac{n}{2}]$)

Claim. $b[i] = \sum_{k=1}^{2i} a[k]$

Proof. By I.H. $b[i] = \sum_{k=1}^{i} b[k]$

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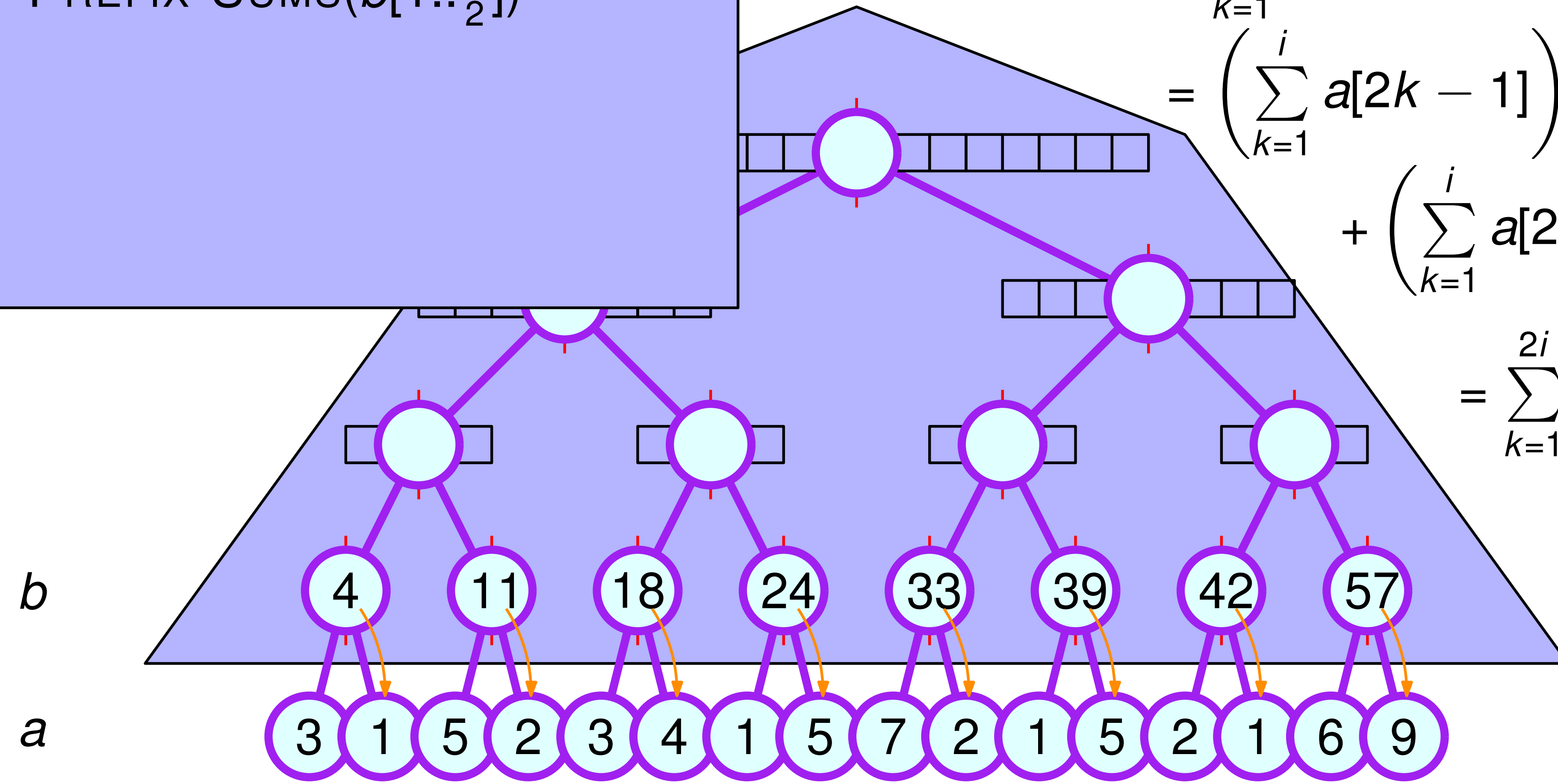

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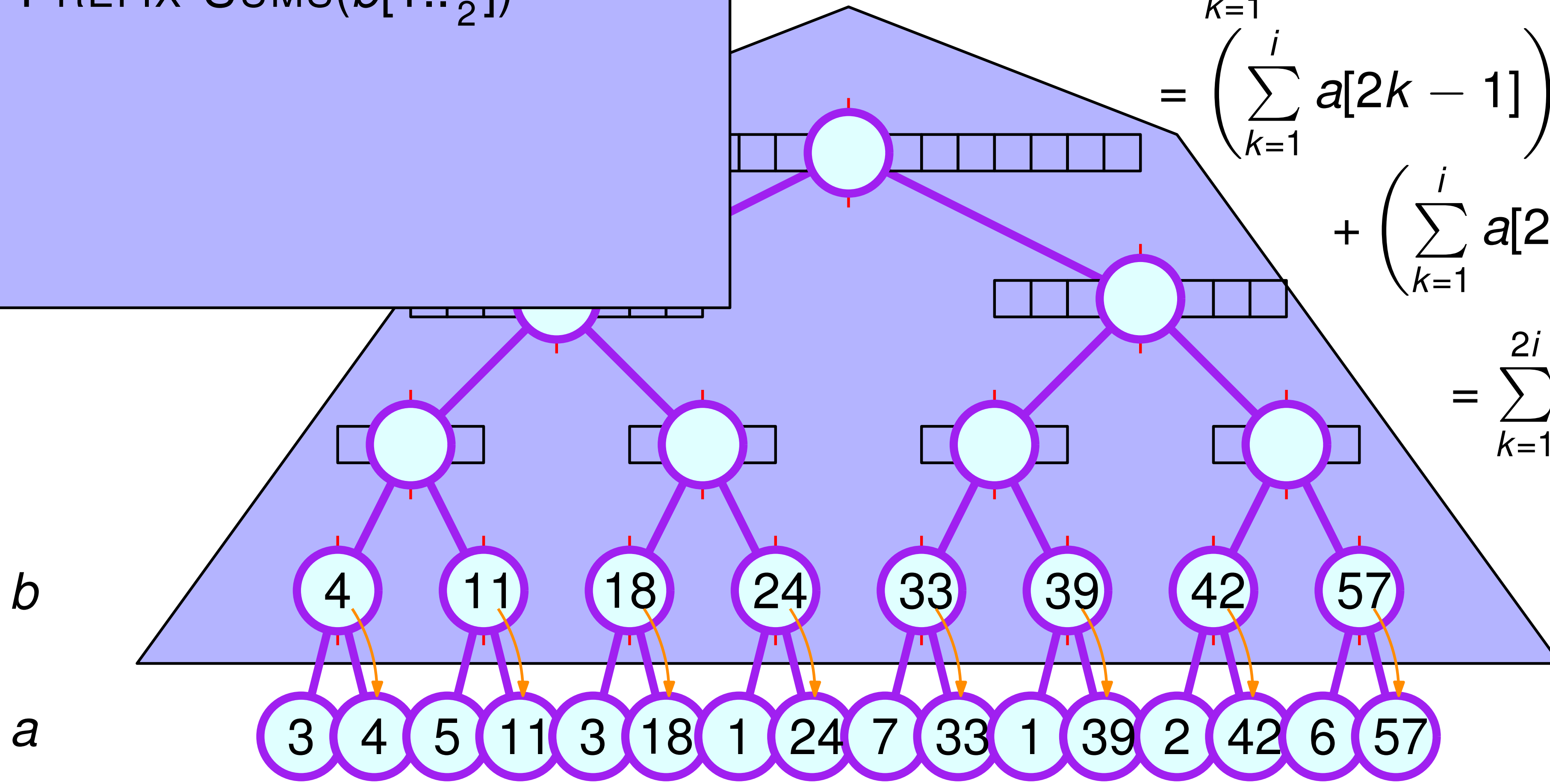

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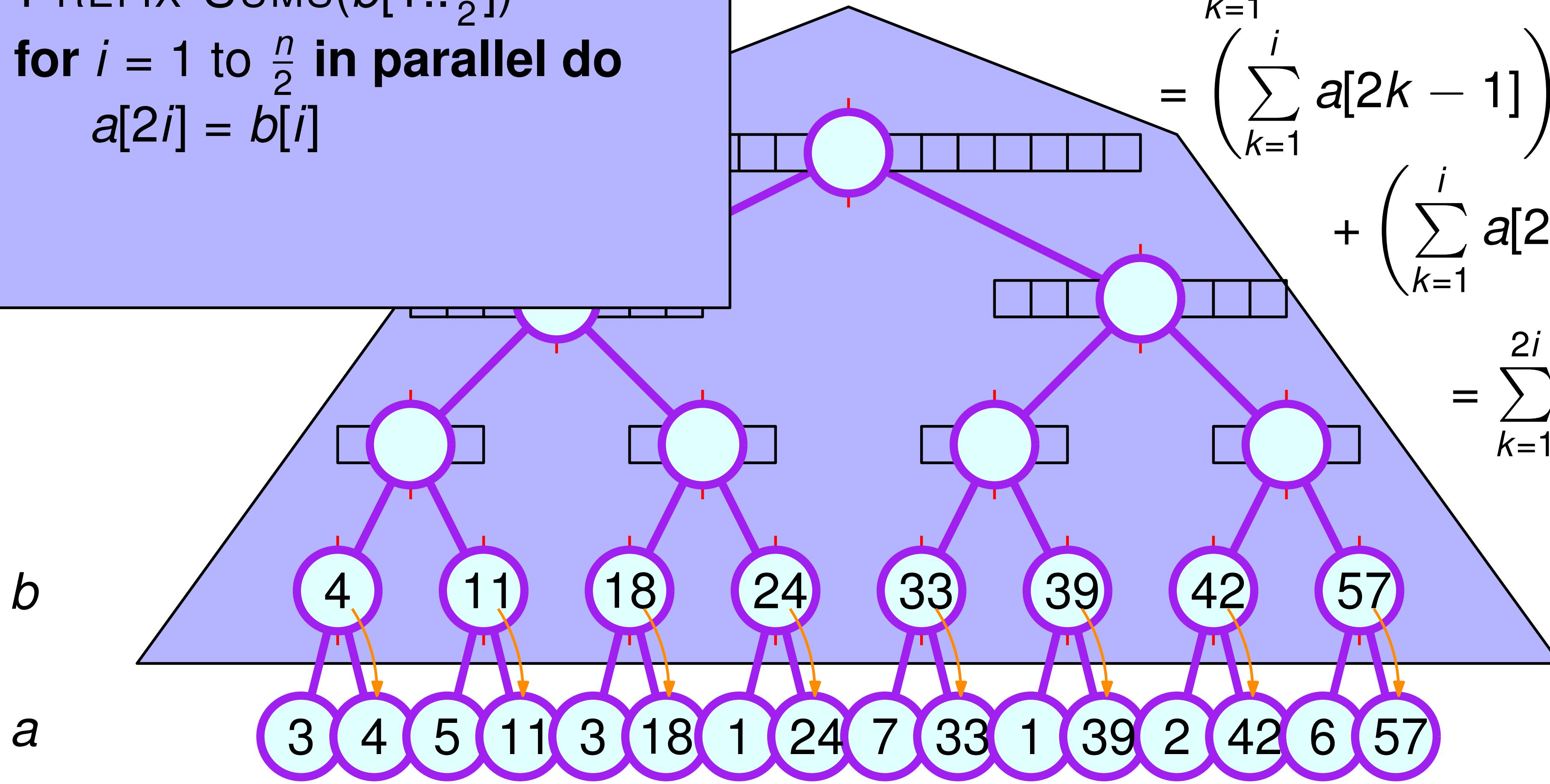

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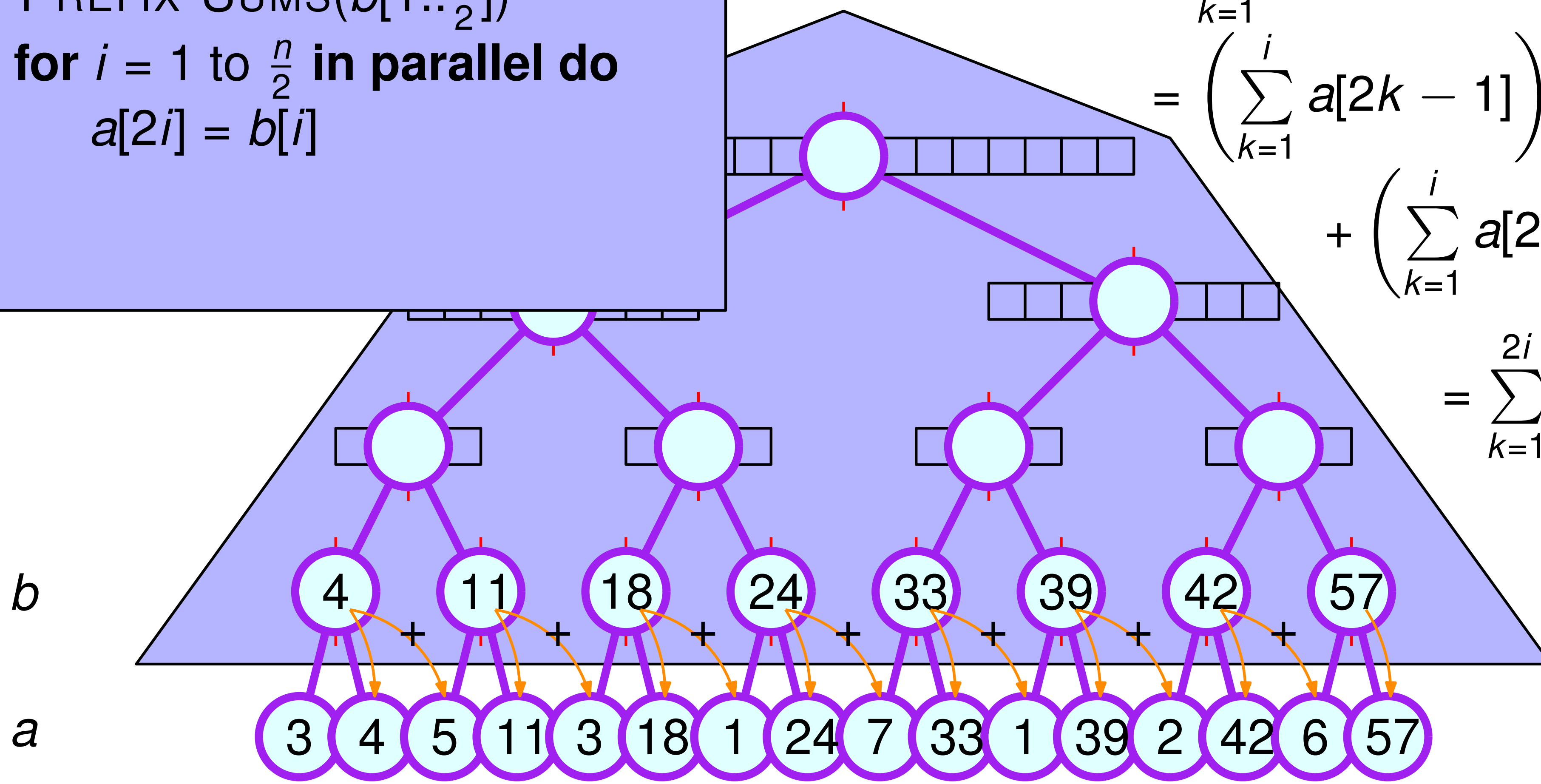

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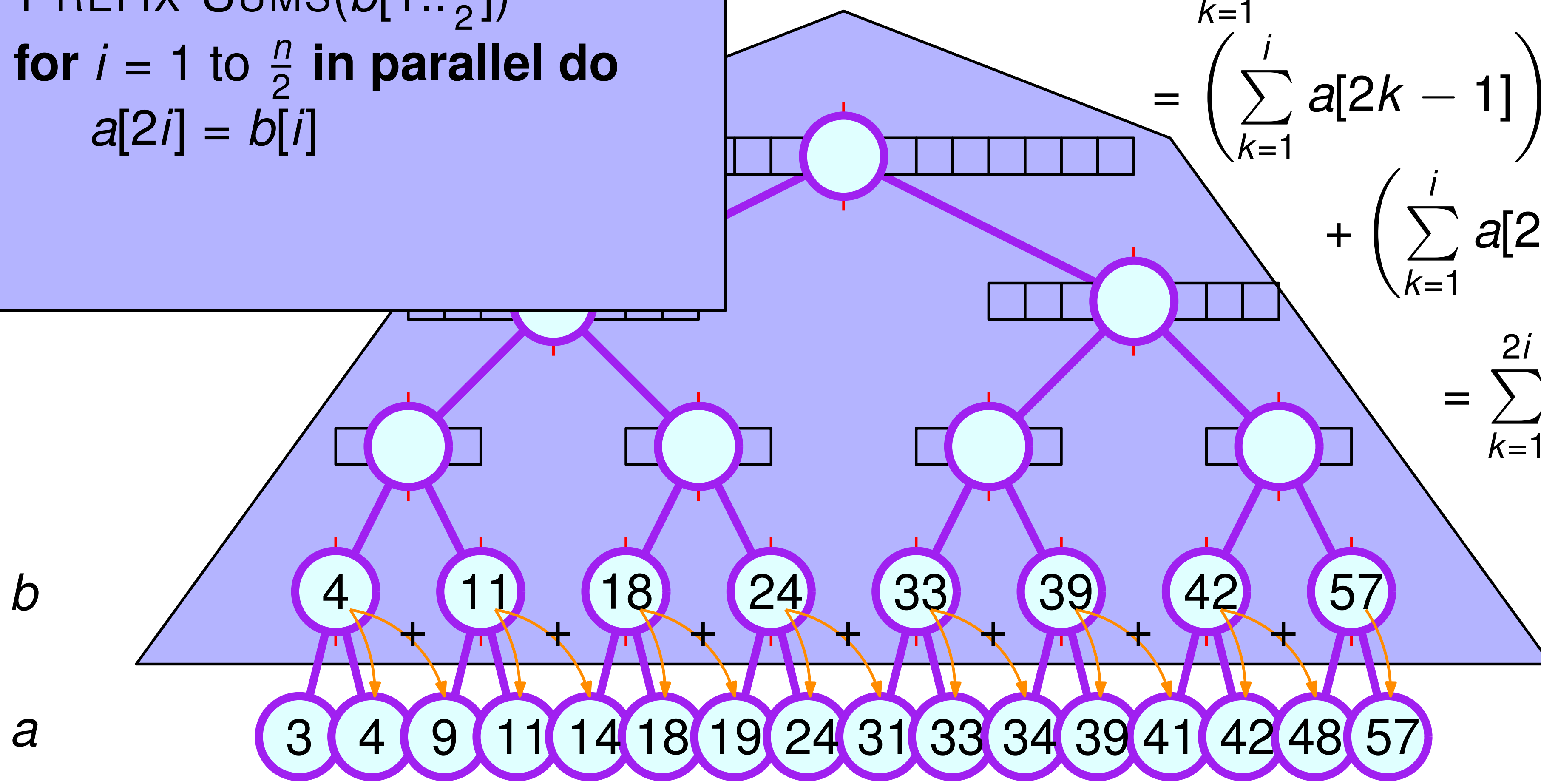

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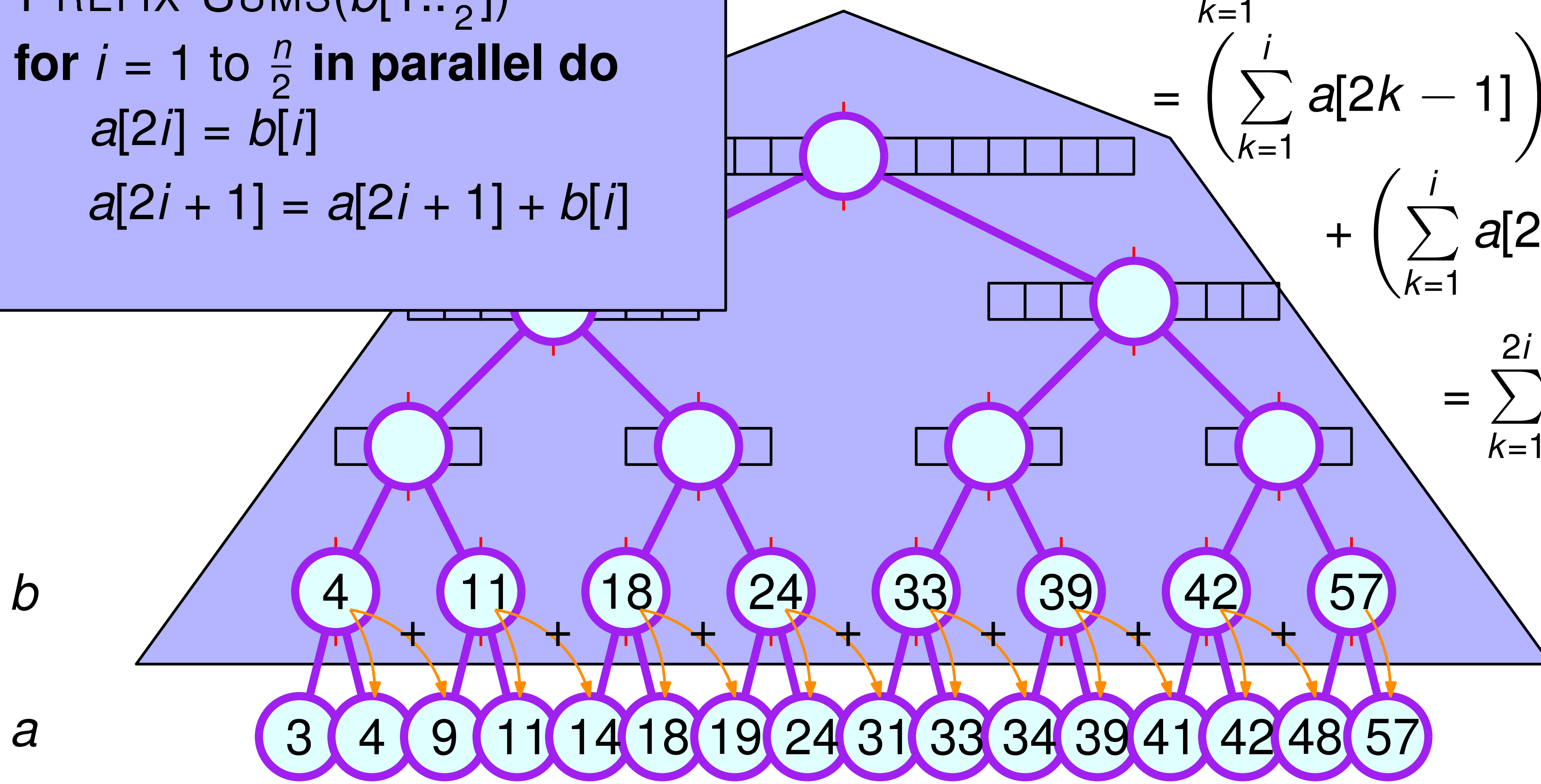

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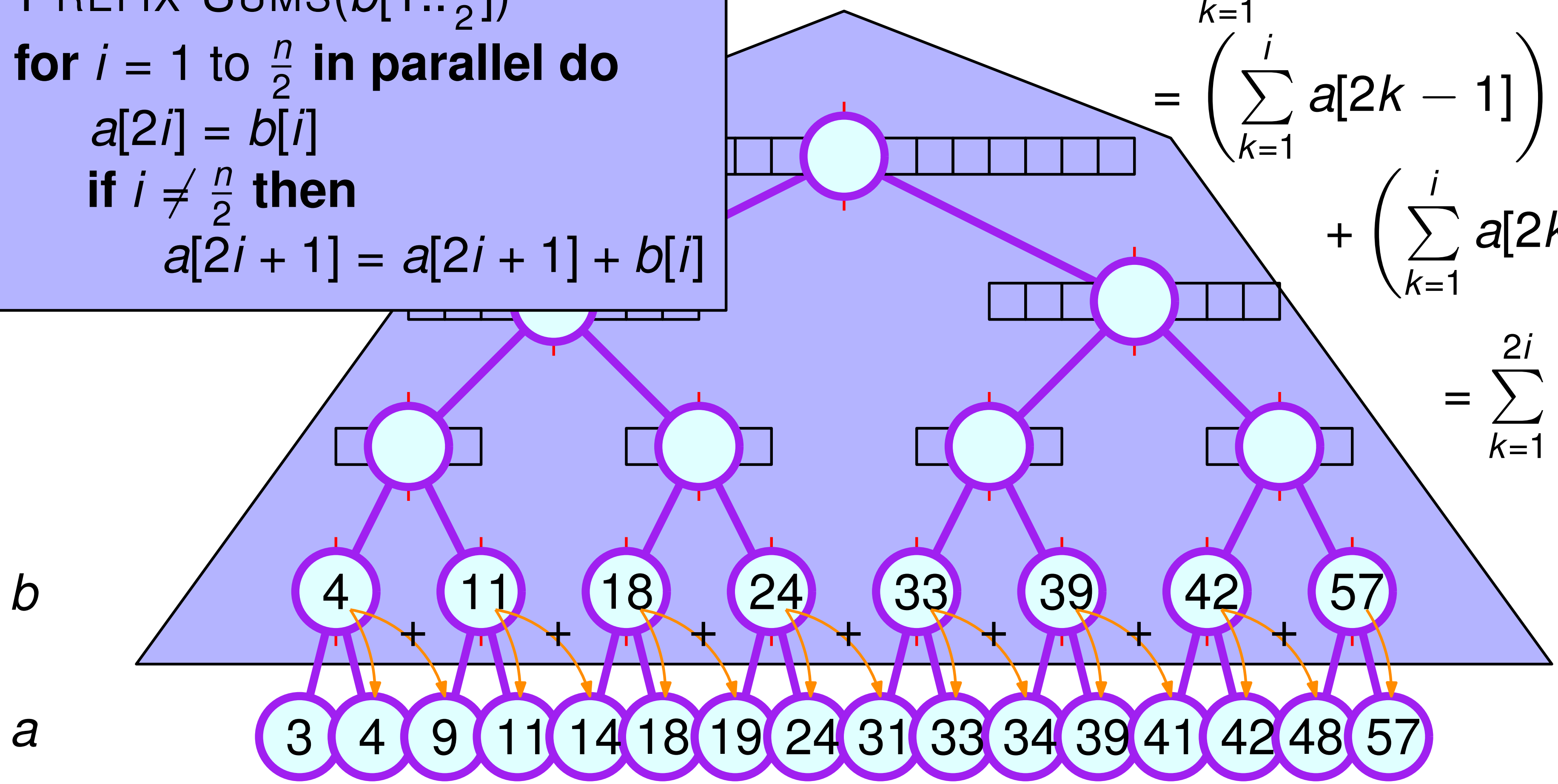

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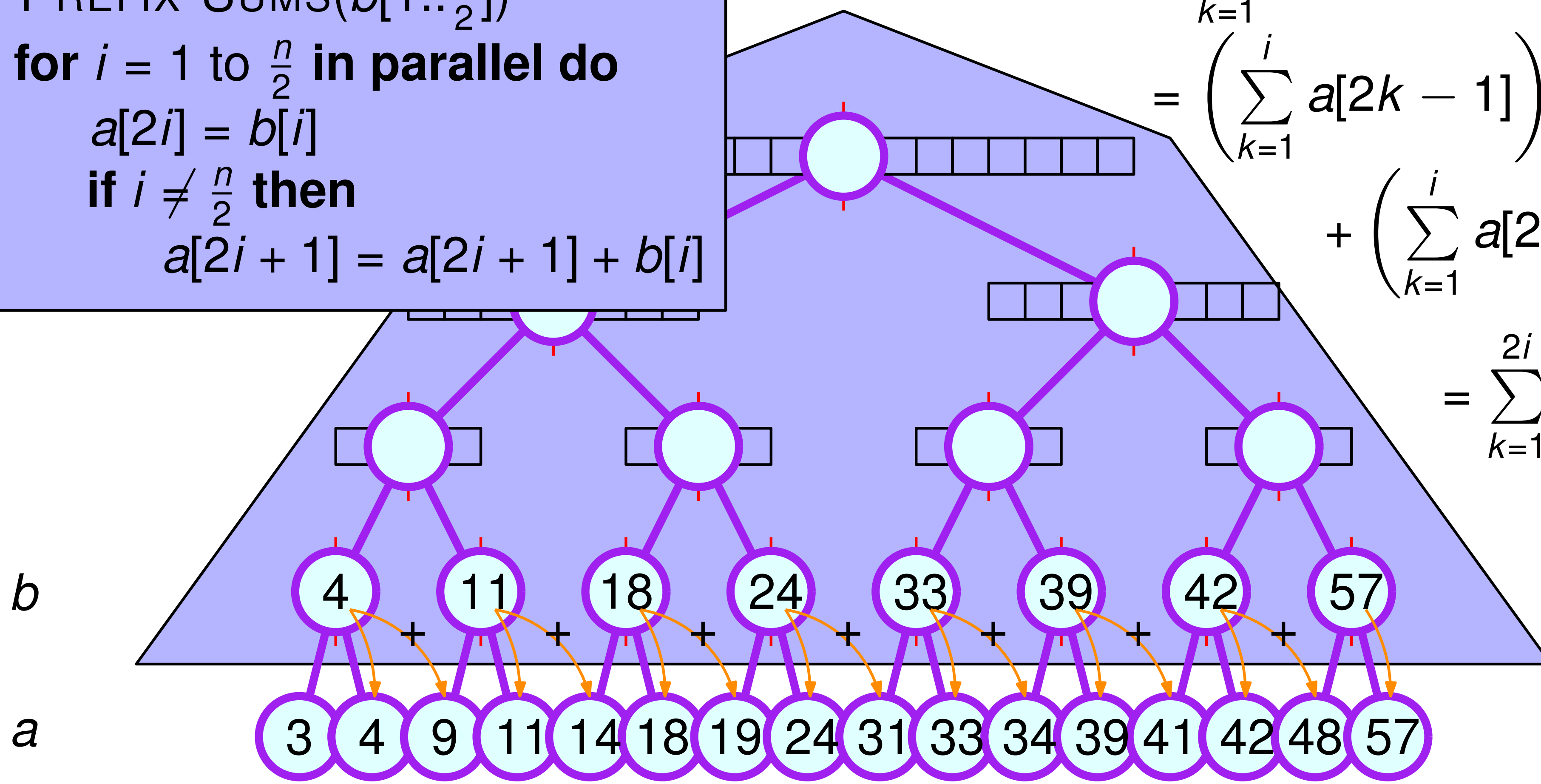

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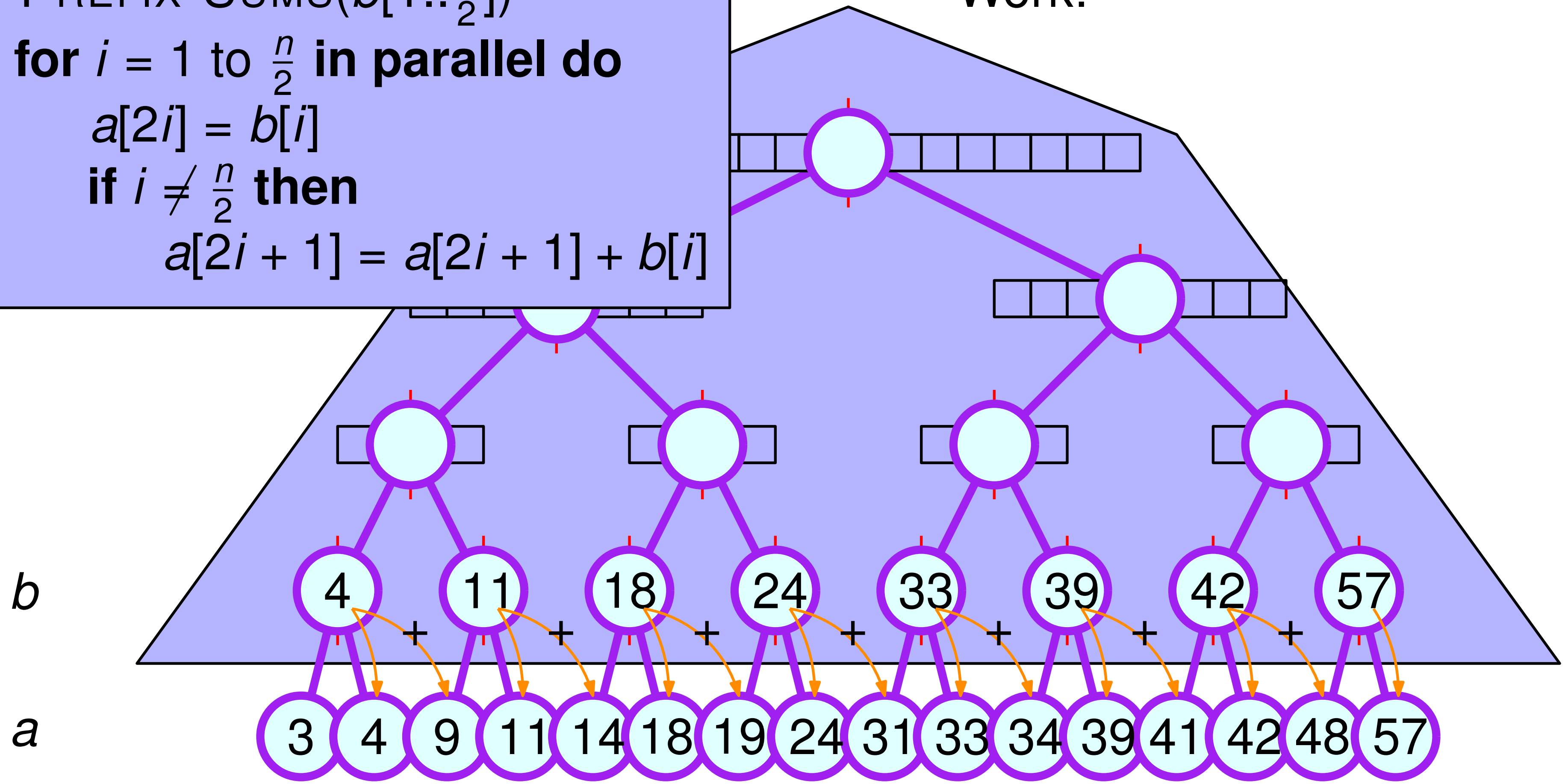
Analysis

Time:

Work:



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    if $i \neq \frac{n}{2}$ **then**
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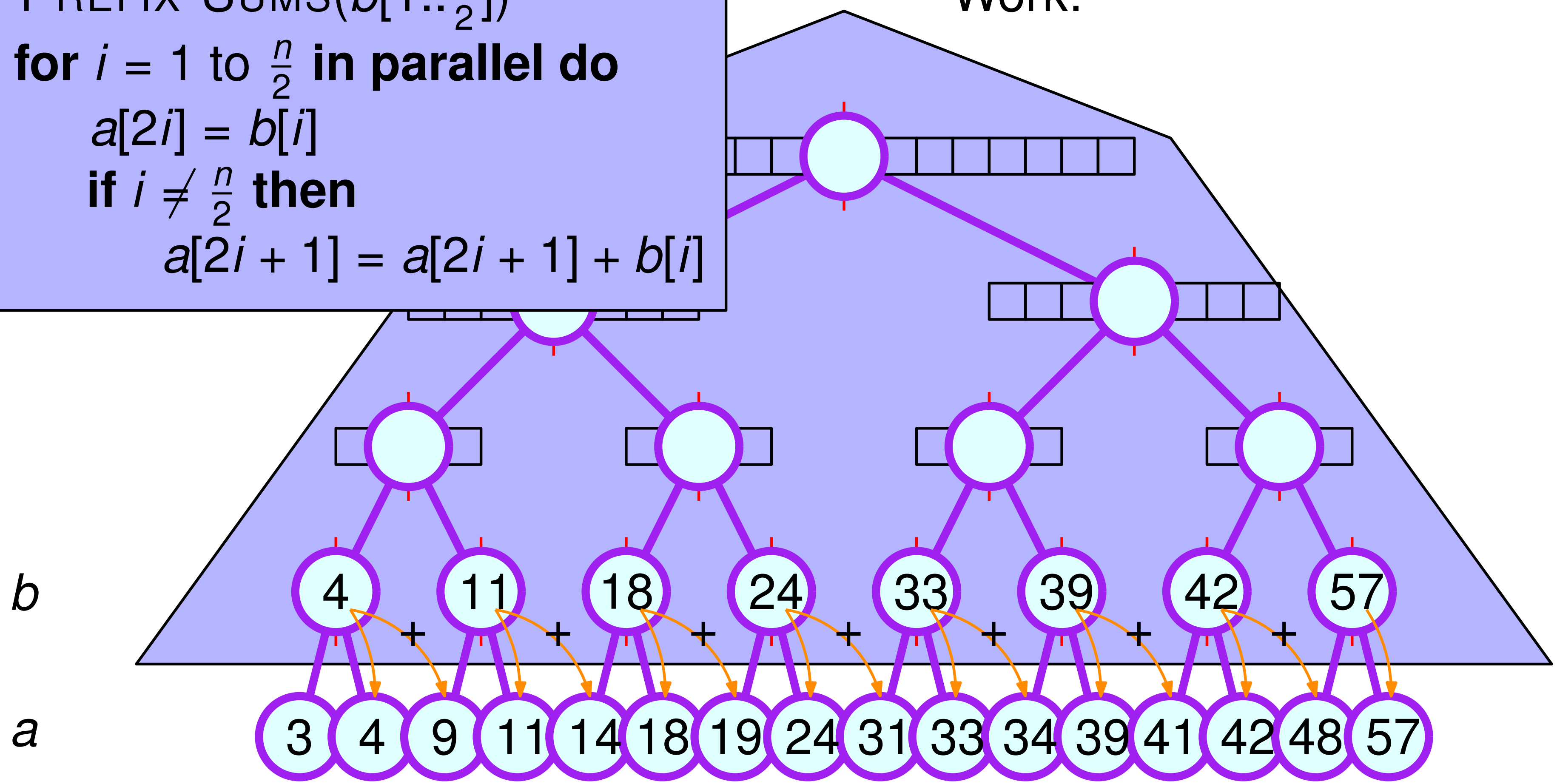
Analysis

Time: $T(n) = T(n/2) + O(1)$

Work:



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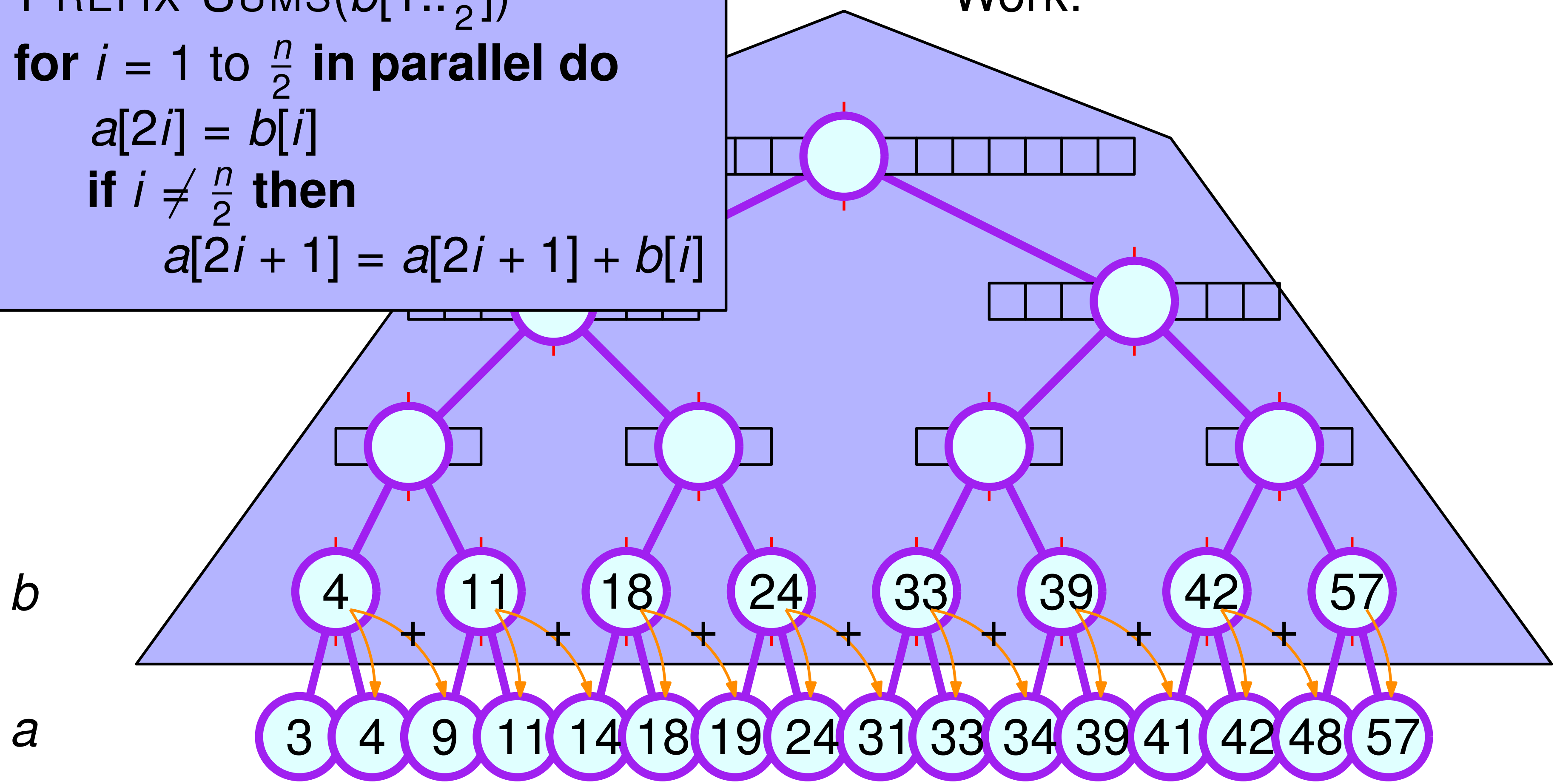
Analysis

Time: $T(n) = T(n/2) + O(1) = O(\log n)$

Work:



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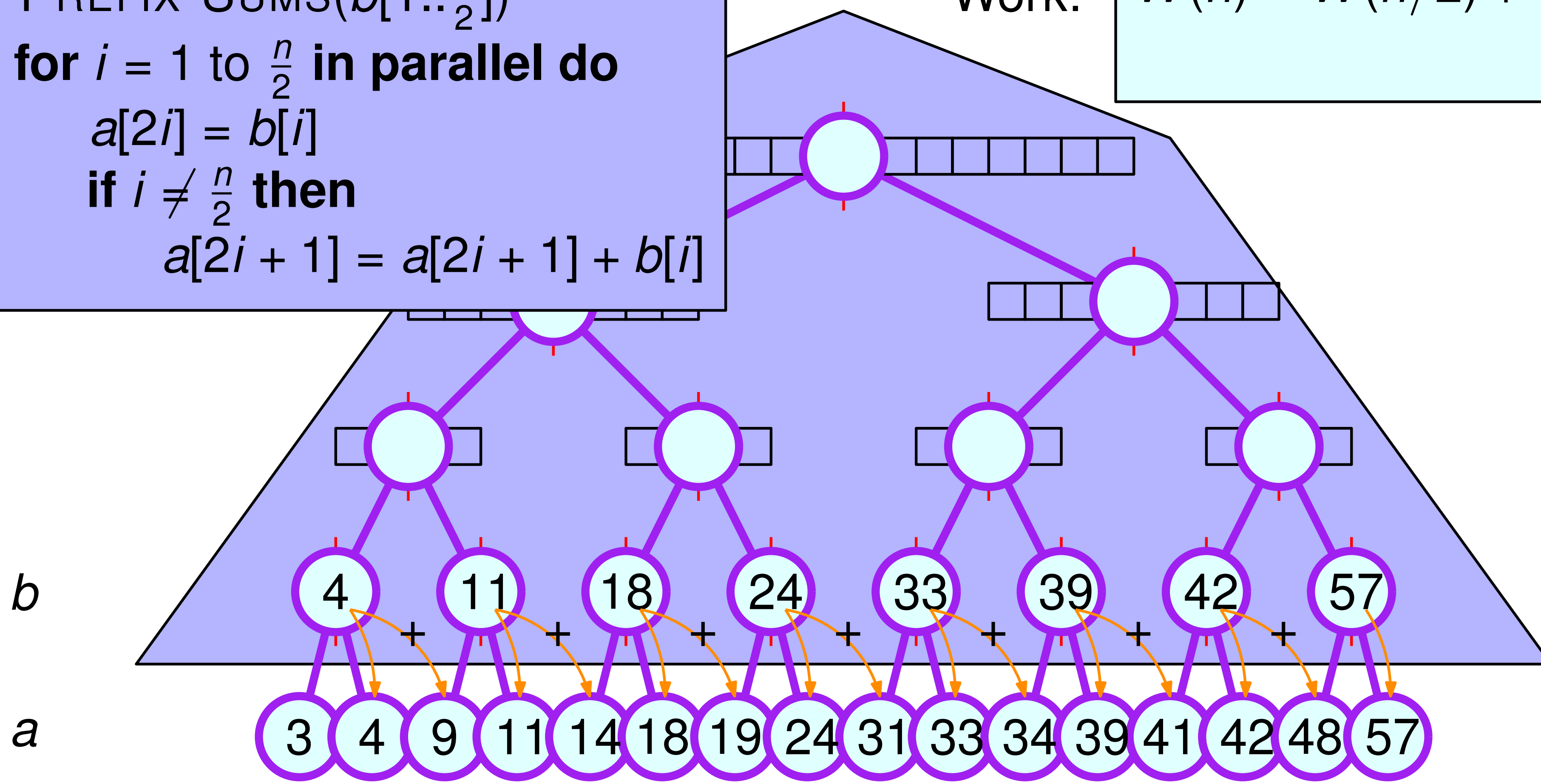
Analysis

Time: $T(n) = T(n/2) + O(1) = O(\log n)$

Work: $W(n) = W(n/2) + O(n)$



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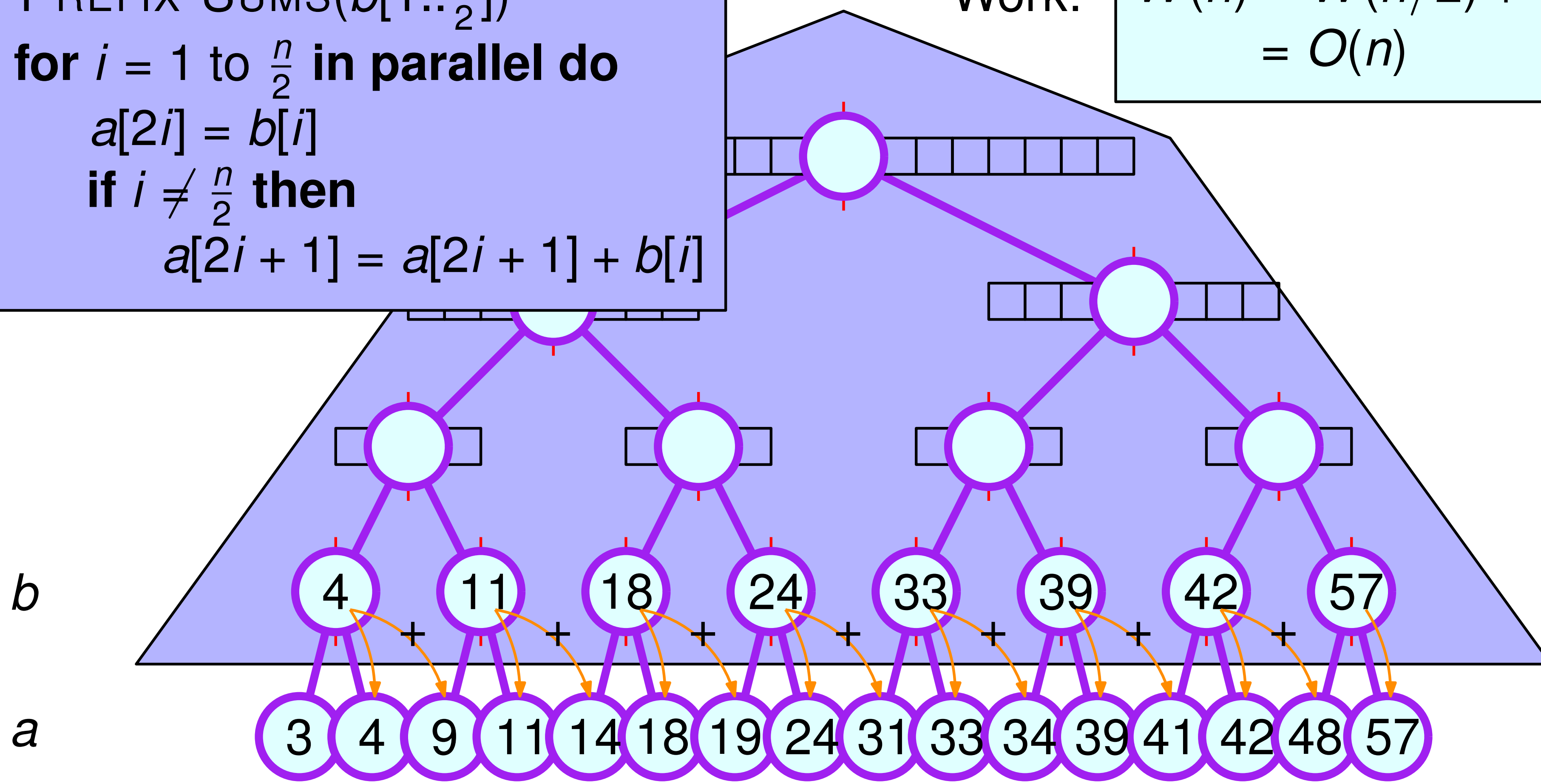

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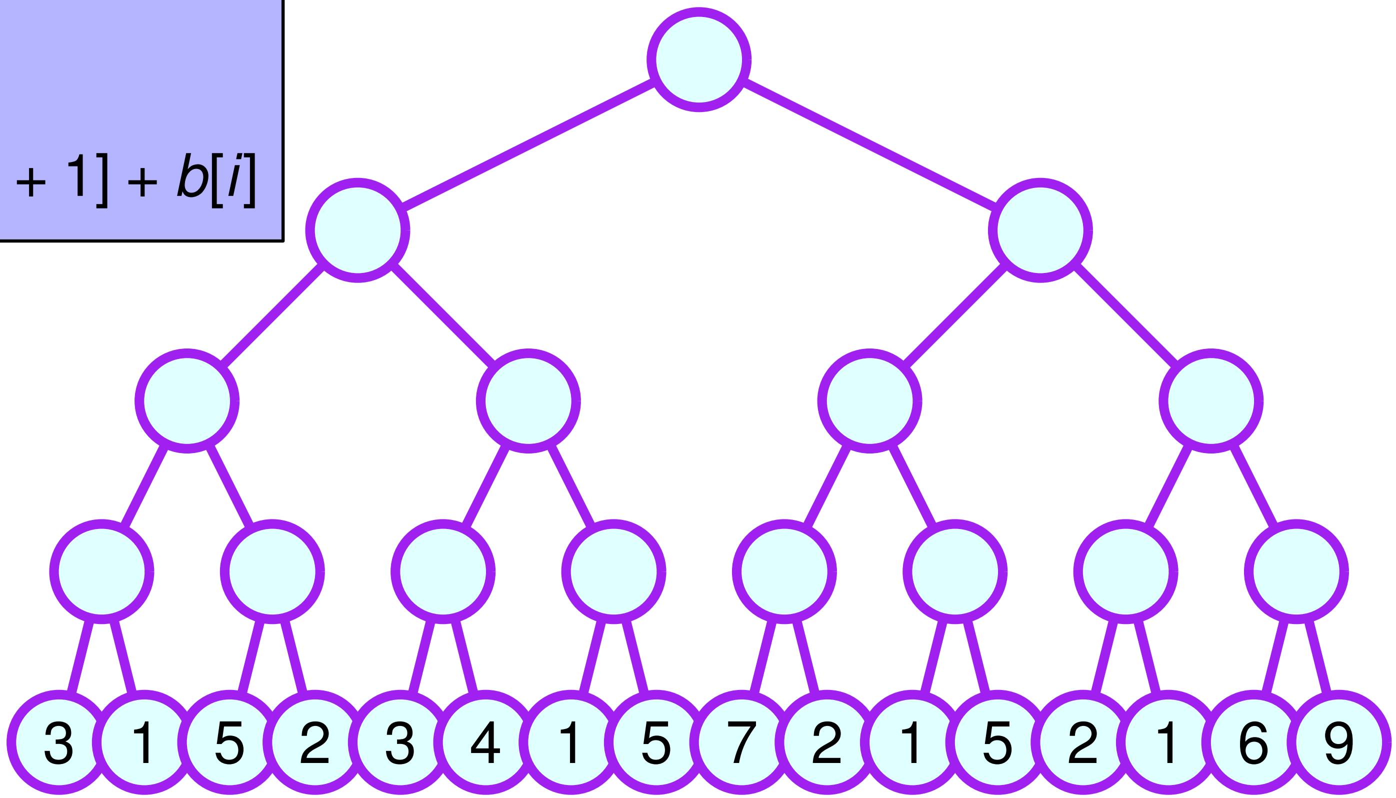

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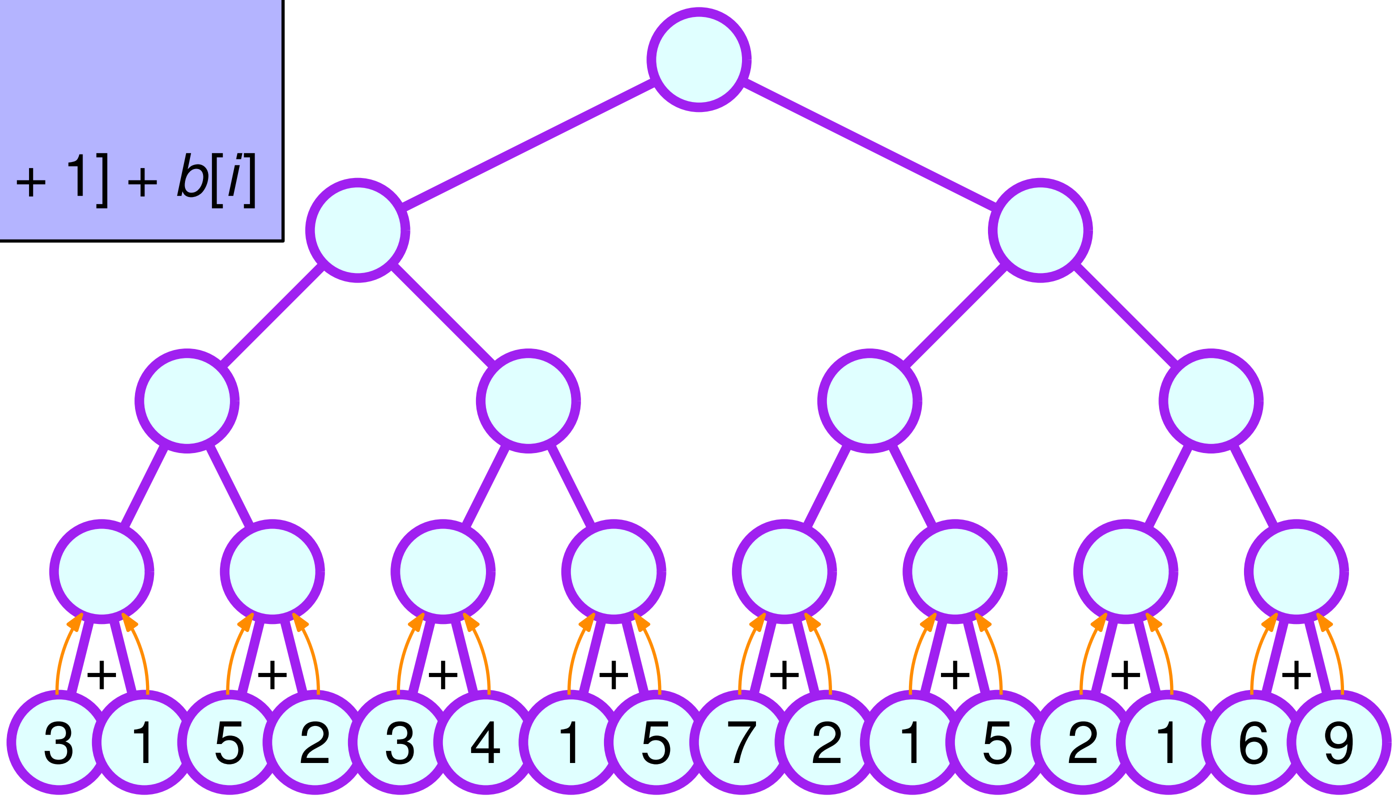
Up-sweep



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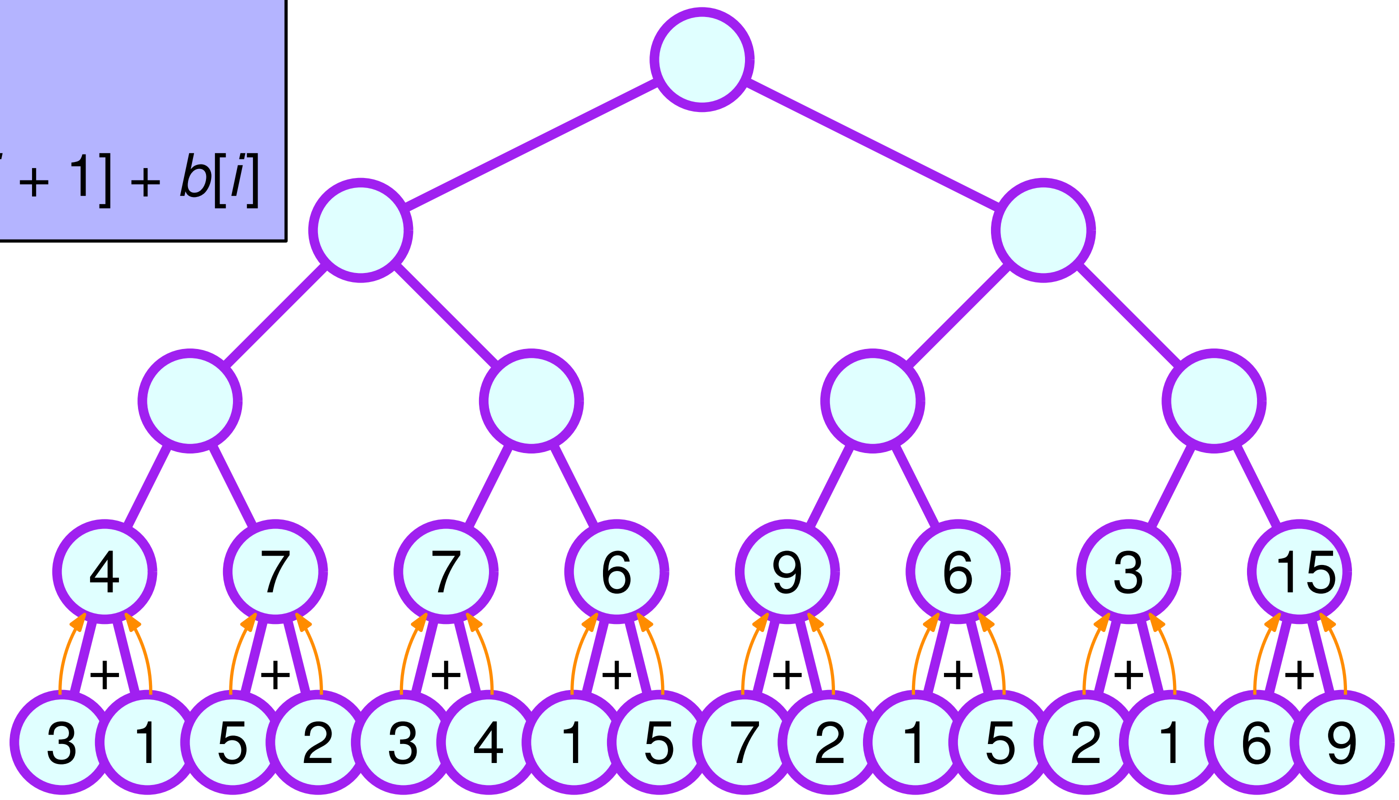

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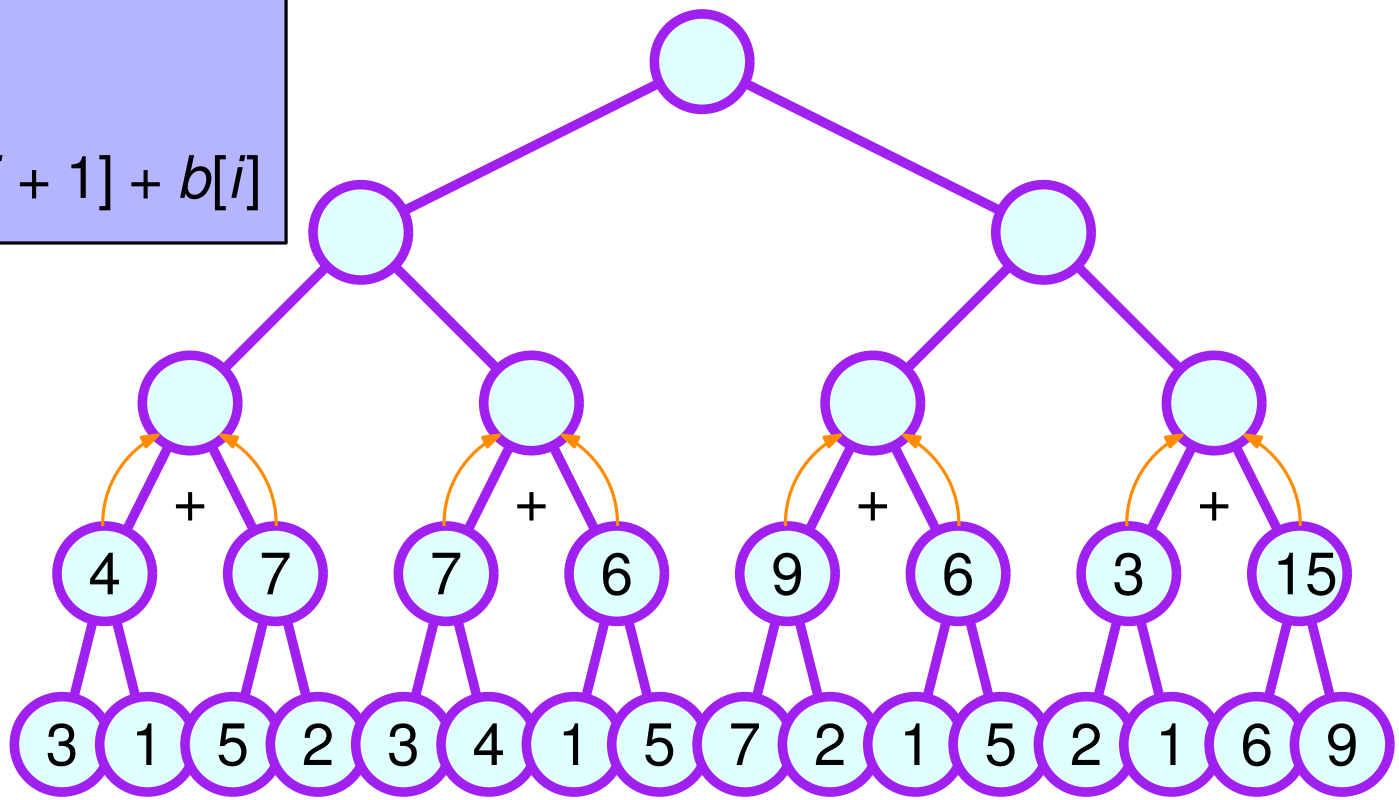

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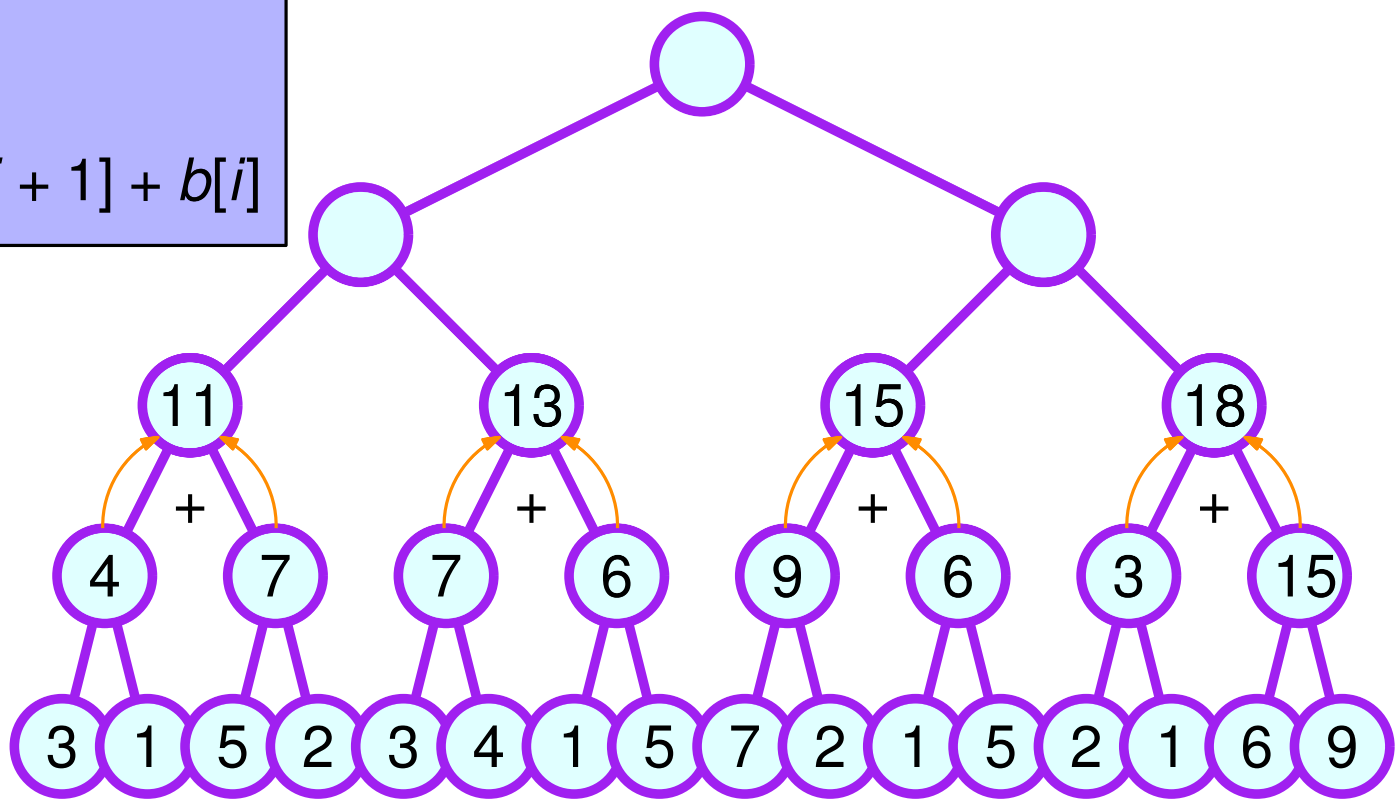

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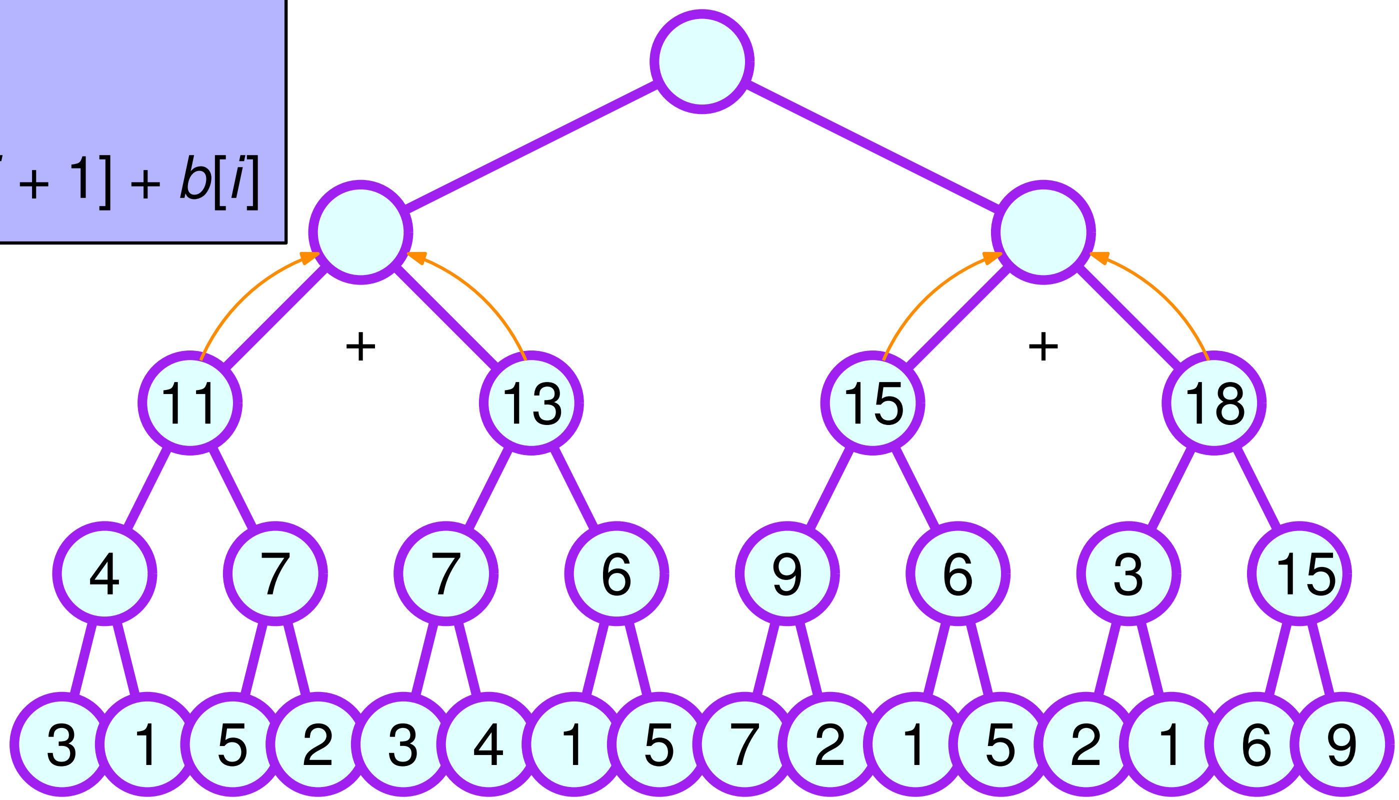

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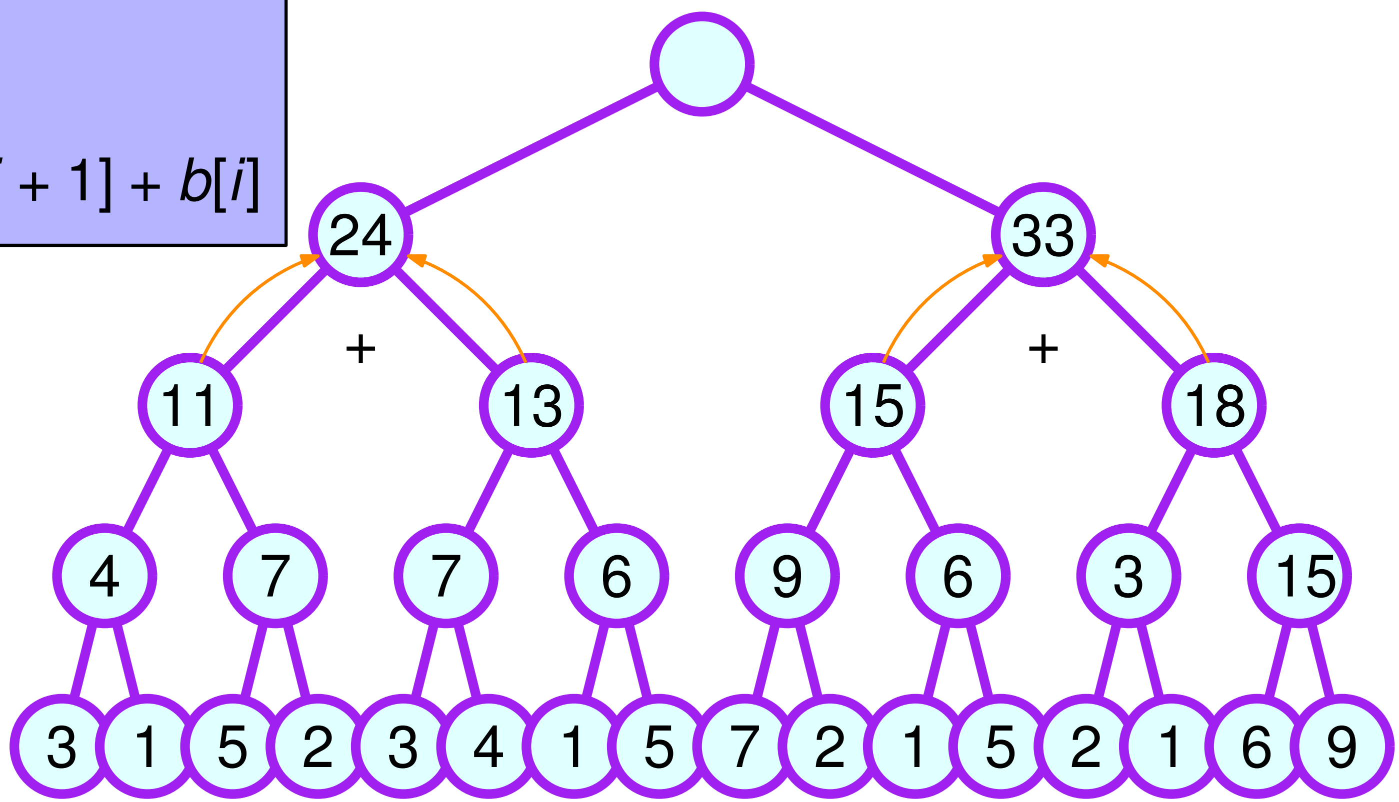

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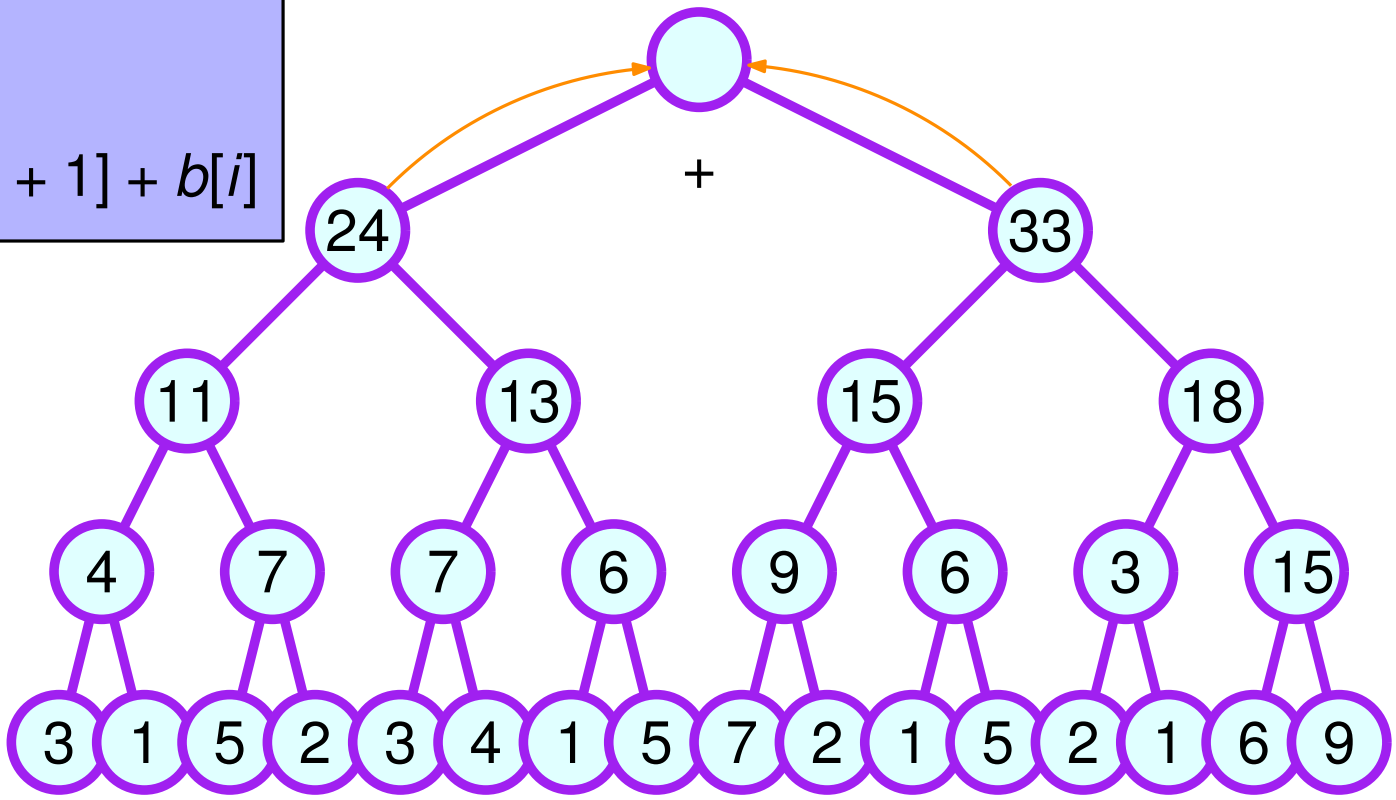

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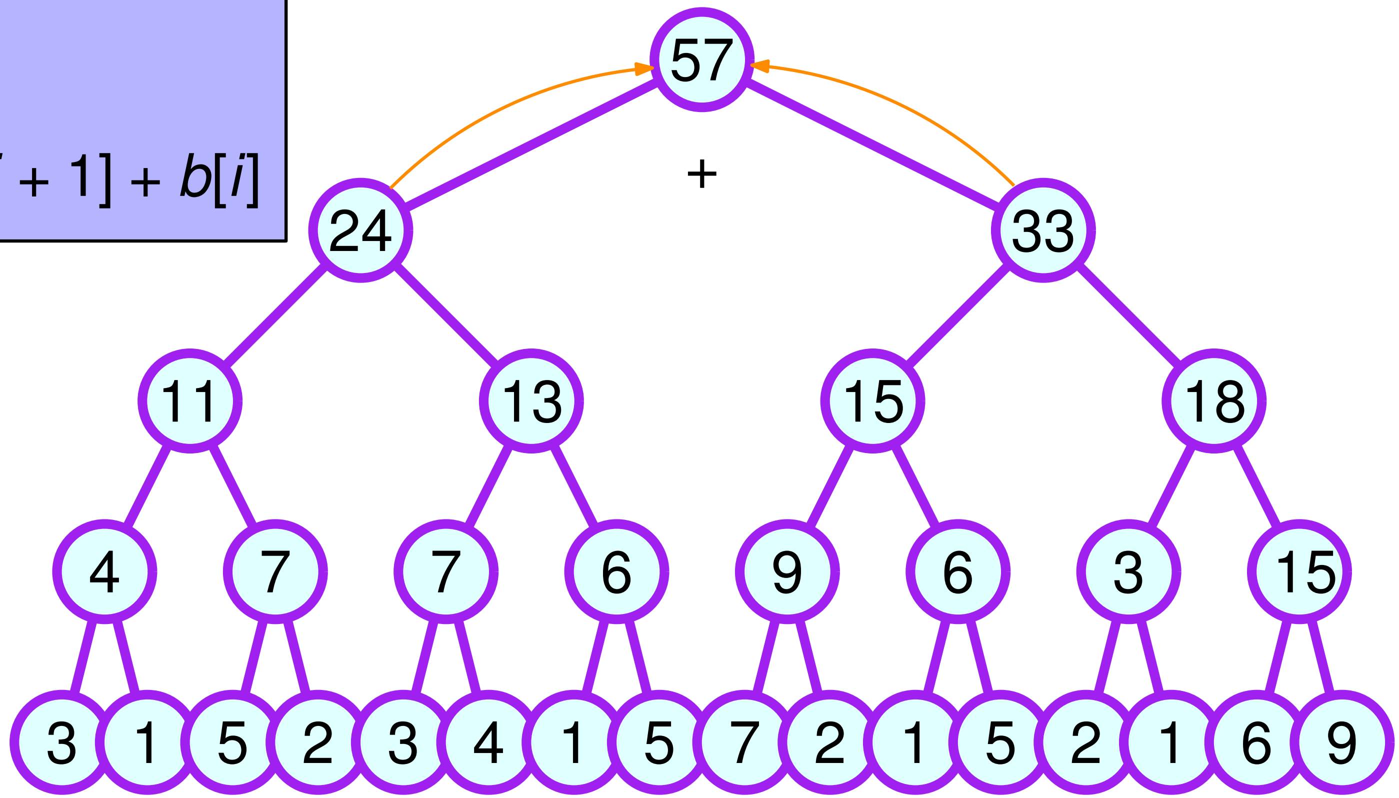

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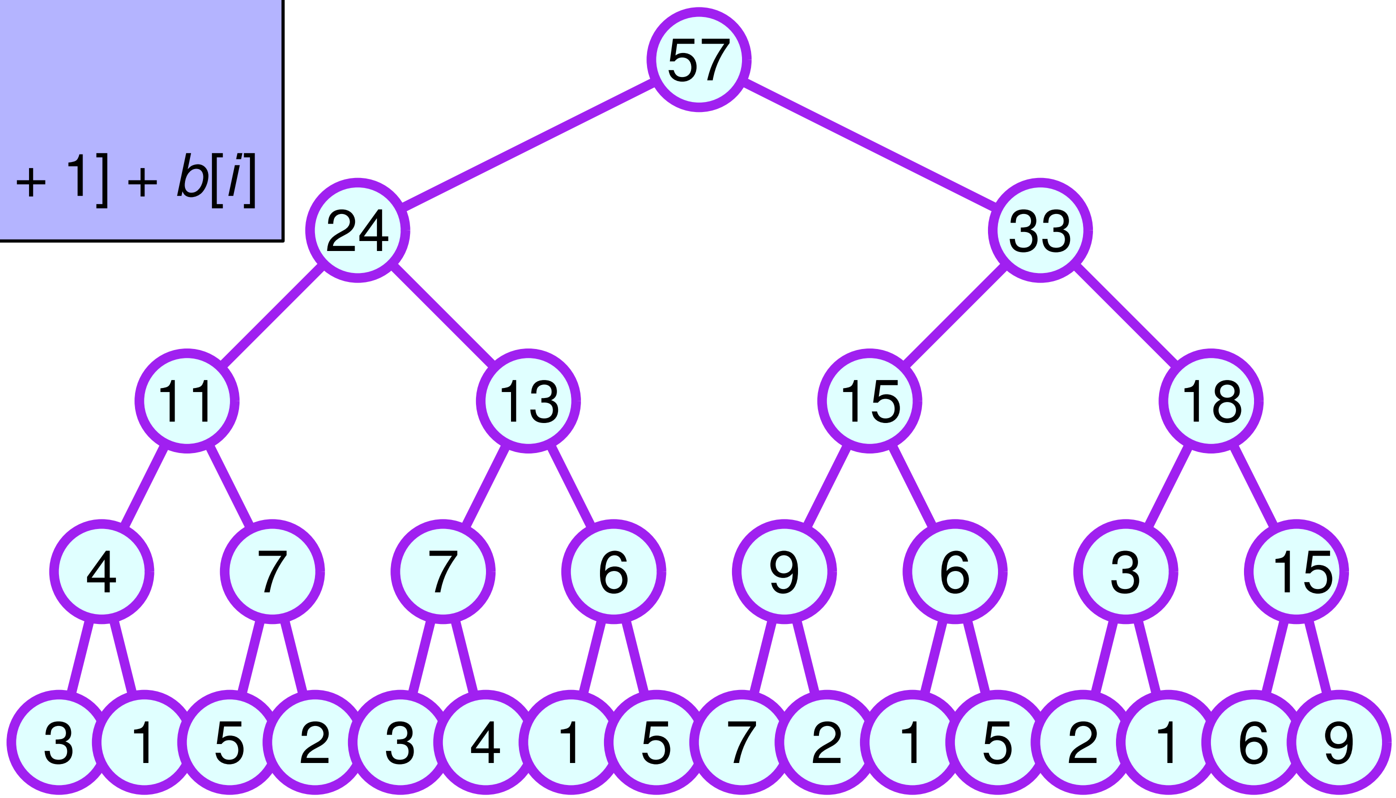

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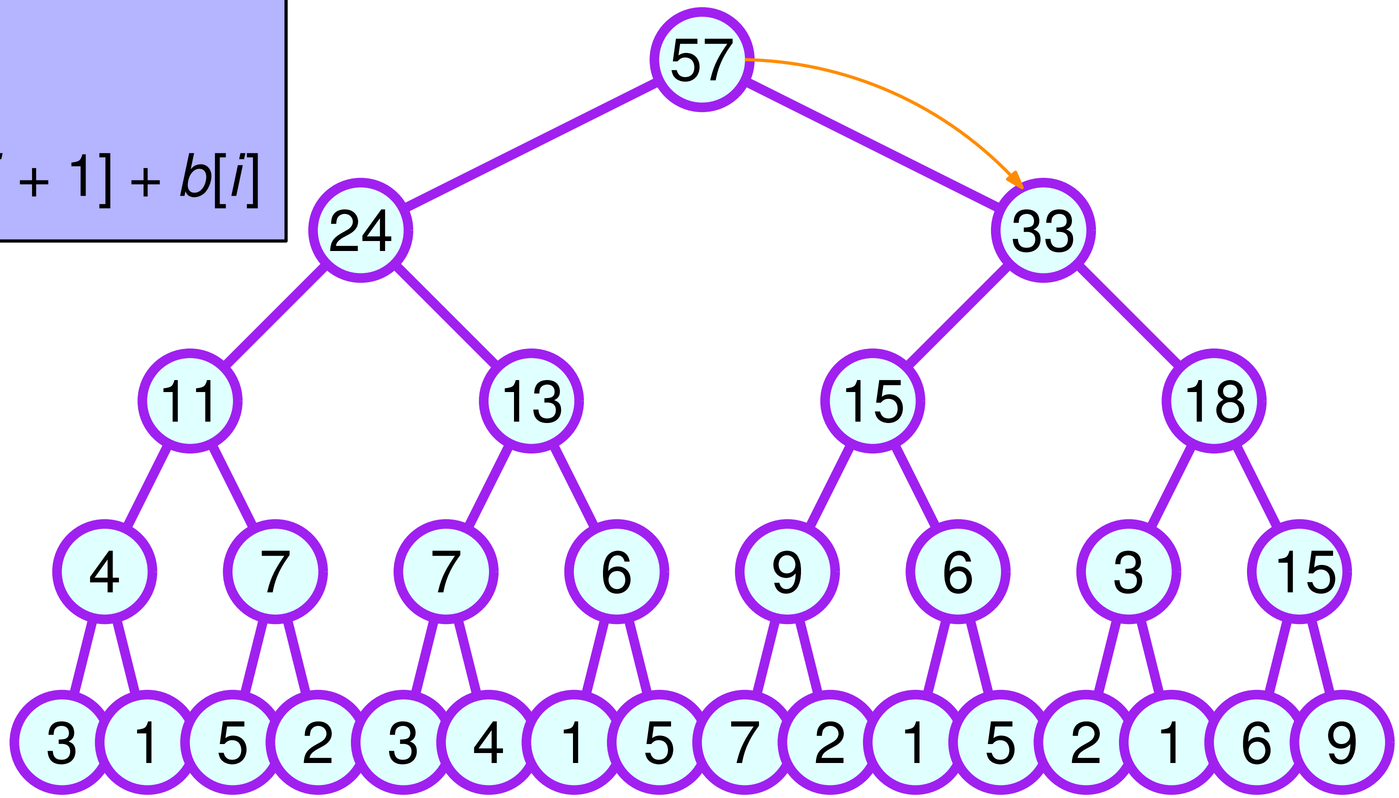
Down-sweep



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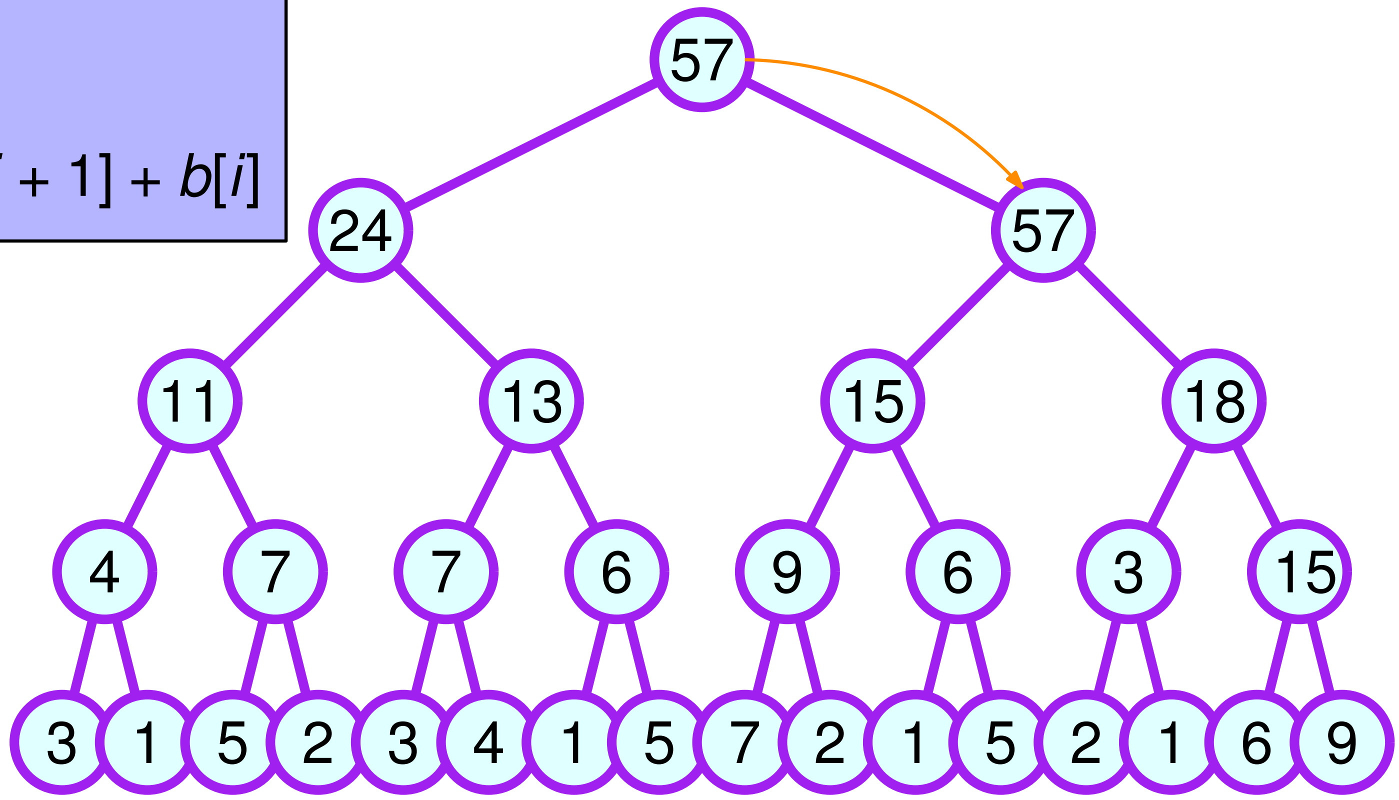

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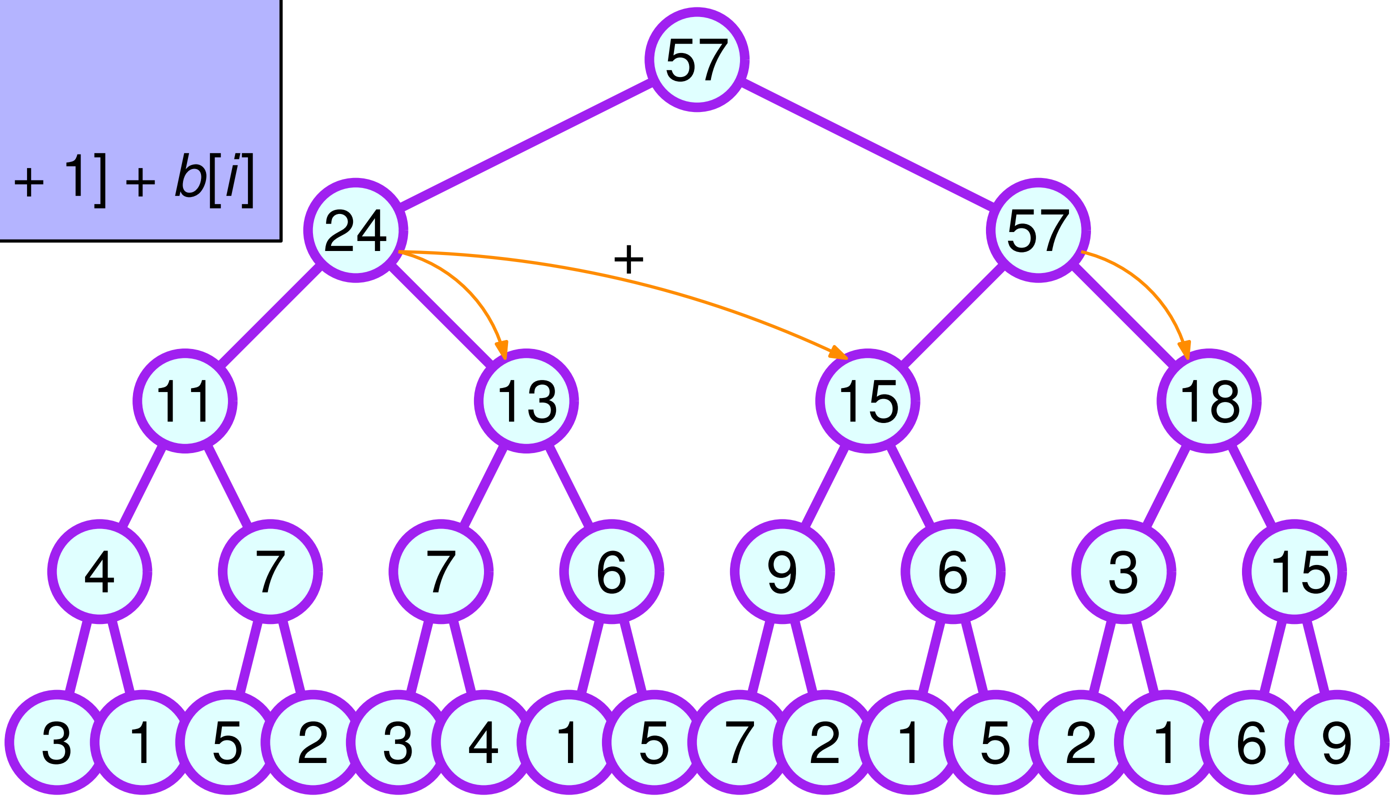
    if $i \neq \frac{n}{2}$ **then**
      $a[2i + 1] = a[2i + 1] + b[i]$

Down-sweep



Balanced-Tree Technique

procedure PREFIX-SUMS($a[1..n]$)
  if $n \leq 1$ **then return**
  for $i = 1$ to $\frac{n}{2}$ **in parallel do**
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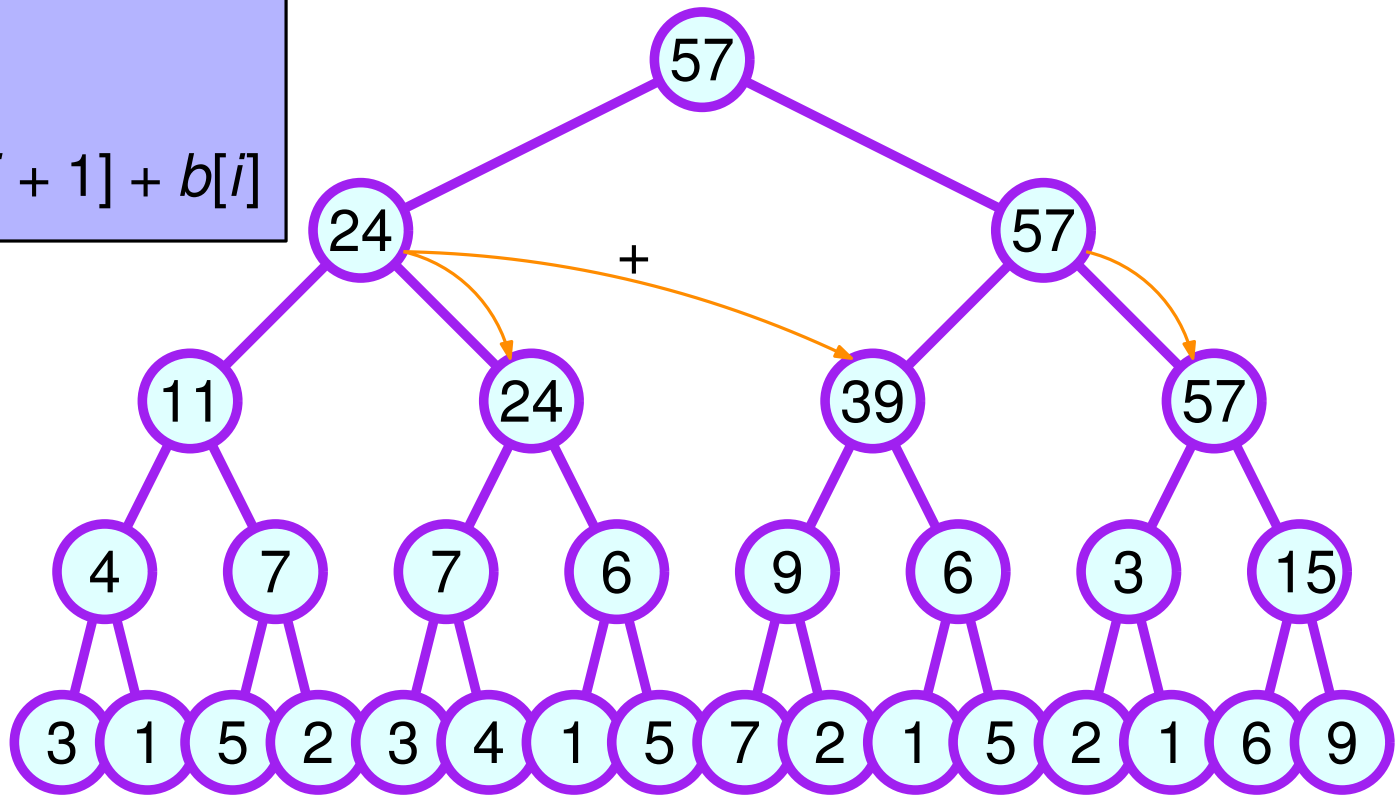

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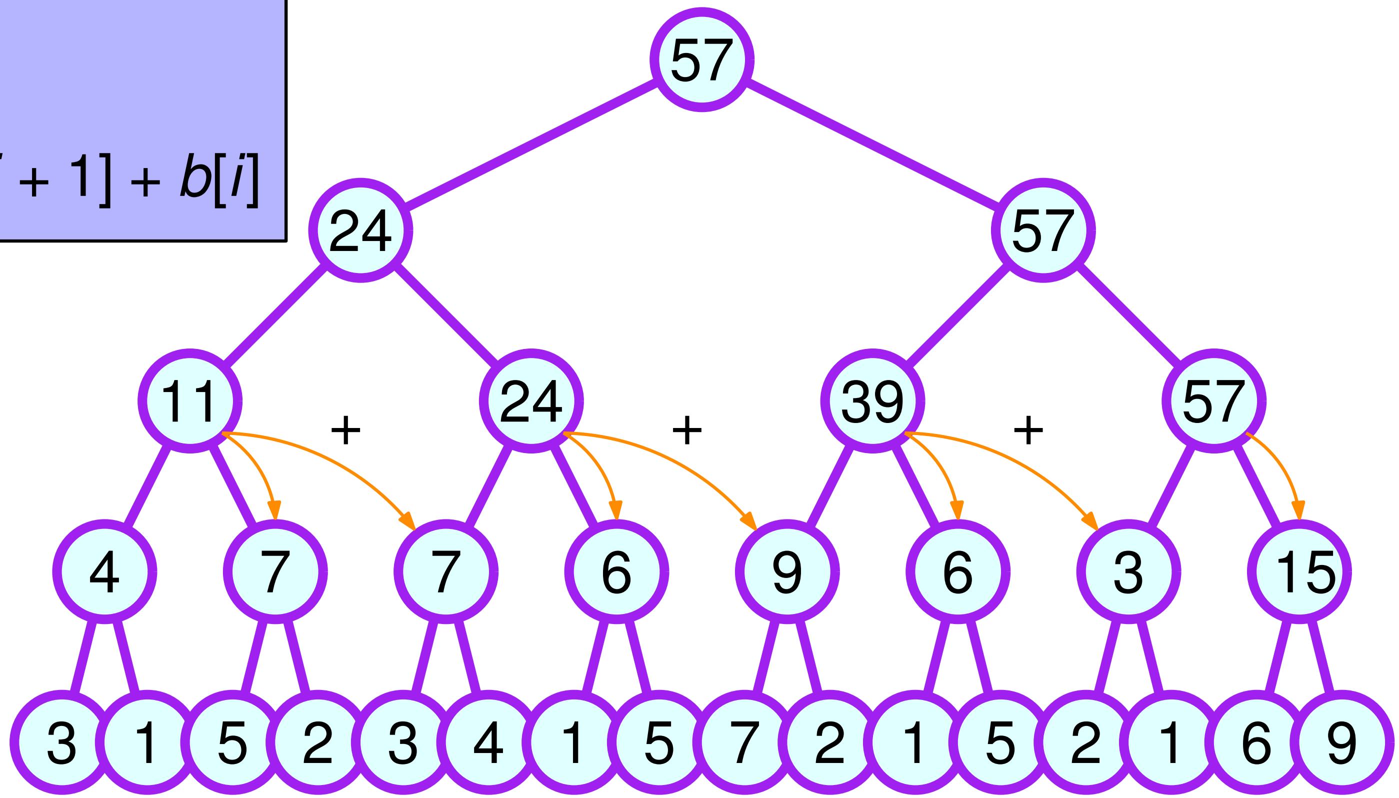

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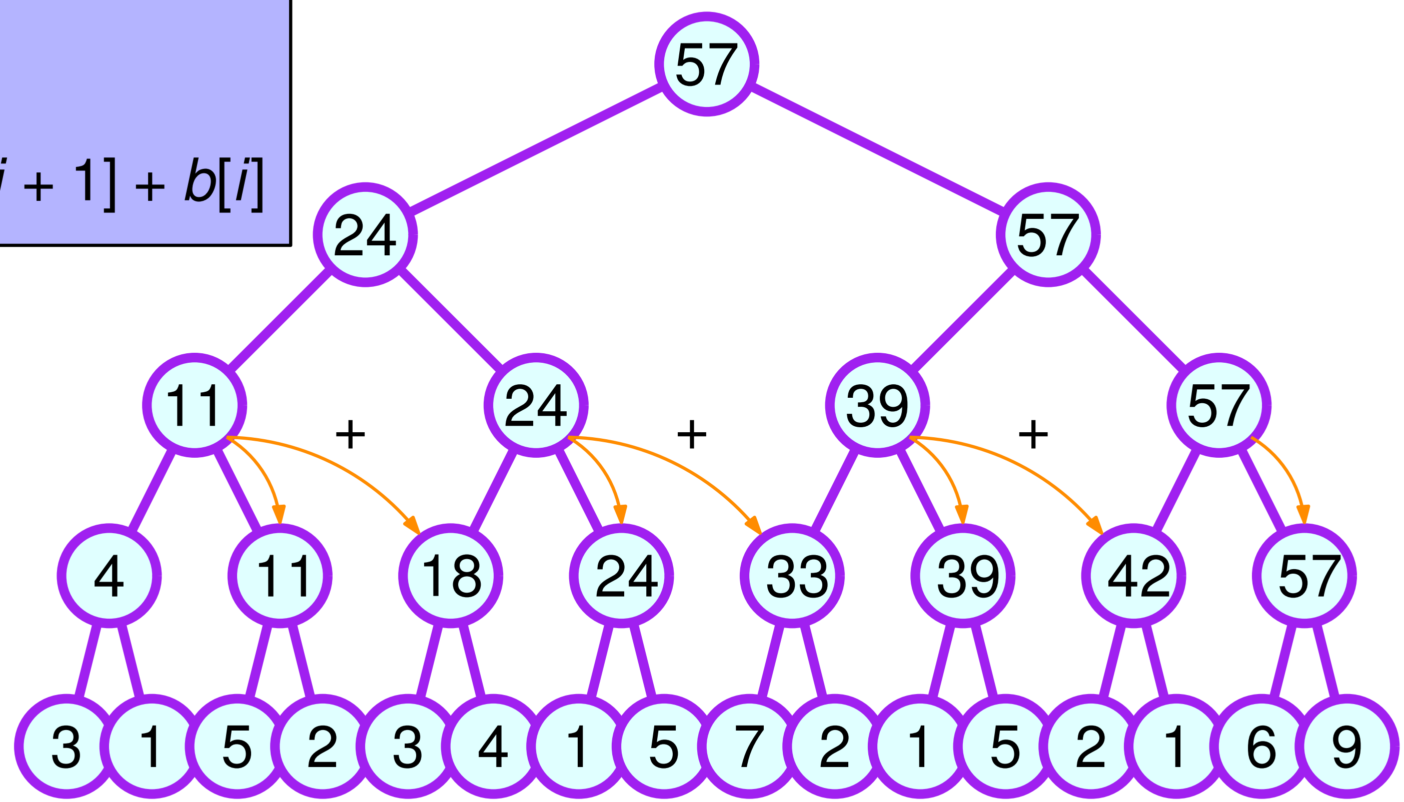

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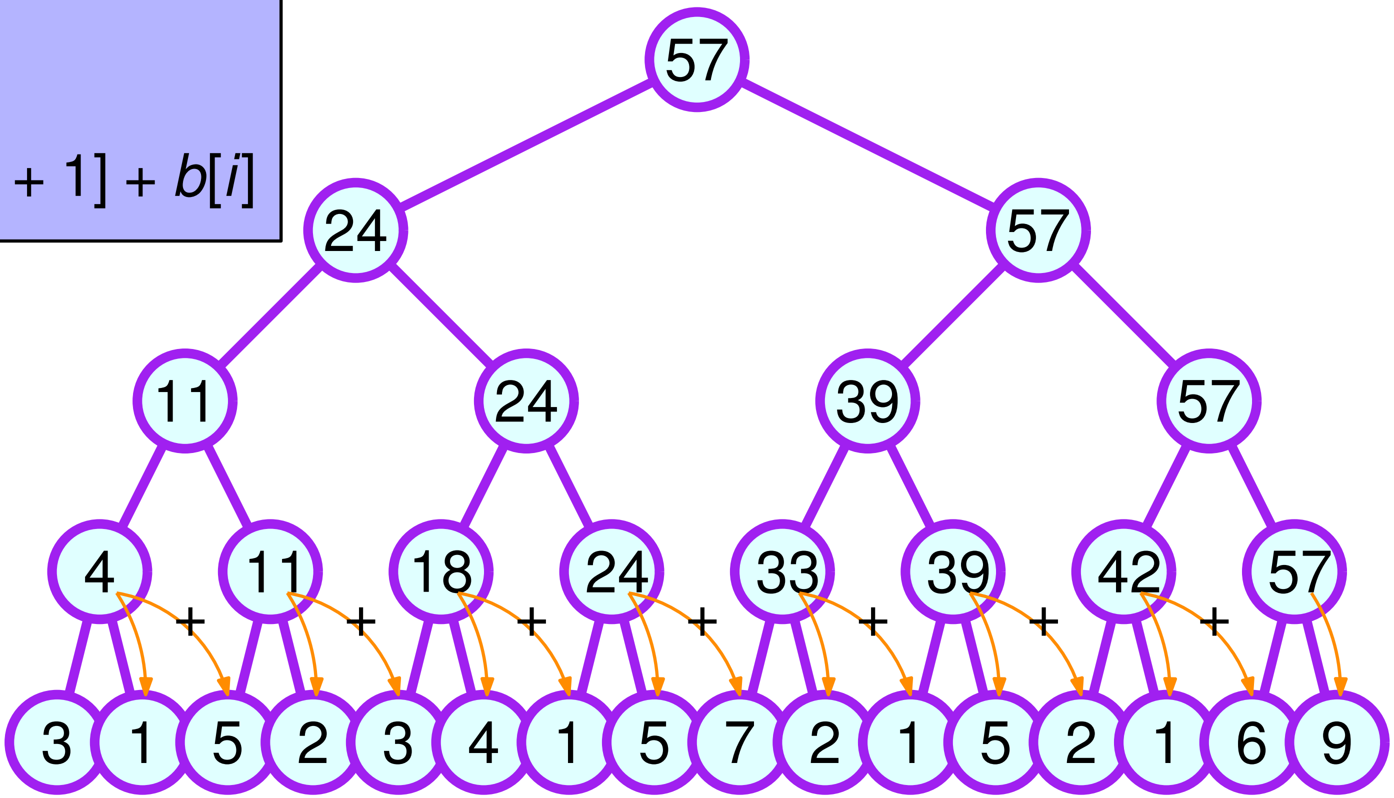

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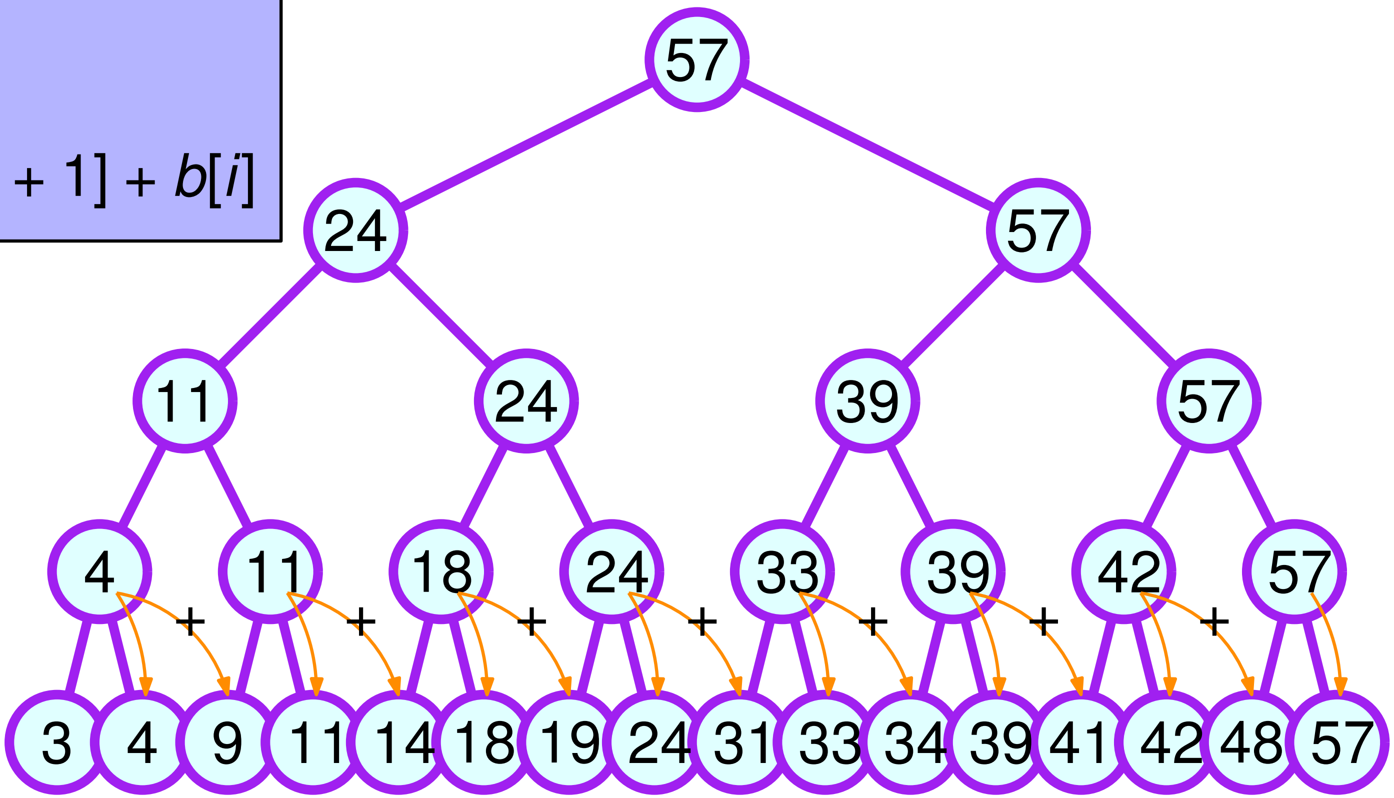
Down-sweep



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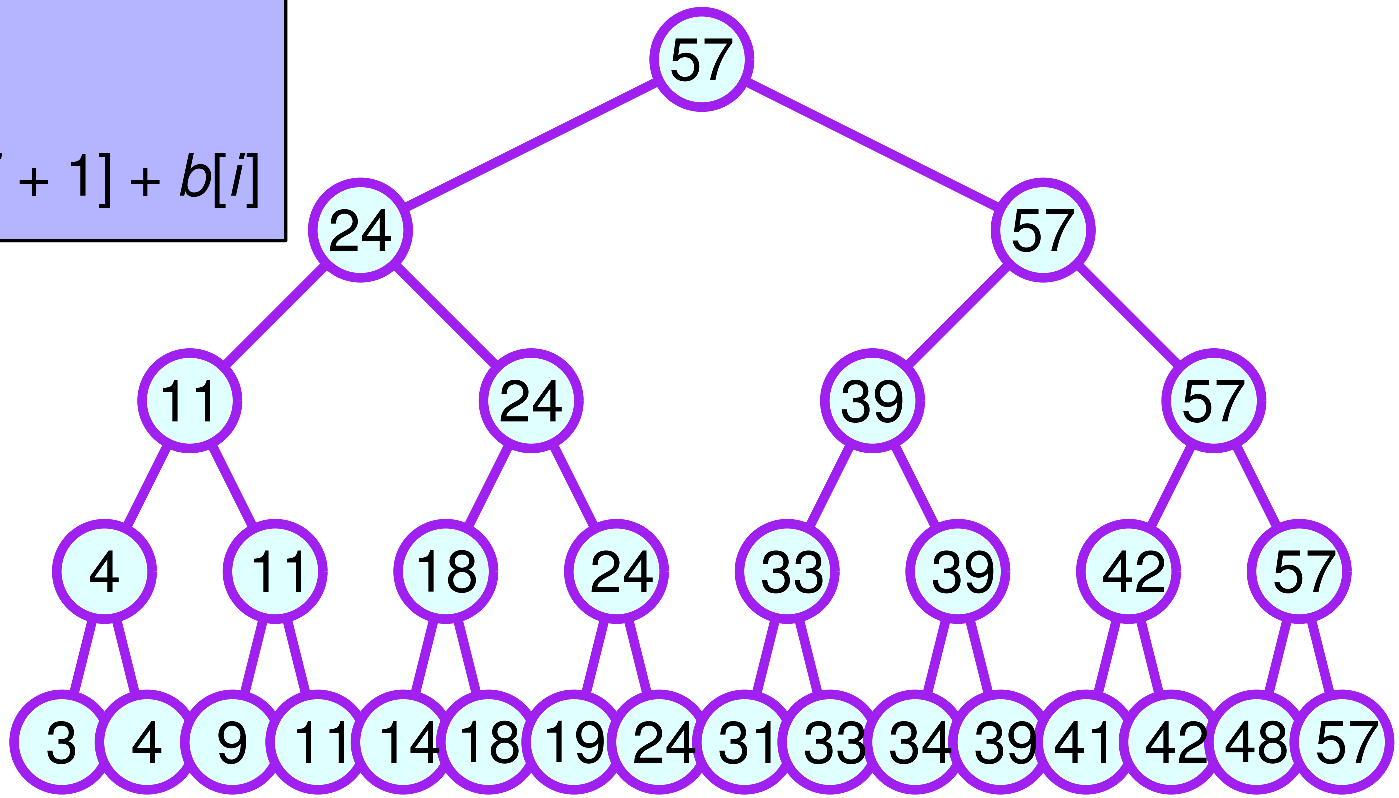

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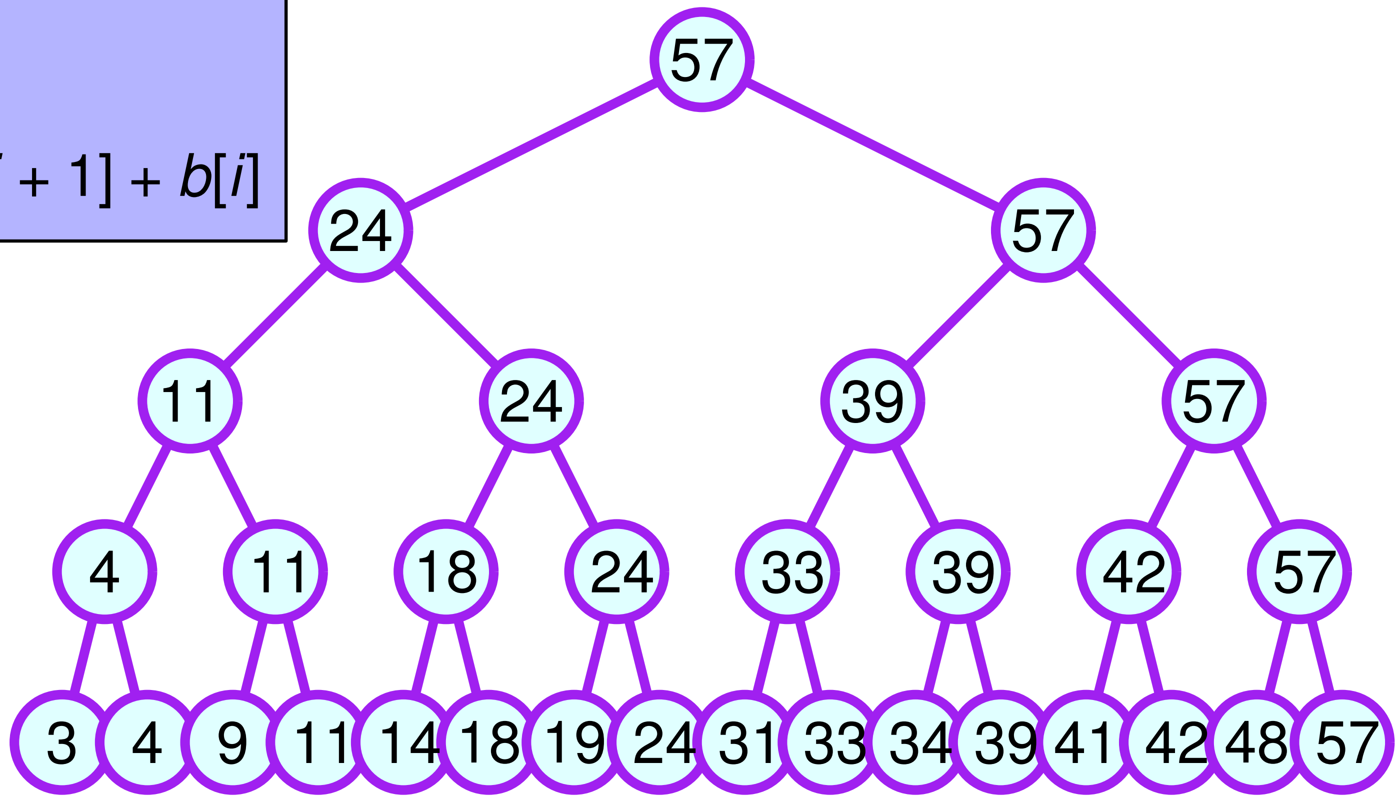
Works with any ***associative*** operation



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Works with any ***associative*** operation

$$(a + b) + c = a + (b + c)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$\min(\min(a, b), c) = \min(a, \min(b, c))$$