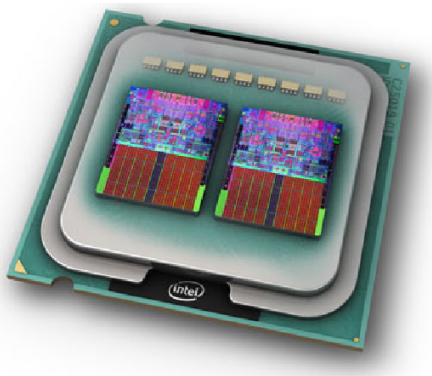
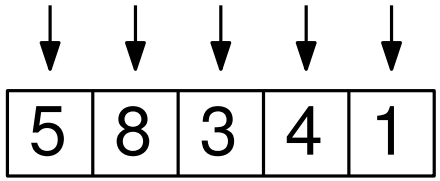
ICS 443: Parallel Algorithms

Prof. Nodari Sitchinava





Lecture 3: Brent's Scheduling Principle

Last Time: EREW Minimum

procedure MIN($a[i..j]$)

  if $i = j$ **then**

    return $a[i]$

  else

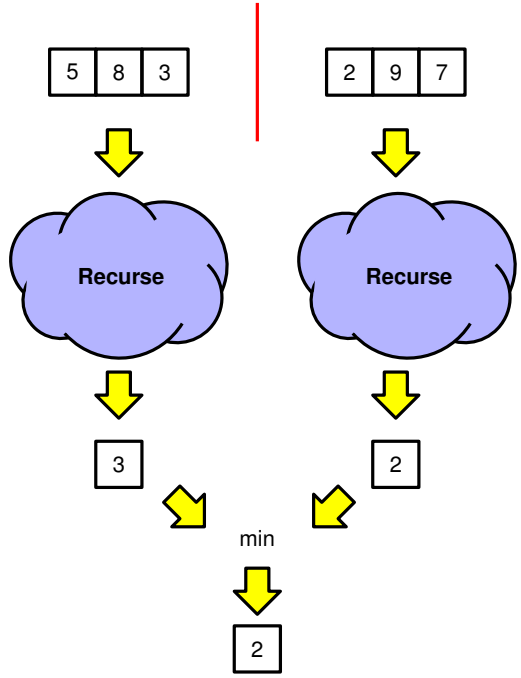
    $mid = \left\lfloor \frac{i+j}{2} \right\rfloor$; $rmid = mid + 1$ — $O(1)$

    in parallel do — $T(n)$

      $left$ = MIN($a[i..mid]$) — $T(n/2)$

      $right$ = MIN($a[rmid..j]$) — $T(n/2)$

    return min($left, right$) — $O(1)$



Runtime of MIN($a[1..n]$)?

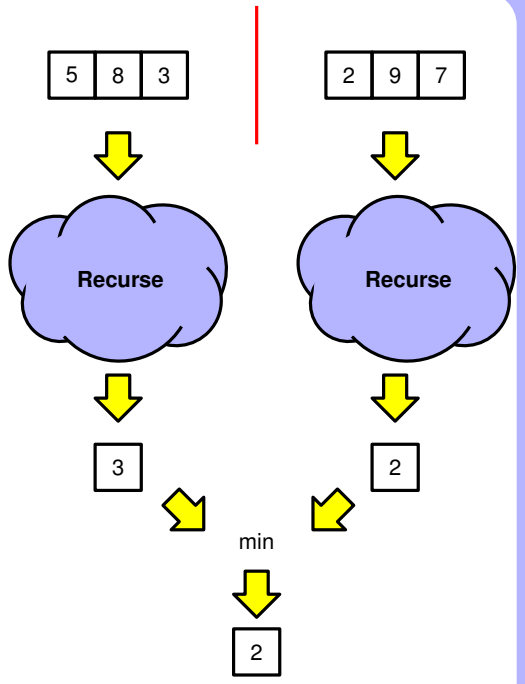
$$T(n) = O(1) + \max \left\{ \begin{array}{c} T(n/2) \\ T(n/2) \end{array} \right\} + O(1)$$

$$= T(n/2) + O(1)$$

$$= \Theta(\log n)$$

Saving Space

```
procedure MIN(a[i..j])
  if i = j then
      return a[i]
  else
      mid = ⌊(i+j)/2⌋; rmid = mid + 1
      in parallel do
          left = MIN(a[i..mid])
          right = MIN(a[rmid..j])
      return min(left, right)
```



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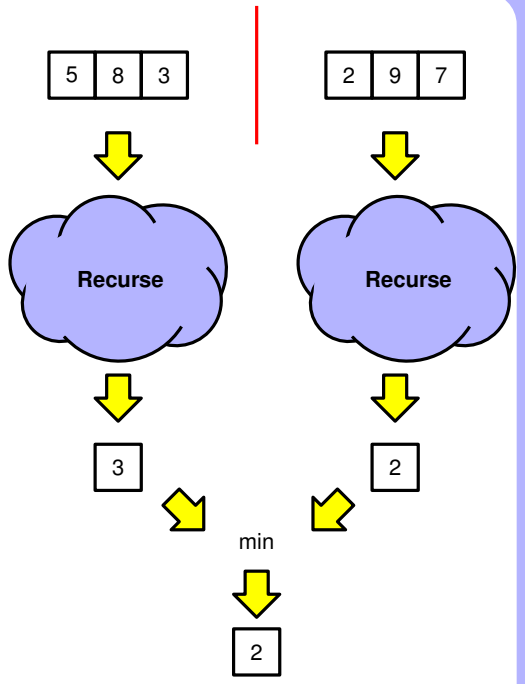

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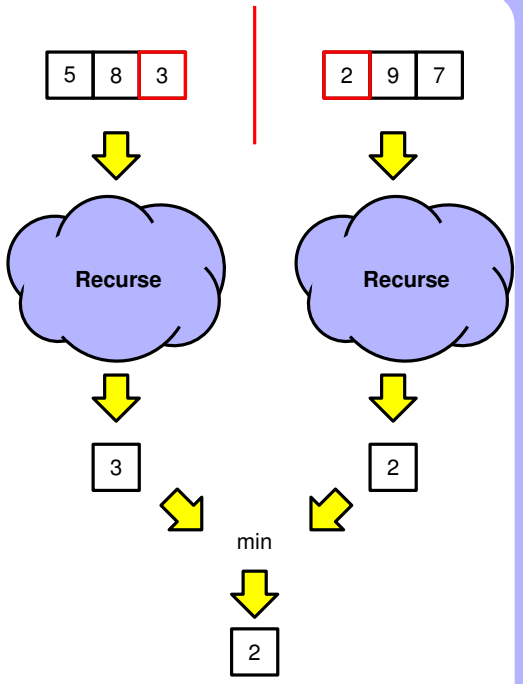

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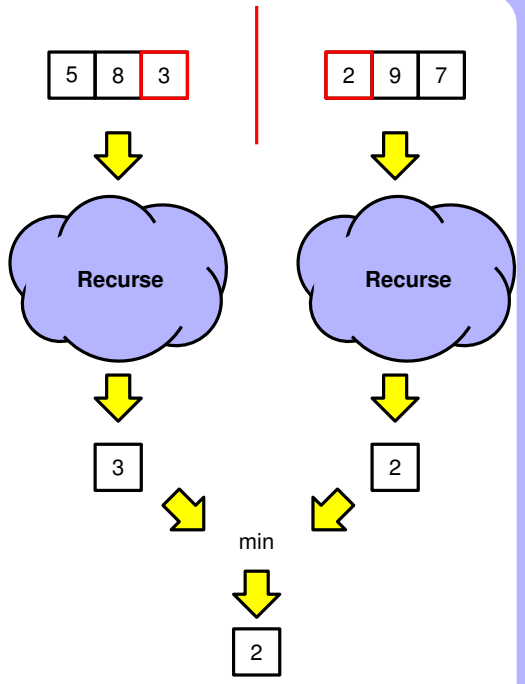

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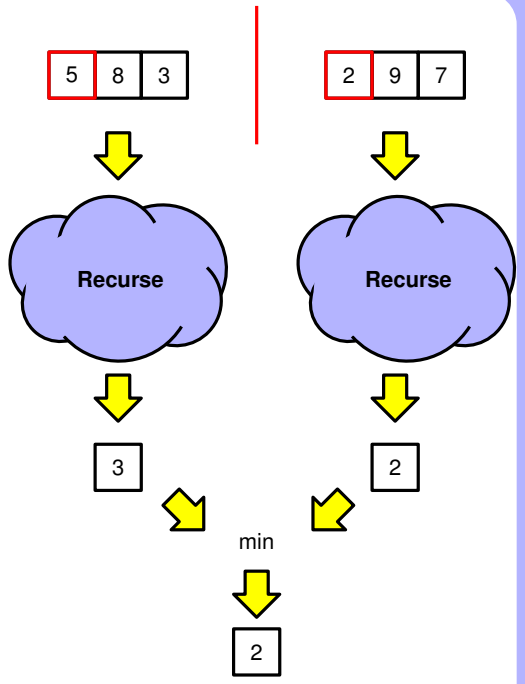

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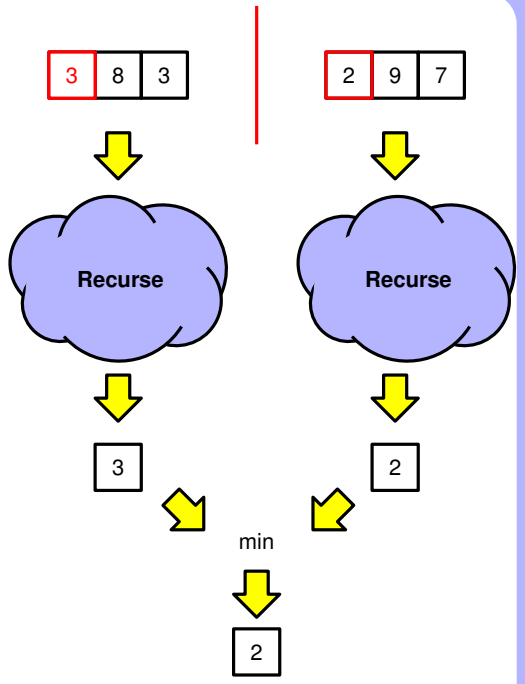

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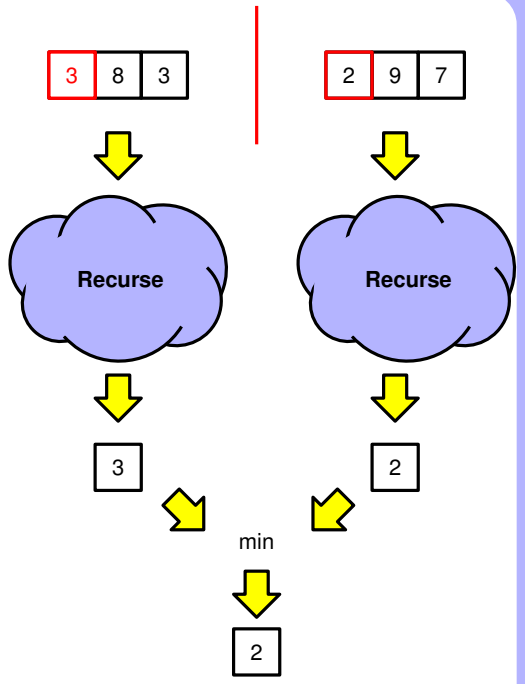
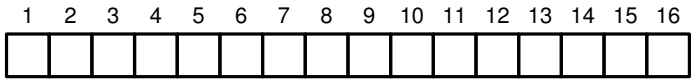


Runtime of MIN($a[1..n]$)?

Can we compute *mid* and *rmid* on the fly?

Computing *mid*



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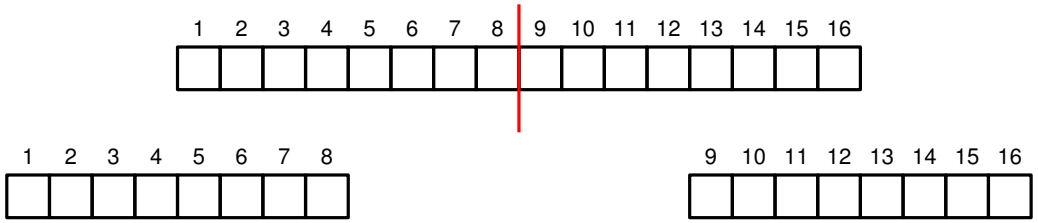

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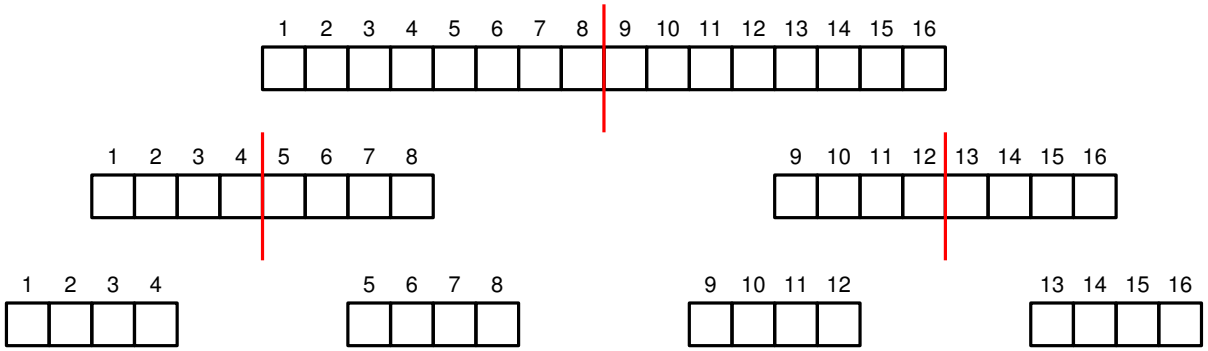

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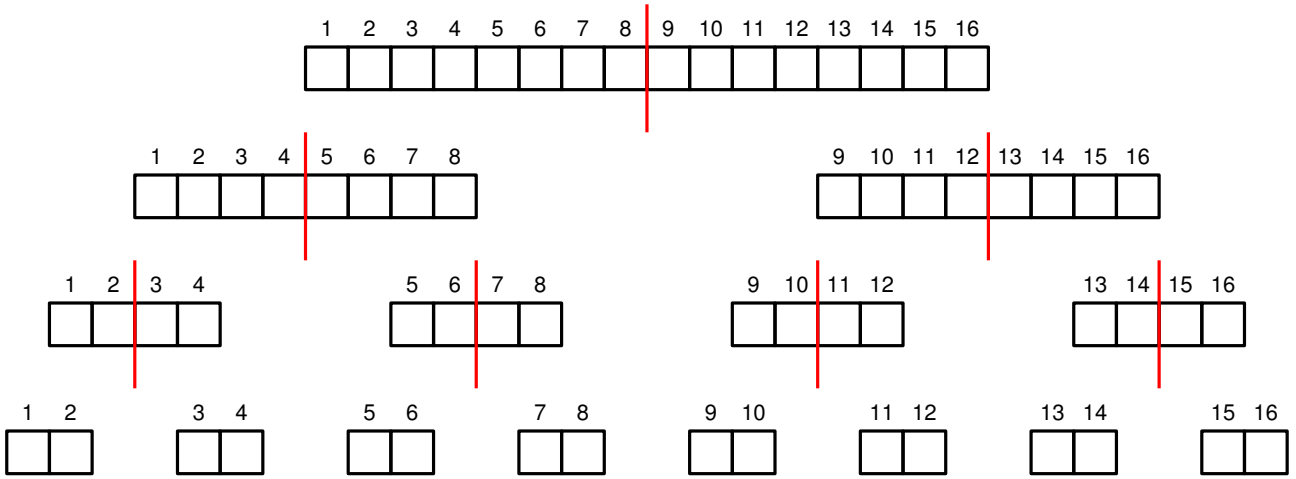

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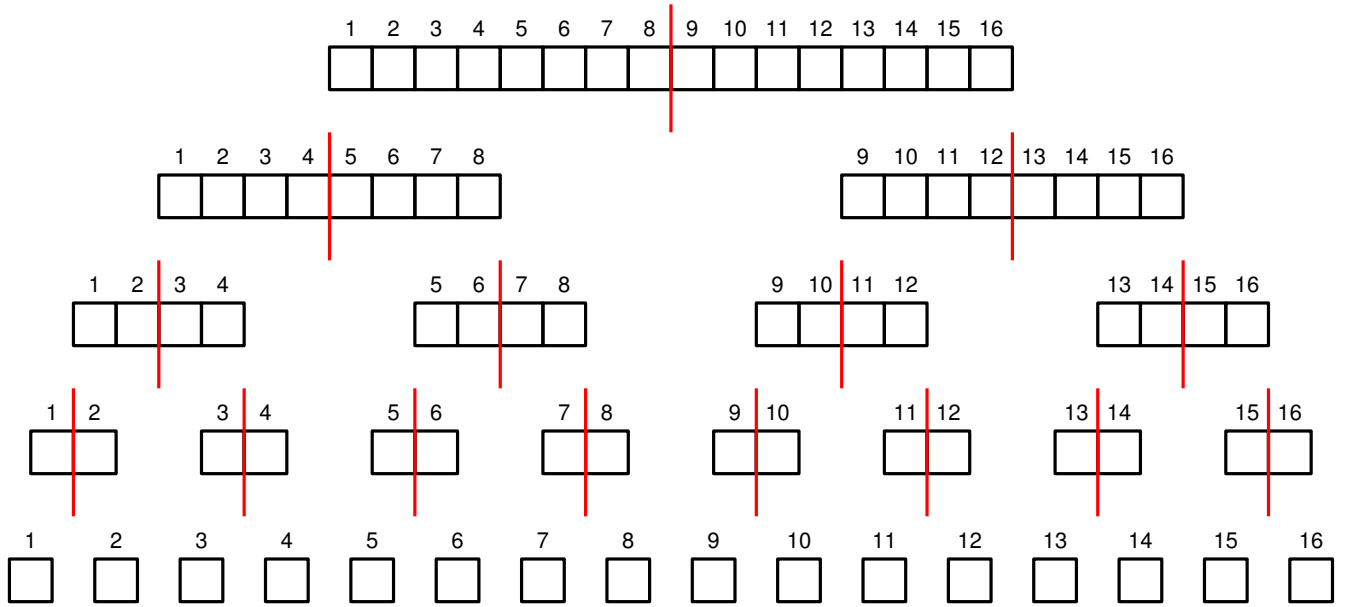

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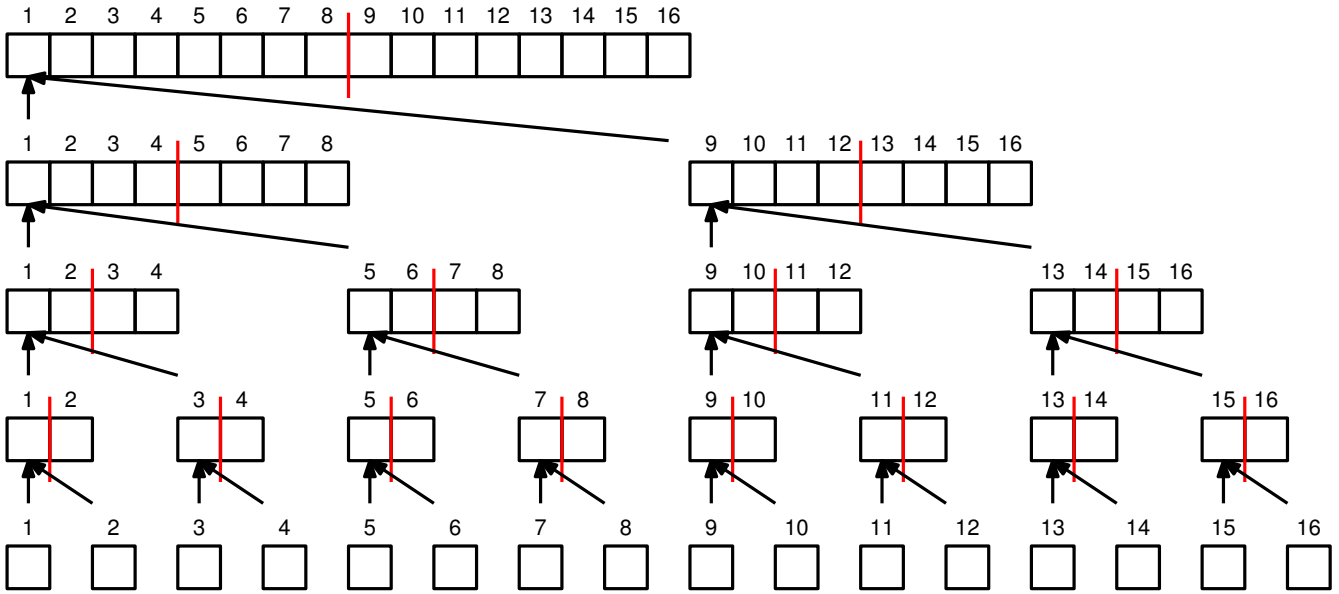

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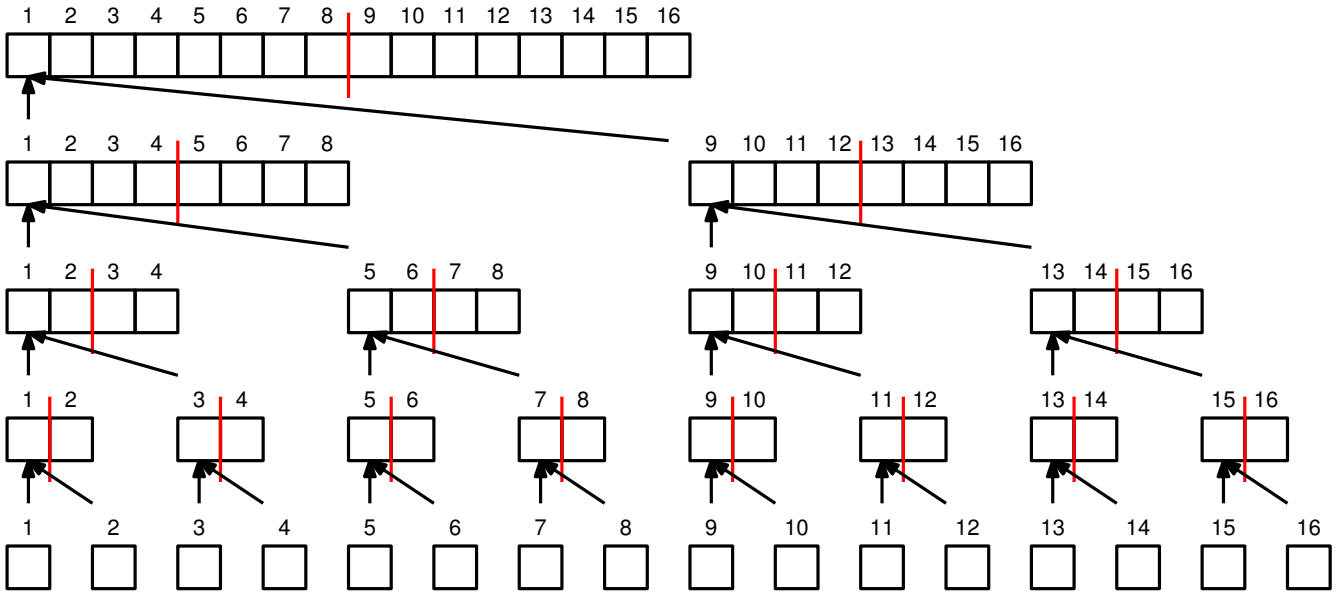
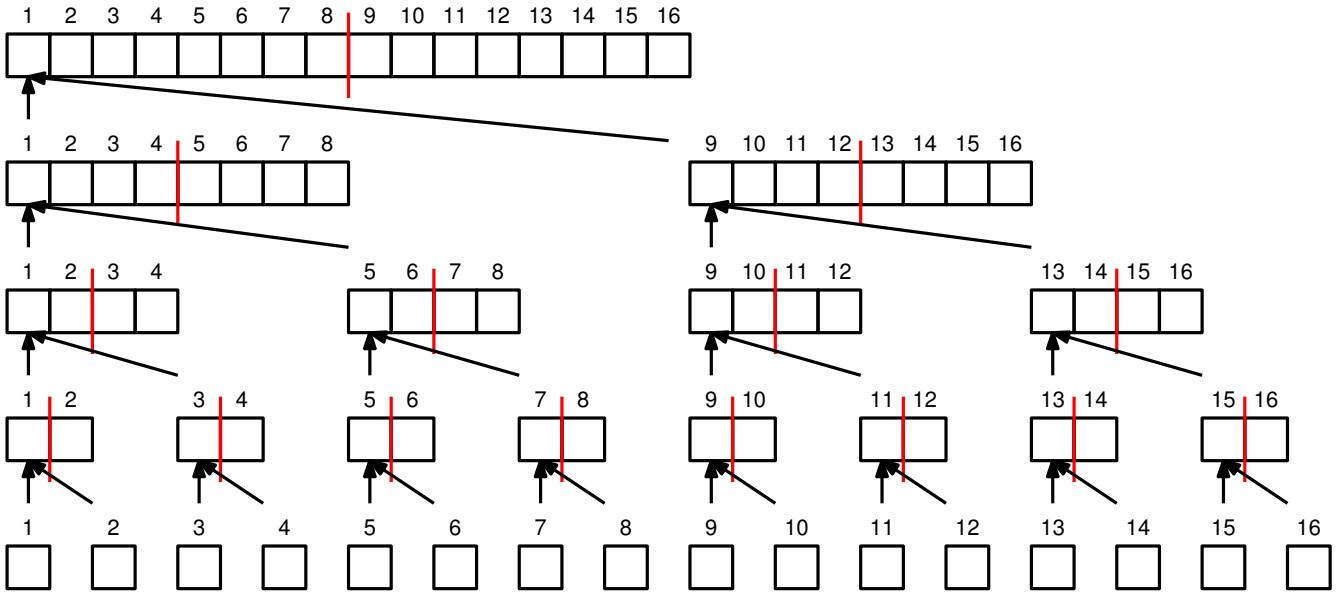


for $i = 1; i \leq n; i = i + 2$ **do**

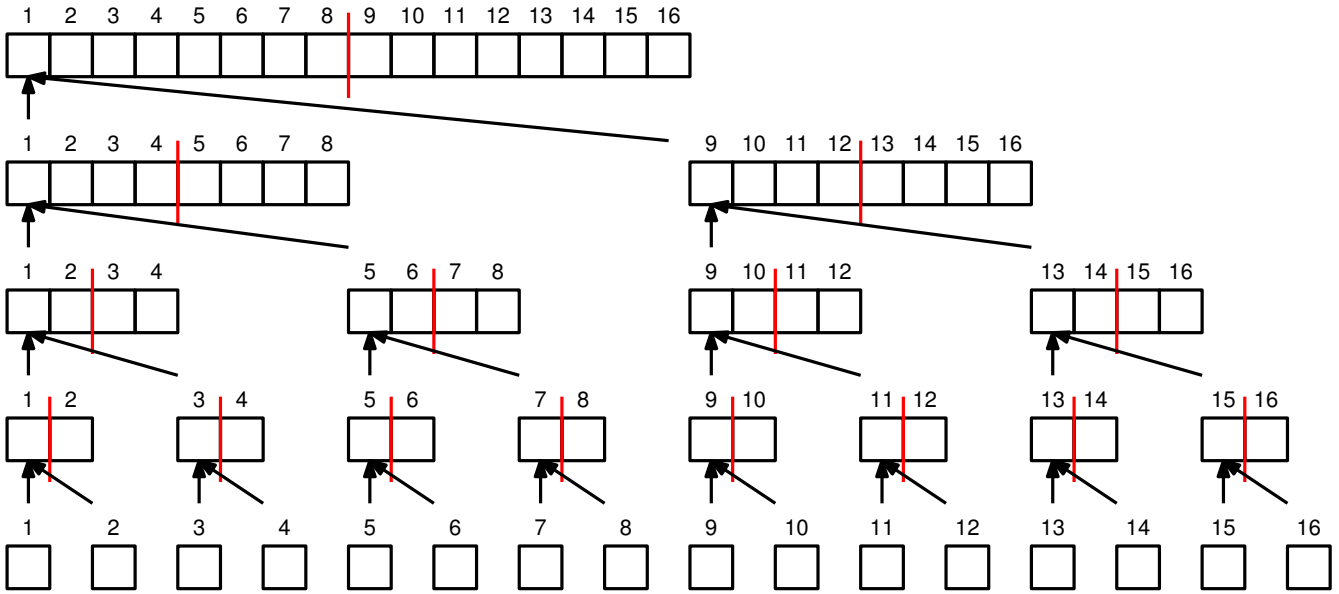
Computing *mid*



for $i = 1; i \leq n; i = i + 4$ **do**

for $i = 1; i \leq n; i = i + 2$ **do**

Computing *mid*



for $i = 1; i \leq n; i = i + 8$ **do**

for $i = 1; i \leq n; i = i + 4$ **do**

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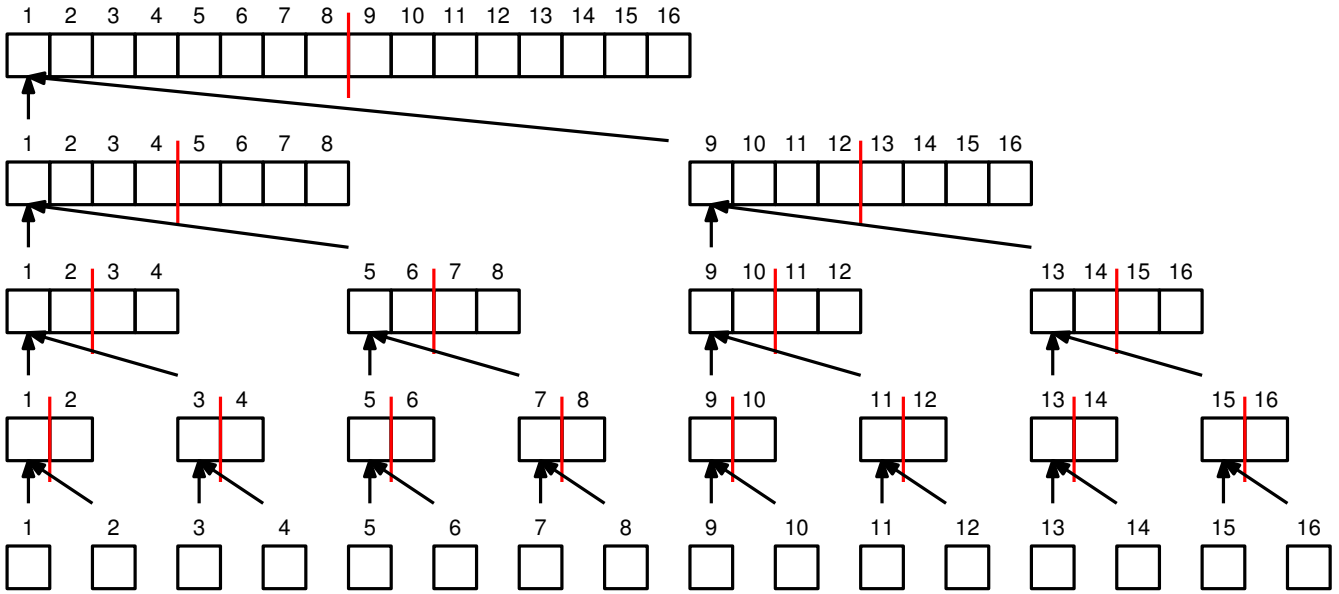

for $i = 1; i \leq n; i = i + 16$ **do**

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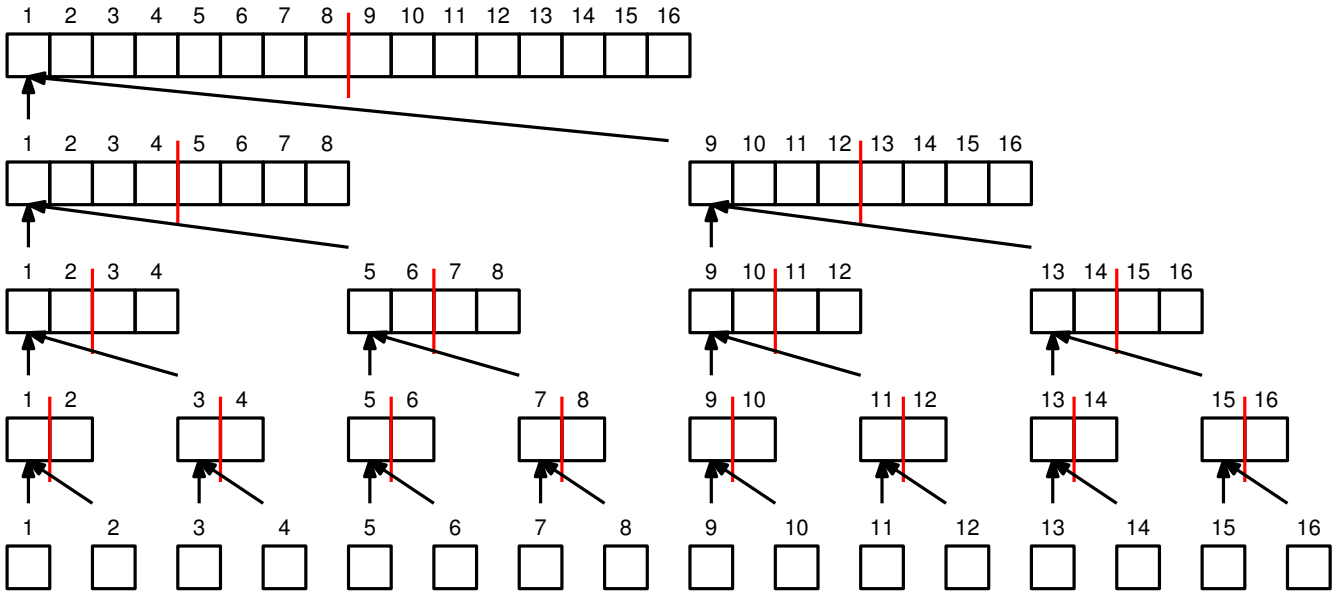

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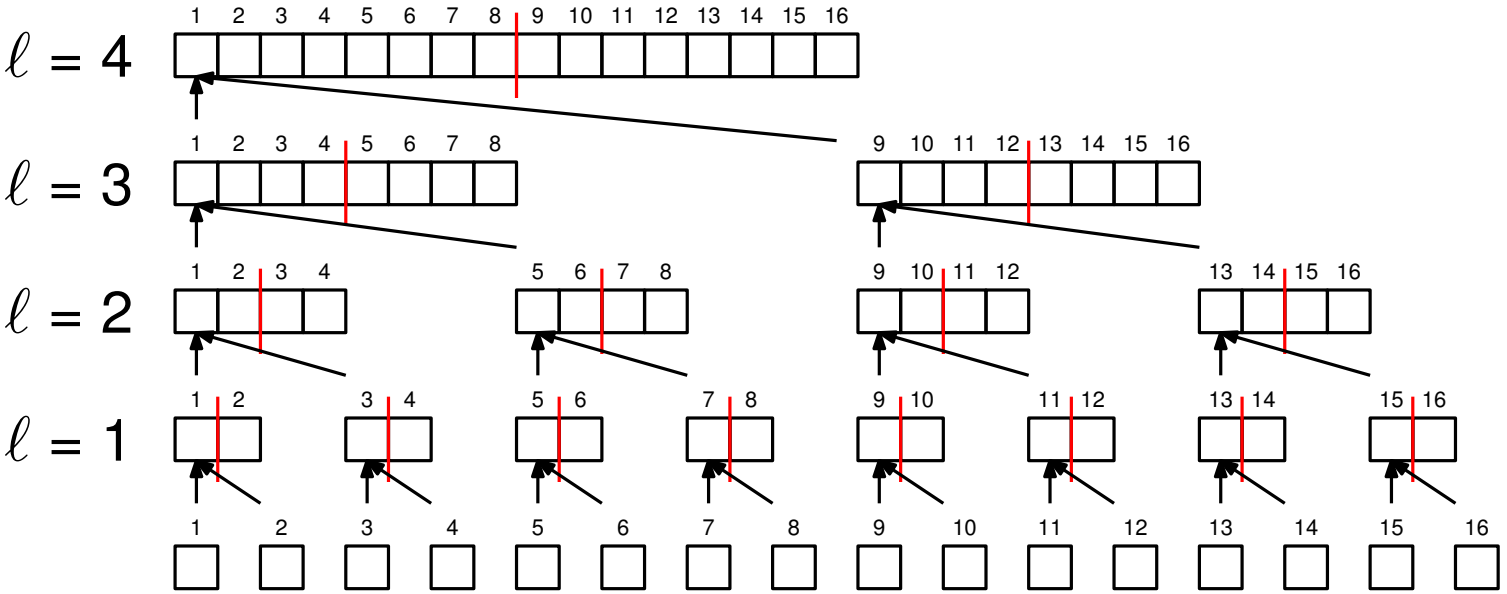

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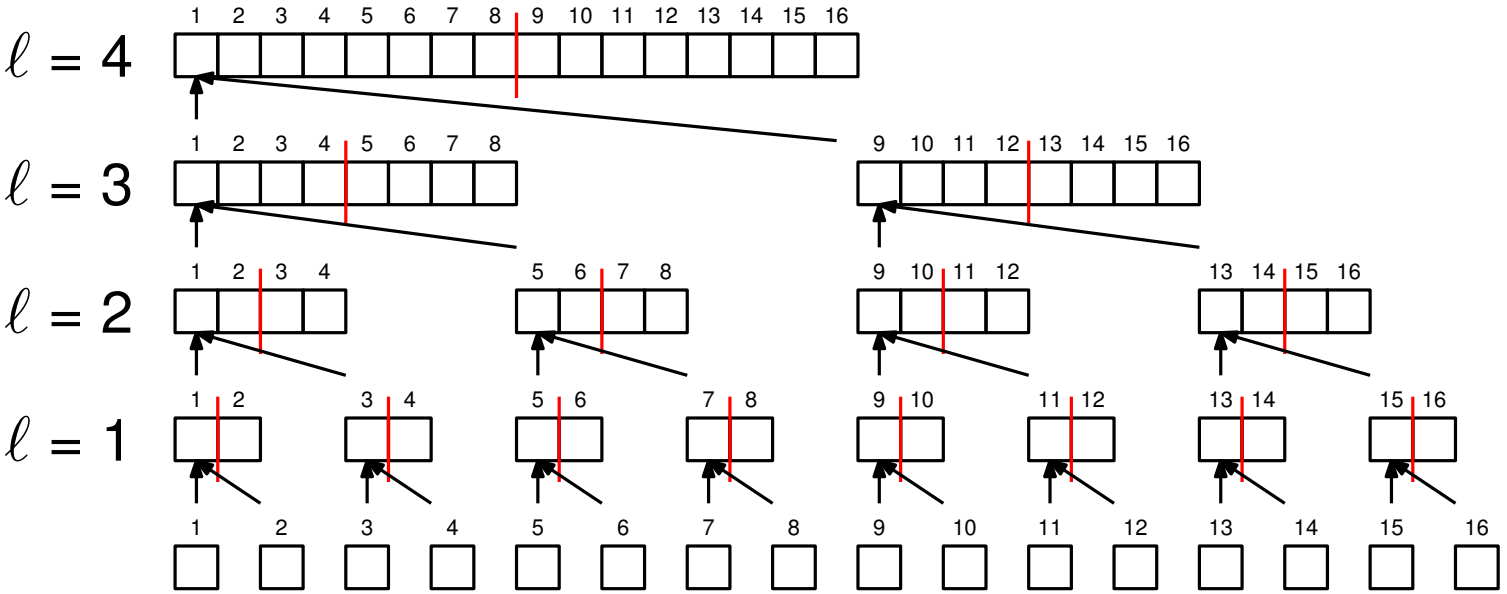

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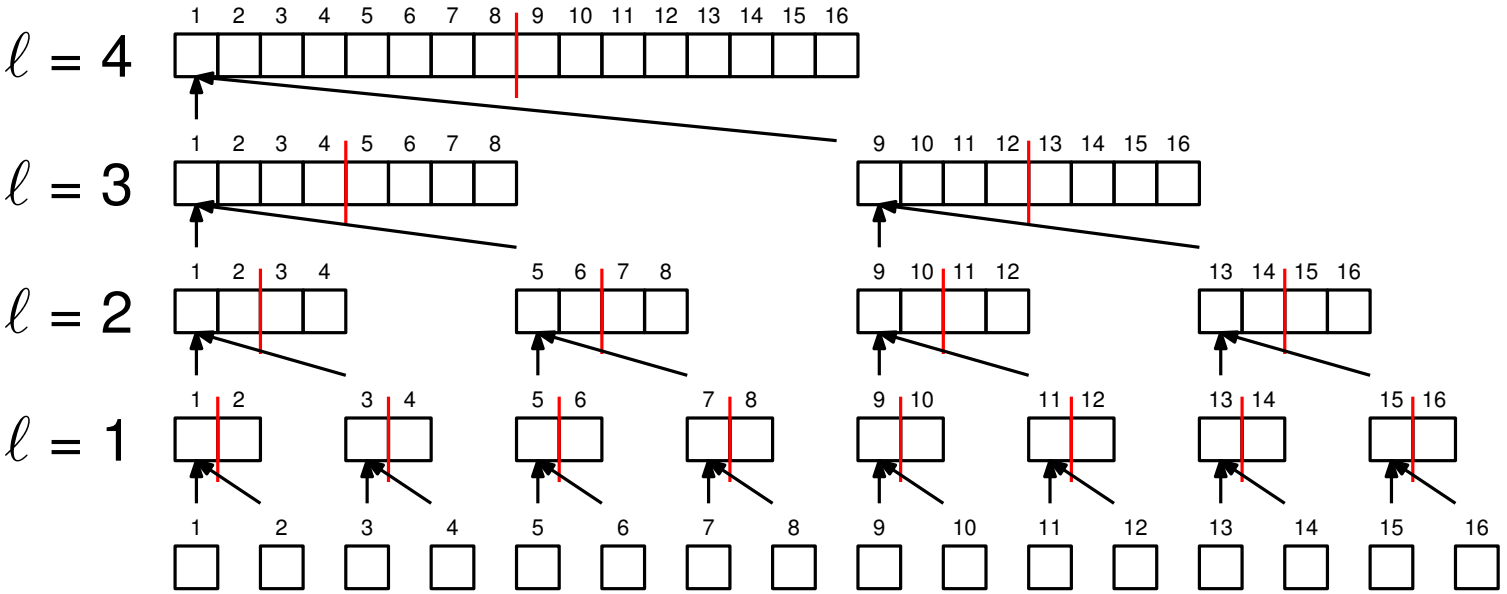

for $\ell = 1$ to $\log n$ **do**

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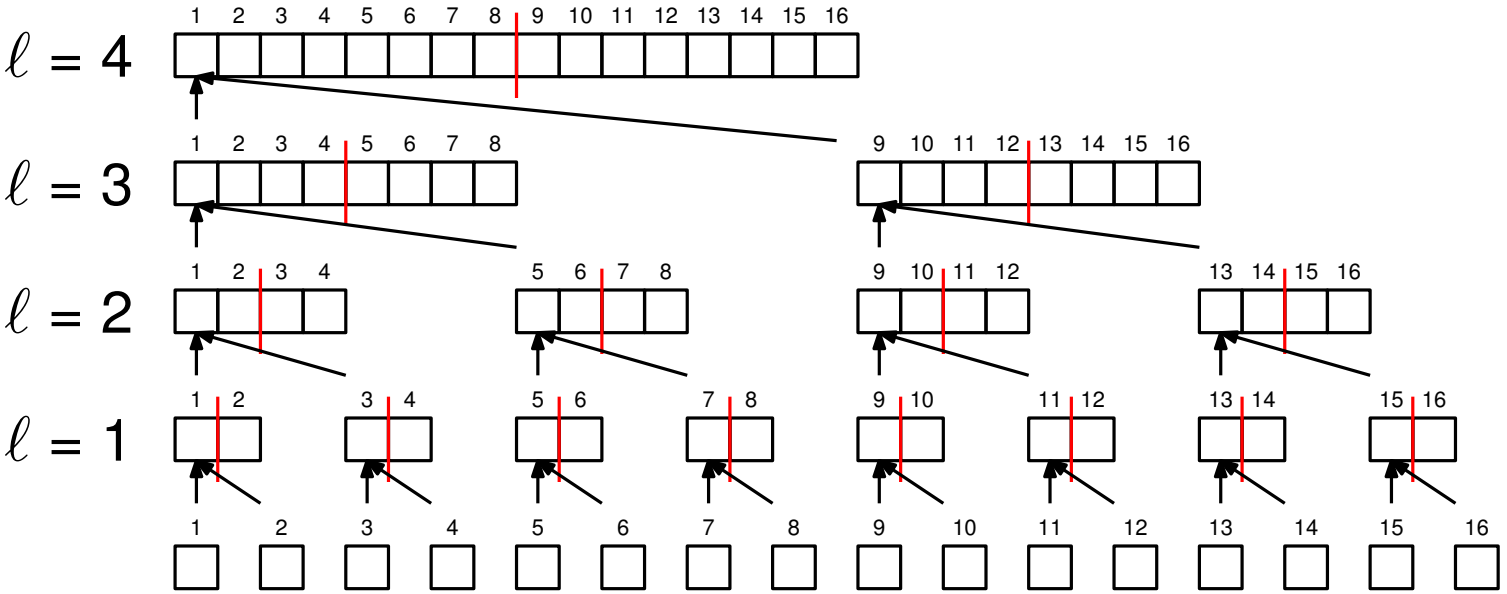

for $\ell = 1$ to $\log n$ **do**

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    if $i + 2^{\ell-1} \leq n$ **then**

      $a[i] = \min(a[i], a[i + 2^{\ell-1}])$

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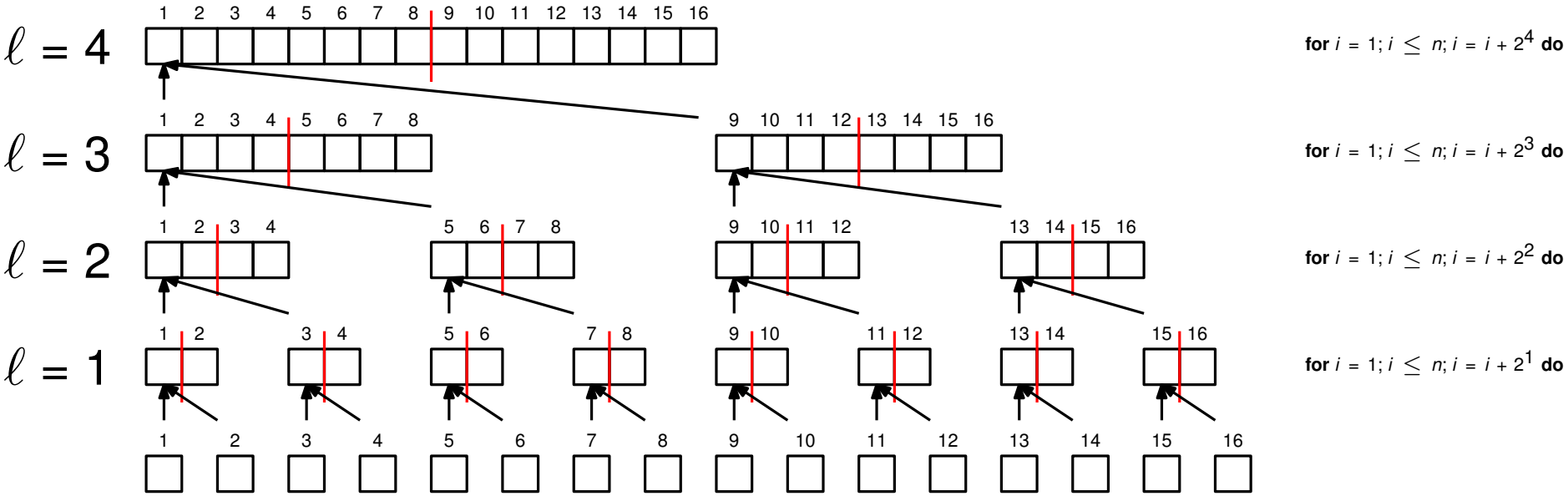

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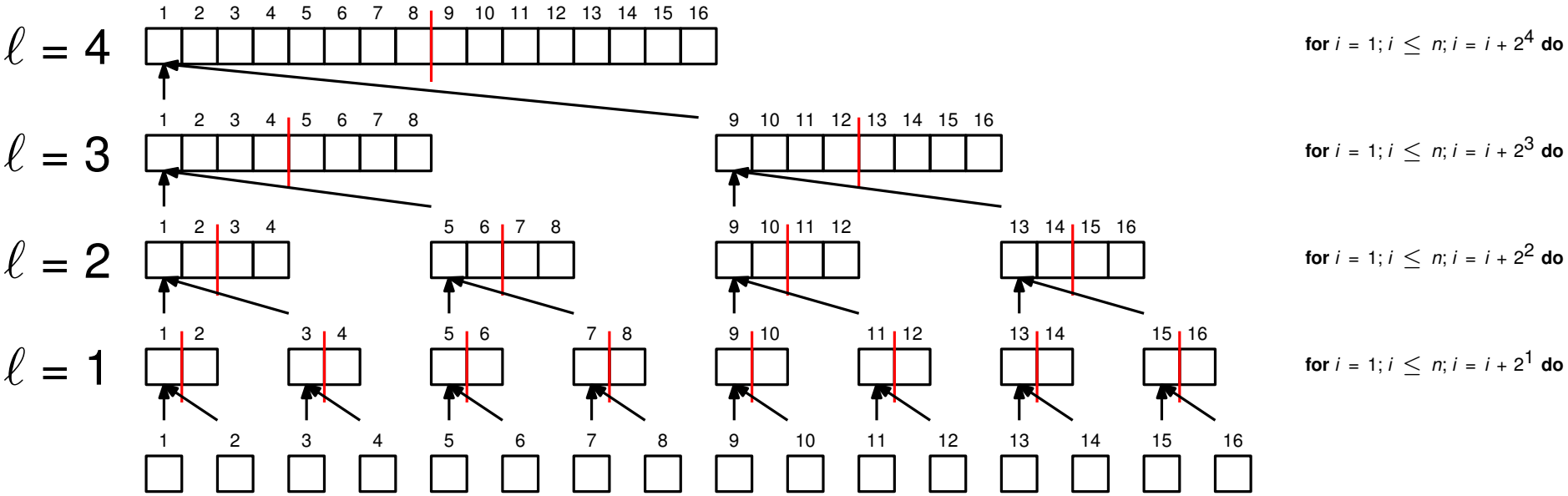

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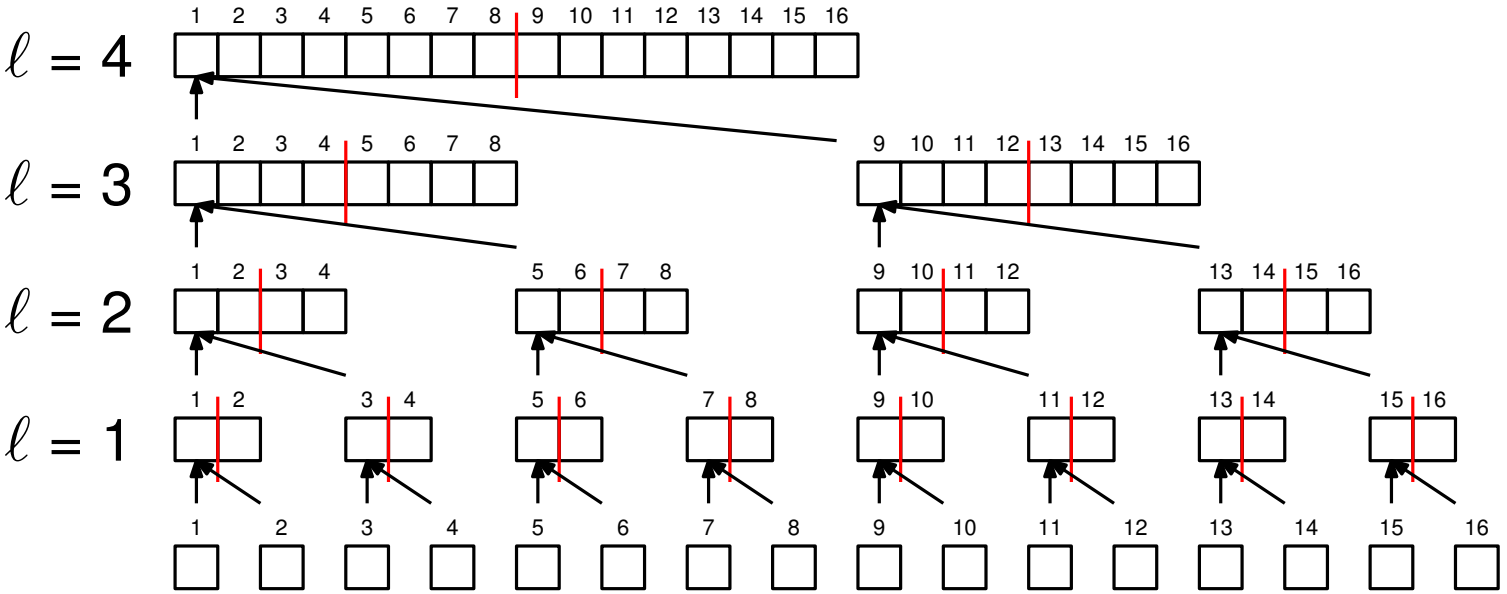

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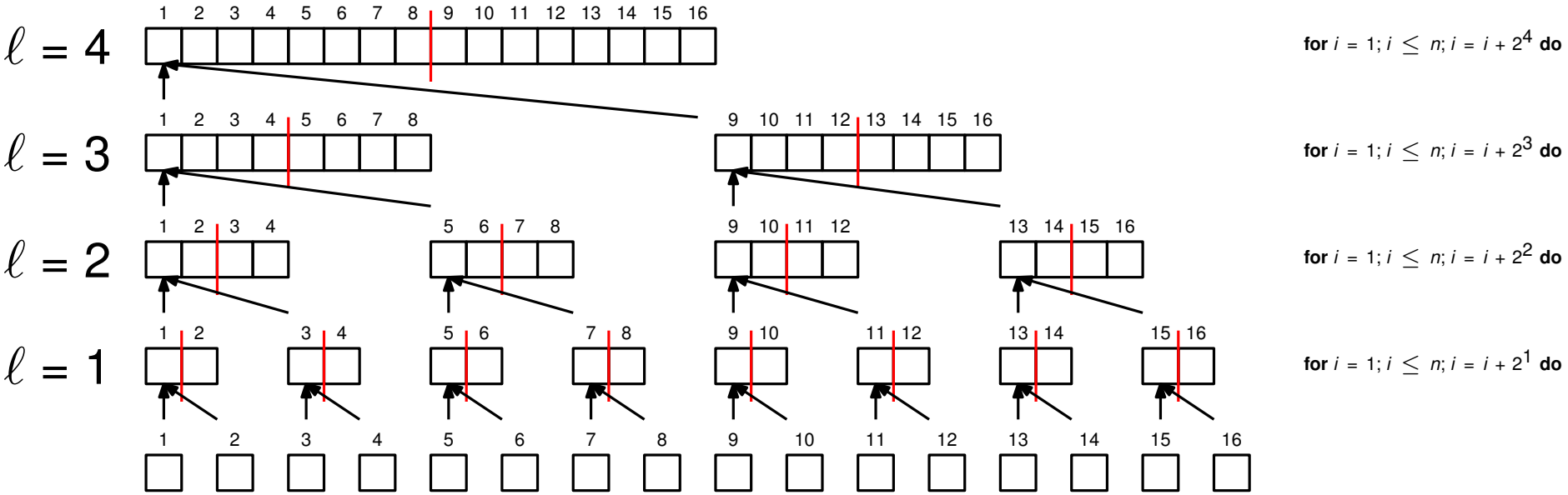

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$O(1)$

$\sum_{\ell=1}^{\log n} O(1) = O(\log n)$

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Recursion vs. parallel **for** loop

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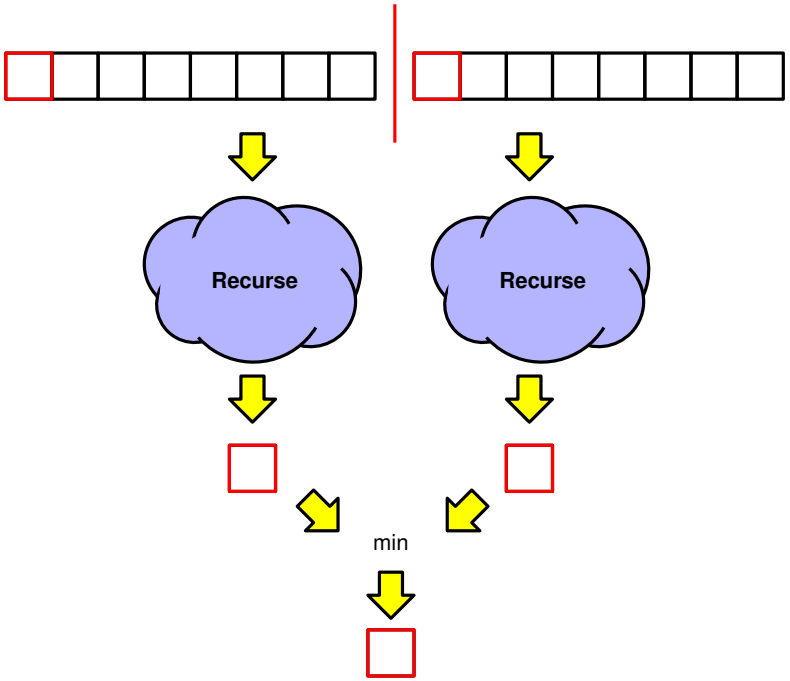


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procedure MIN(a[1..n])
   for ℓ = 1 to log n do
      for i = 1; i ≤ n; i = i + 2^ℓ in parallel do
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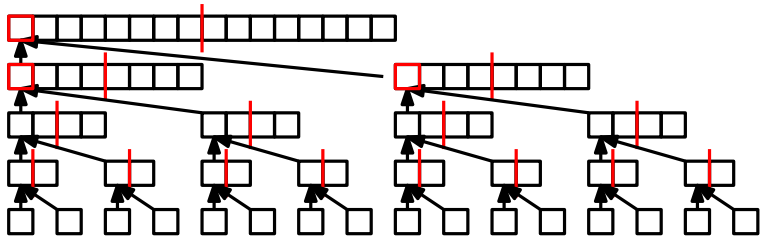

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- Recursion is often easier to design and understand
- Analysis of recursive algorithm requires solving recurrences
- A programming language might support one more efficiently than another
- Spawning n threads might be slow (for large n)
  - Hidden cost of the parallel **for** loop
  - At most $O(\log n)$, but often negligible

Fewer than n processors?

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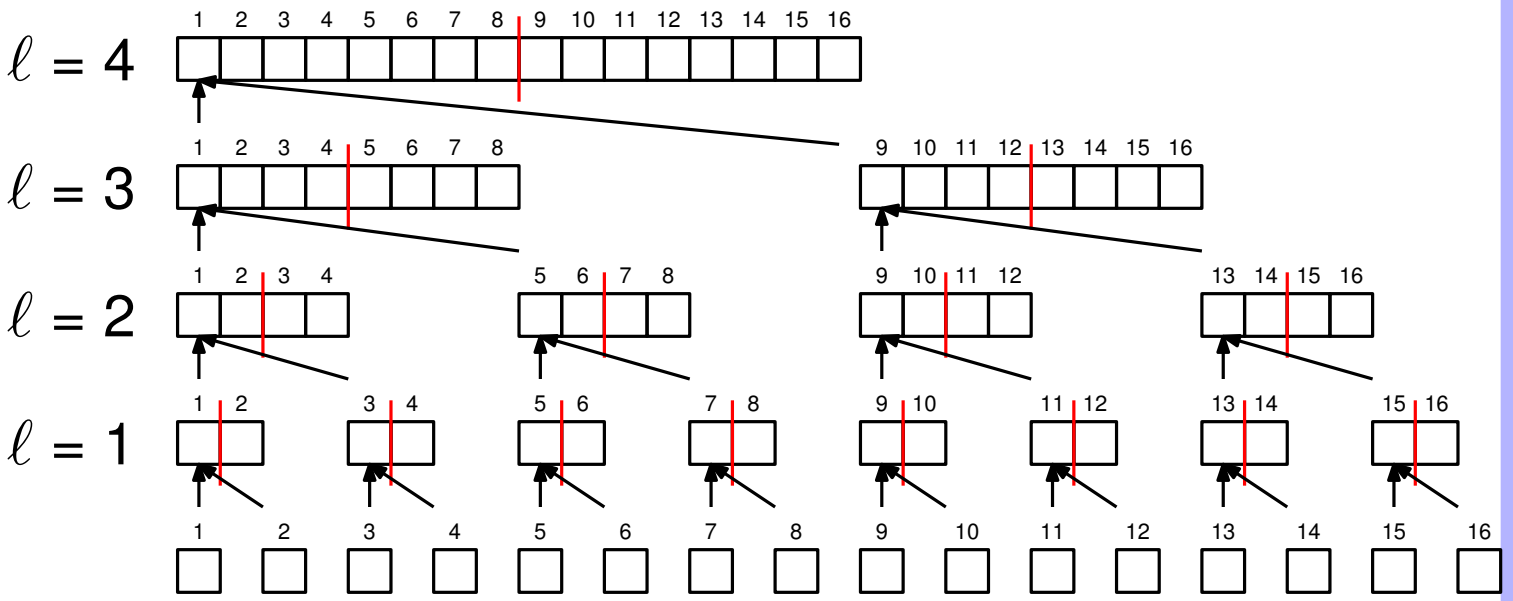
$n = 16$



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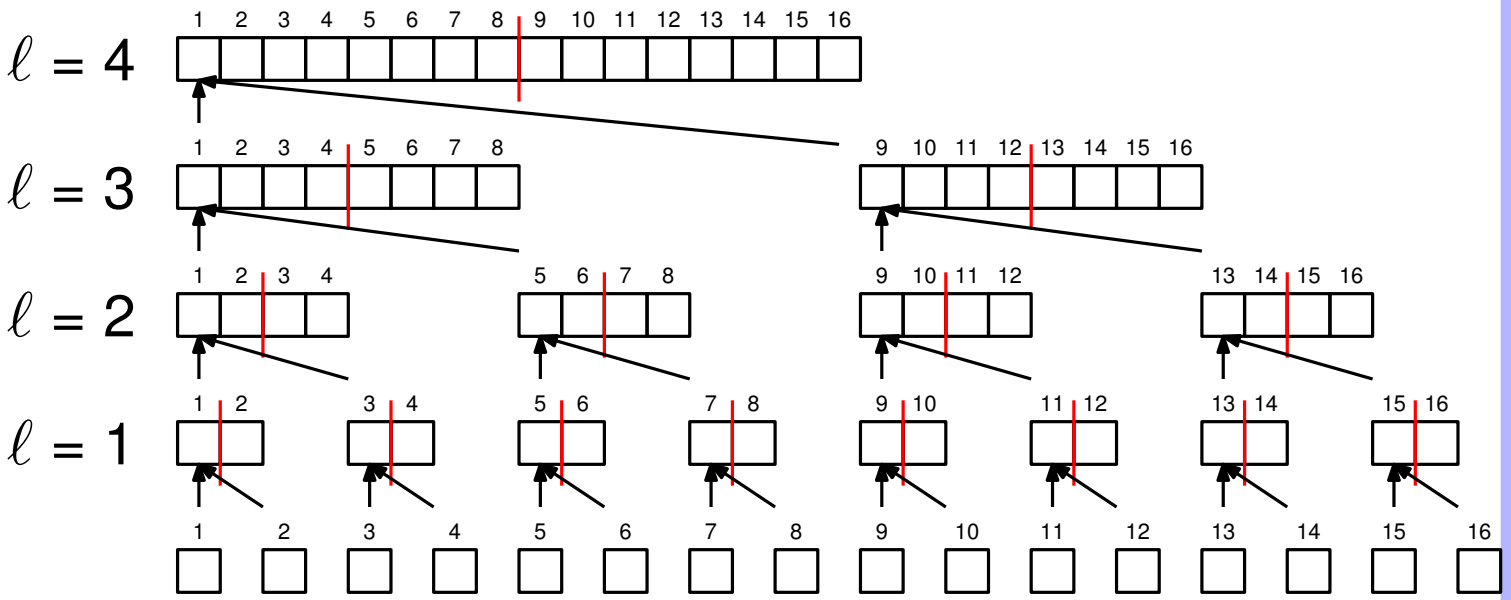
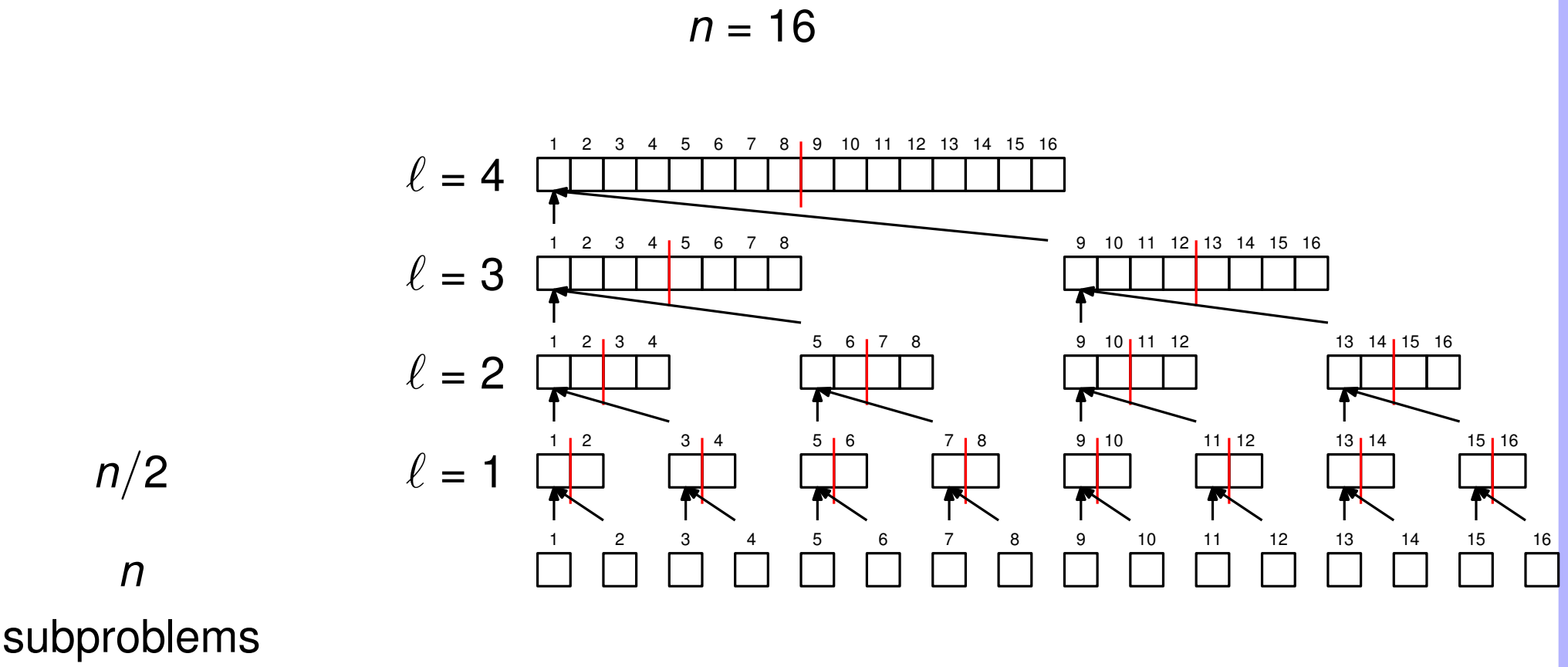


n subproblems

Fewer than n processors?



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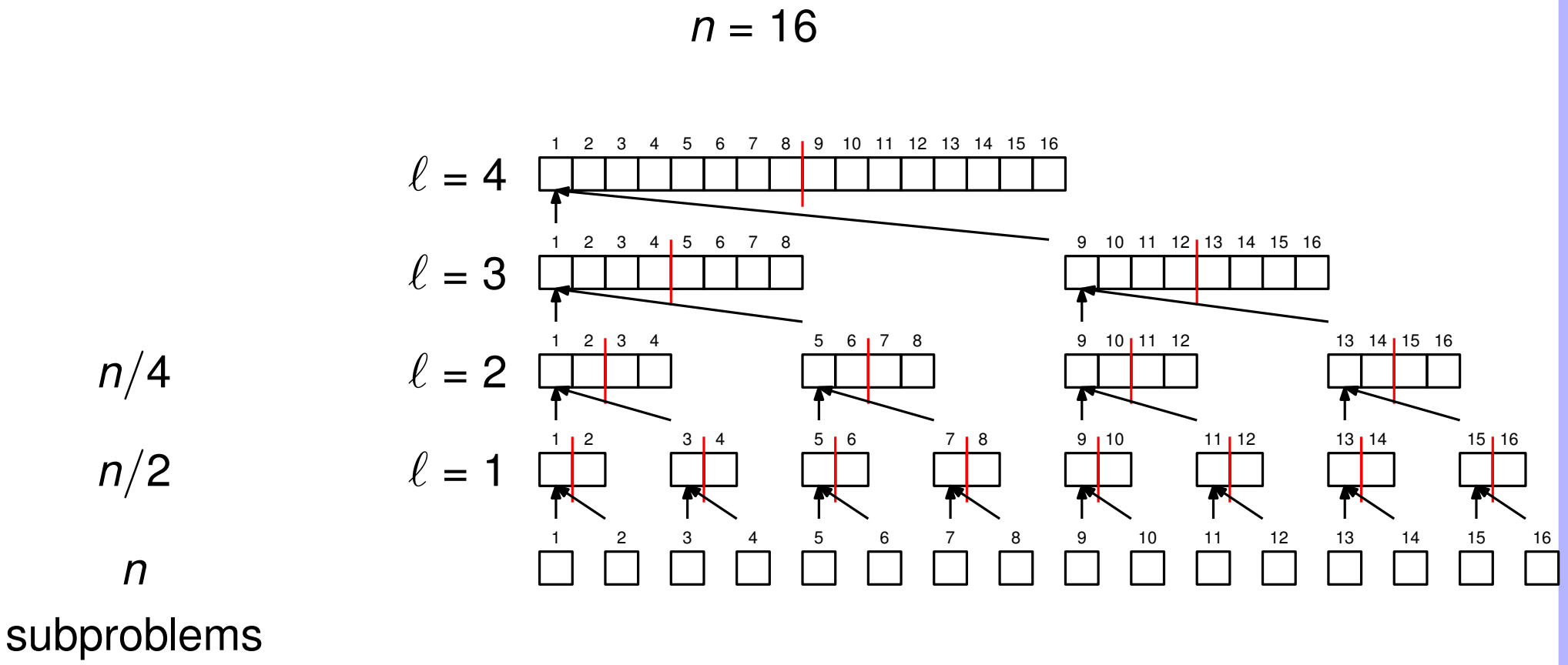

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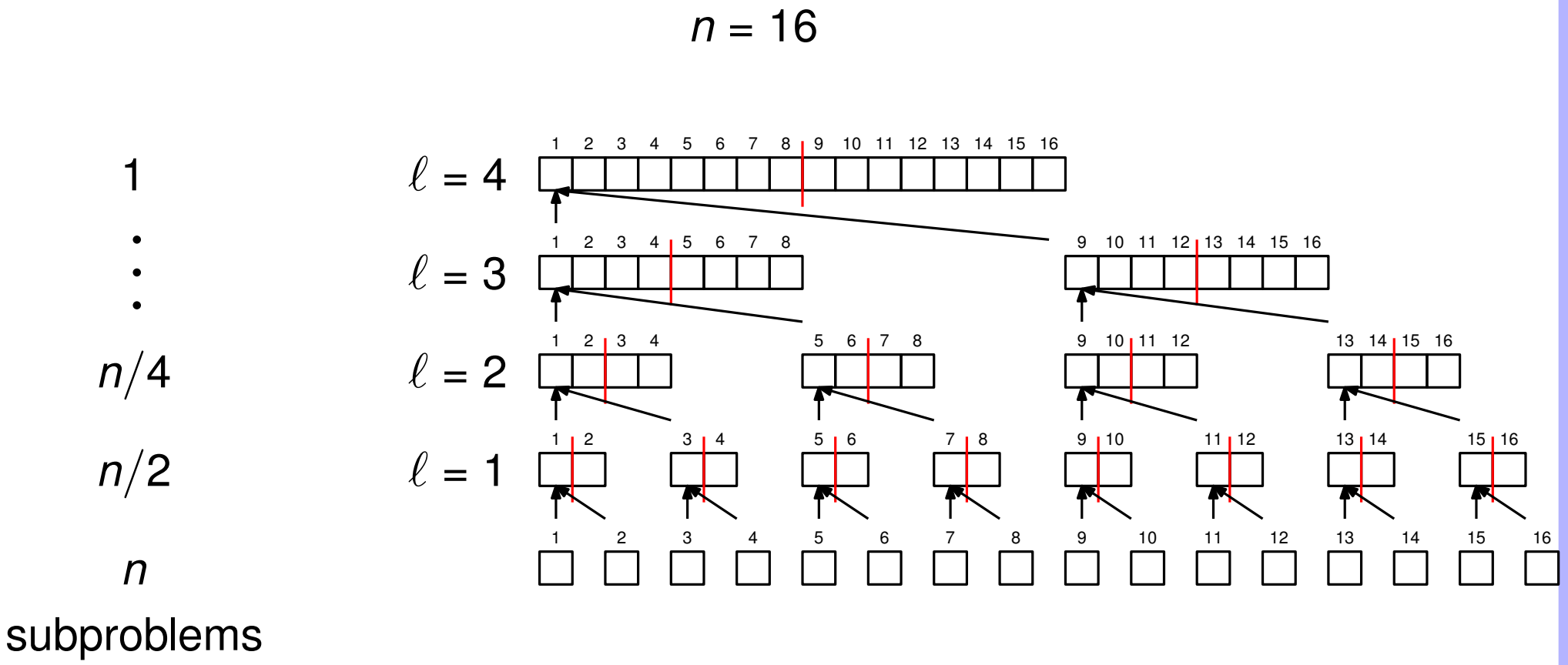

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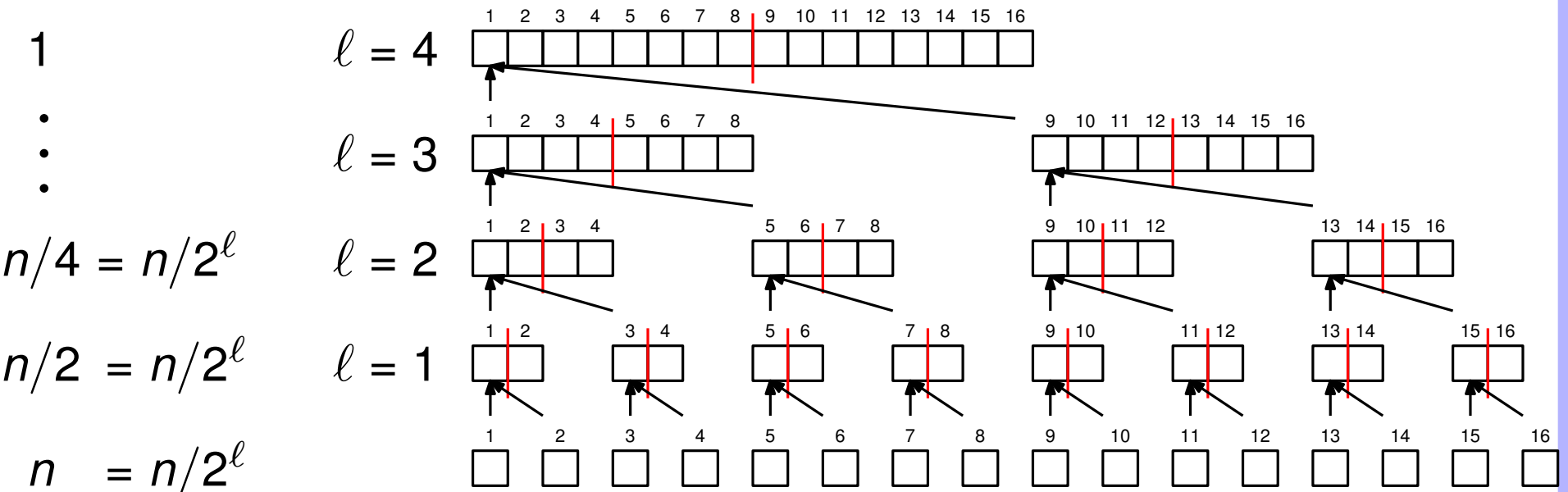

Fewer than n processors?

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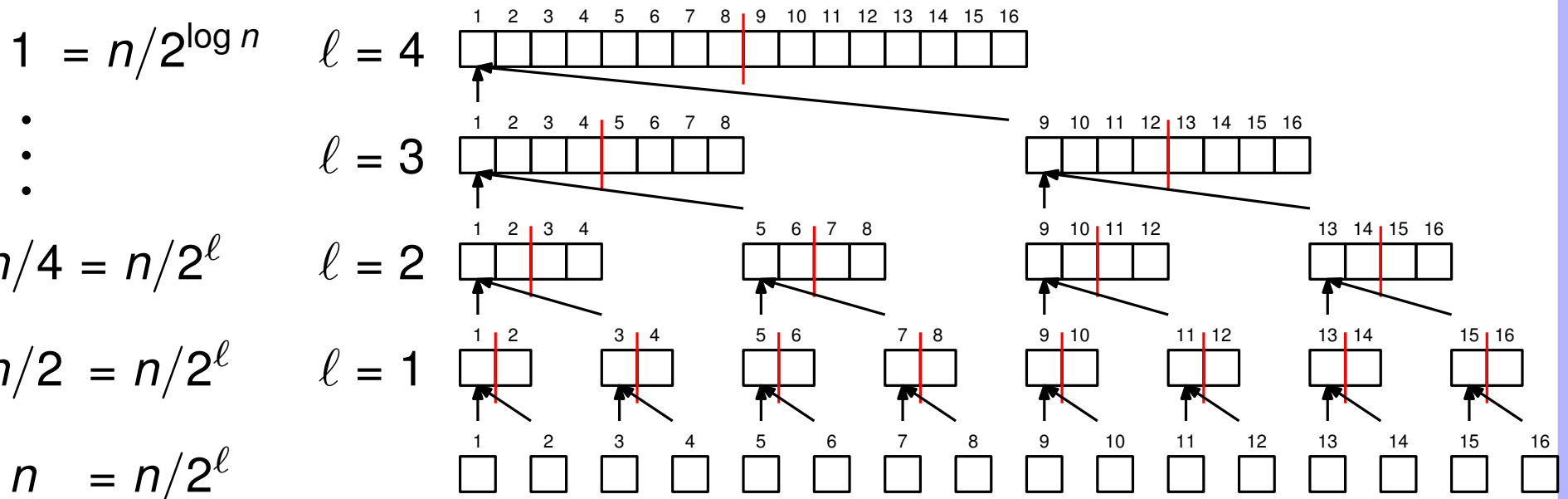


subproblems

Fewer than n processors?

$n = 16$

$P = 1 = n/2^{\log n}$ $\ell = 4$

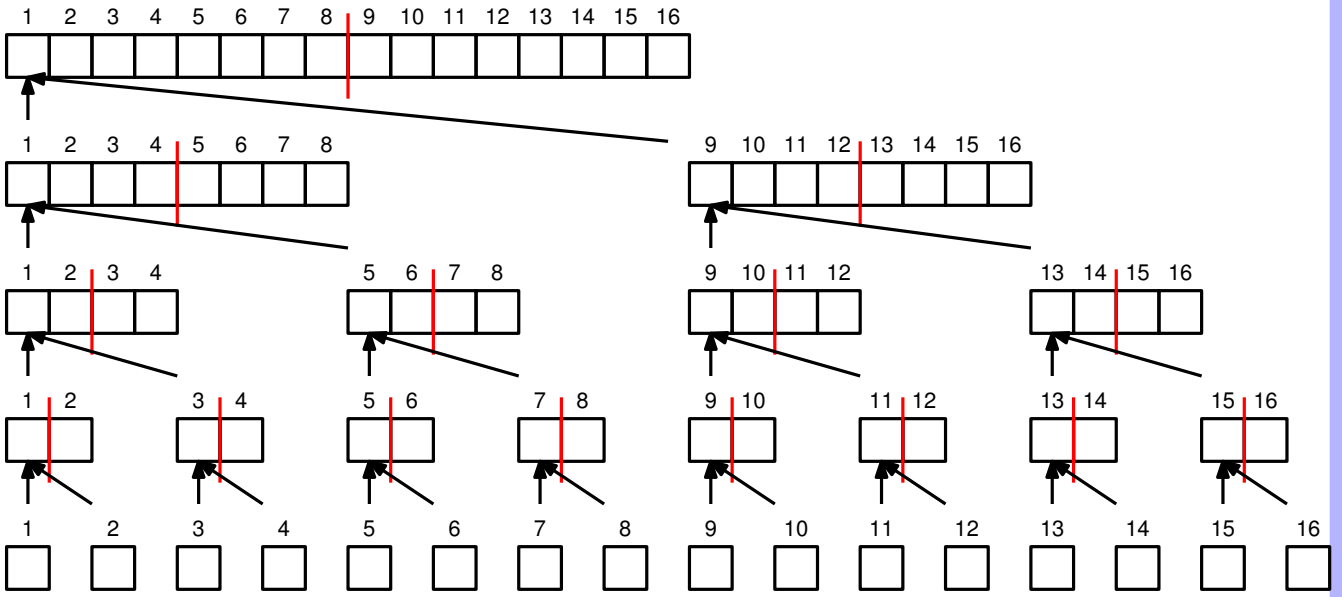
$\vdots$ $\ell = 3$

$P = n/4 = n/2^{\ell}$ $\ell = 2$

$P = n/2 = n/2^{\ell}$ $\ell = 1$

$P = n = n/2^{\ell}$

subproblems



Fewer than n processors?

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$\vdots$ $\quad \ell = 3$

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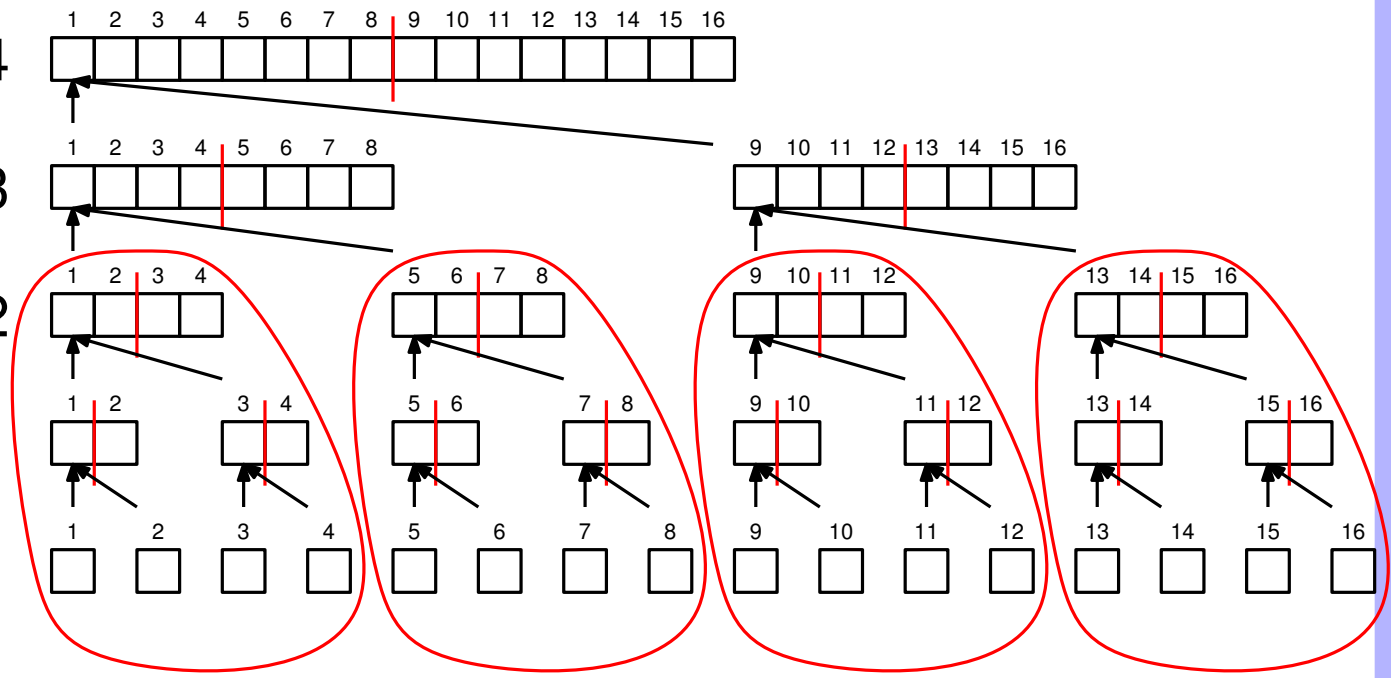
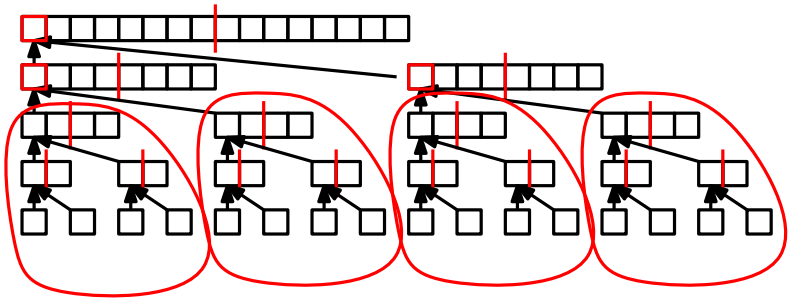
$P = n = n/2^{\ell}$

subproblems



Fewer than n processors?

$P = n/2^{\ell}$

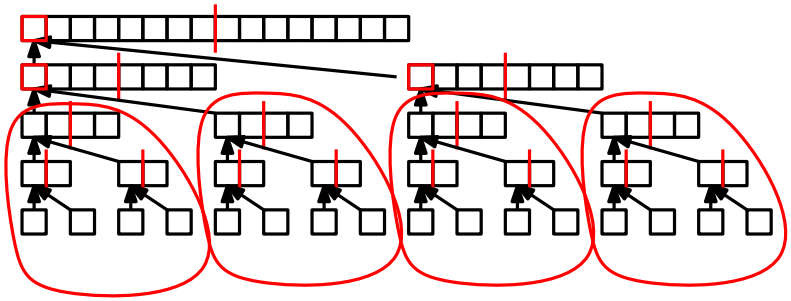


procedure MIN($a[1..n]$)

for $\ell = 1$ to $\log n$ **do**
  for $i = 1; i \leq n; i = i + 2^{\ell}$ **in parallel do**
    if $i + 2^{\ell-1} \leq n$ **then**
      $a[i] = \min(a[i], a[i + 2^{\ell-1}])$
return $a[1]$

Fewer than n processors?

$$\ell = \log \frac{n}{P} \longleftarrow P = n/2^{\ell}$$



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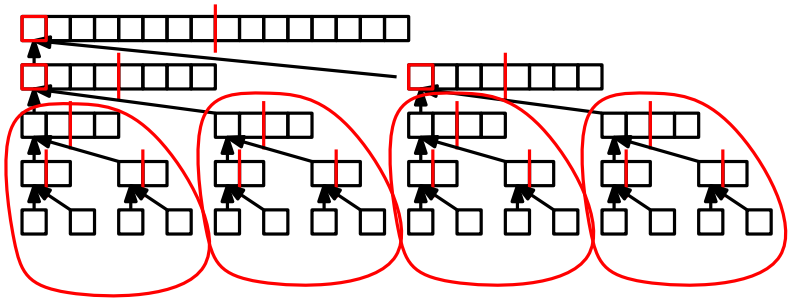
Fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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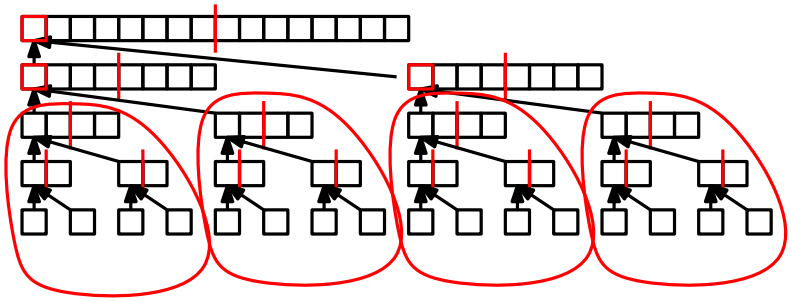
Upper part

Fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



procedure MIN($a[1..n]$)

for $\ell = \log \left\lceil \frac{n}{P} \right\rceil + 1$ to $\log n$ **do**
  for $i = 1; i \leq n; i = i + 2^{\ell}$ **in parallel do**
    if $i + 2^{\ell-1} \leq n$ **then**
      $a[i] = \min(a[i], a[i + 2^{\ell-1}])$
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Upper part

Fewer than n processors?

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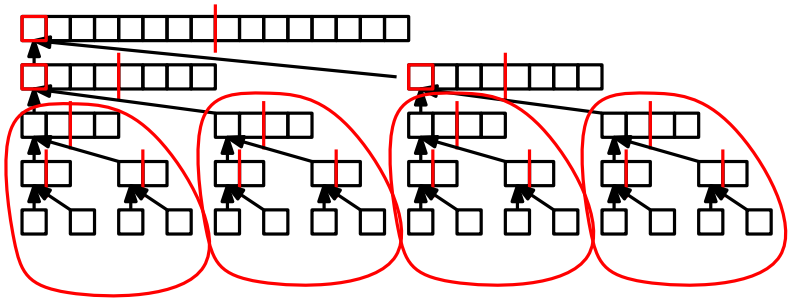

procedure MIN($a[1..n]$)

  for $i = 1$ to P **in parallel do**
    $start = \ldots$
    for $k = start + 1$ to $\ldots$ **do**
      if $a[k] < a[start]$ **then**
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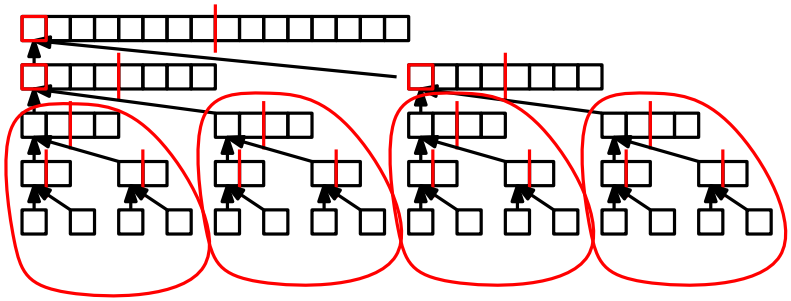
Lower part

Upper part

Fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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Upper part

Fewer than n processors?

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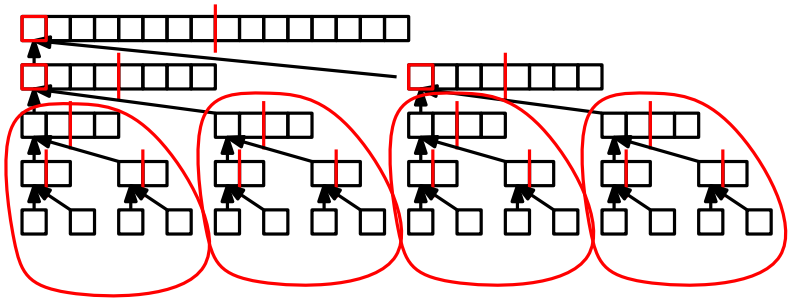

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Lower part

Upper part

Fewer than n processors?

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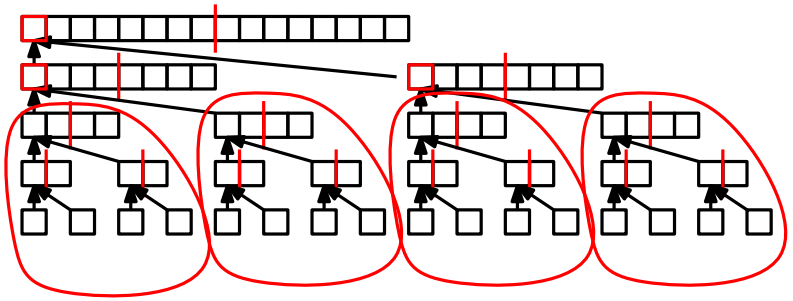

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Runtime of MIN($a[1..n]$)?

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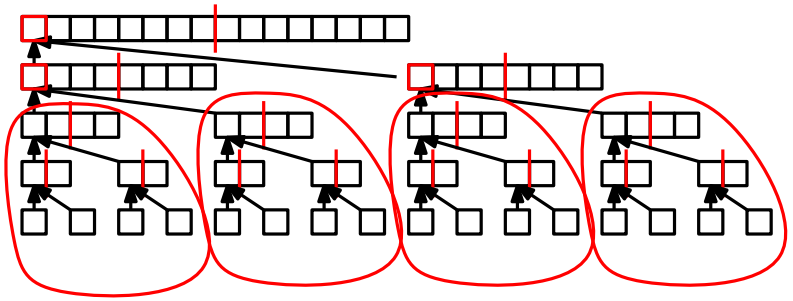

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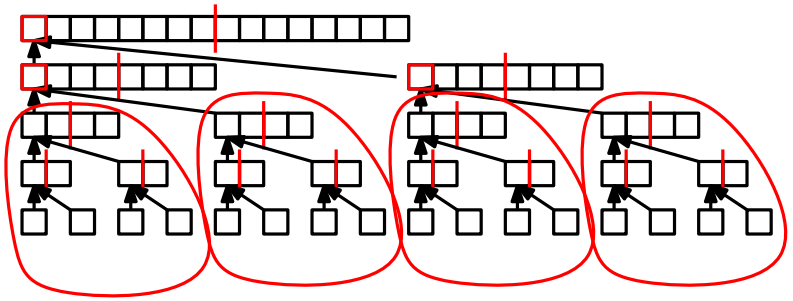
Runtime of MIN($a[1..n]$)?

$O(n/P)$

Fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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Runtime of MIN($a[1..n]$)?

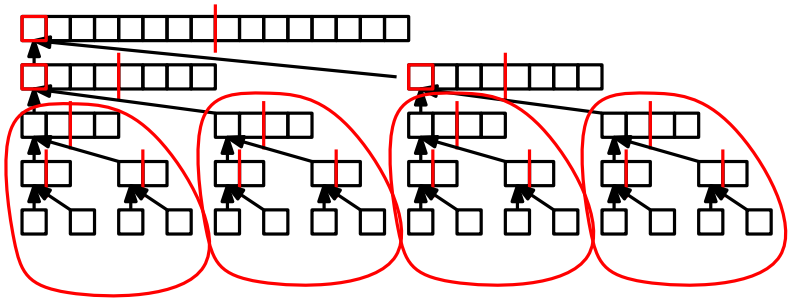
$O(n/P)$

$O\left(\log n - \left(\log \frac{n}{p} + 1\right)\right)$

$= O(\log P)$

Fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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Runtime of MIN($a[1..n]$)?

$O(n/P)$

$O\left(\log n - \left(\log \frac{n}{p} + 1\right)\right)$

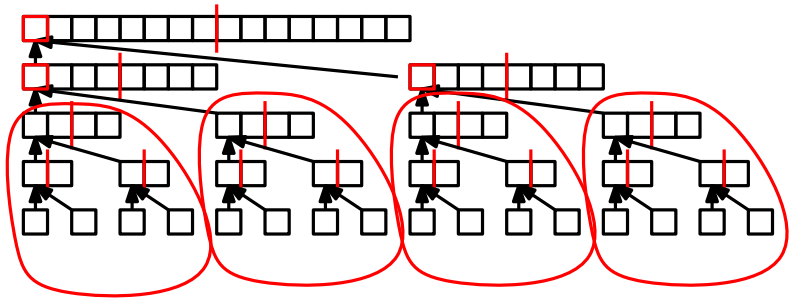
$= O(\log P)$

Total:

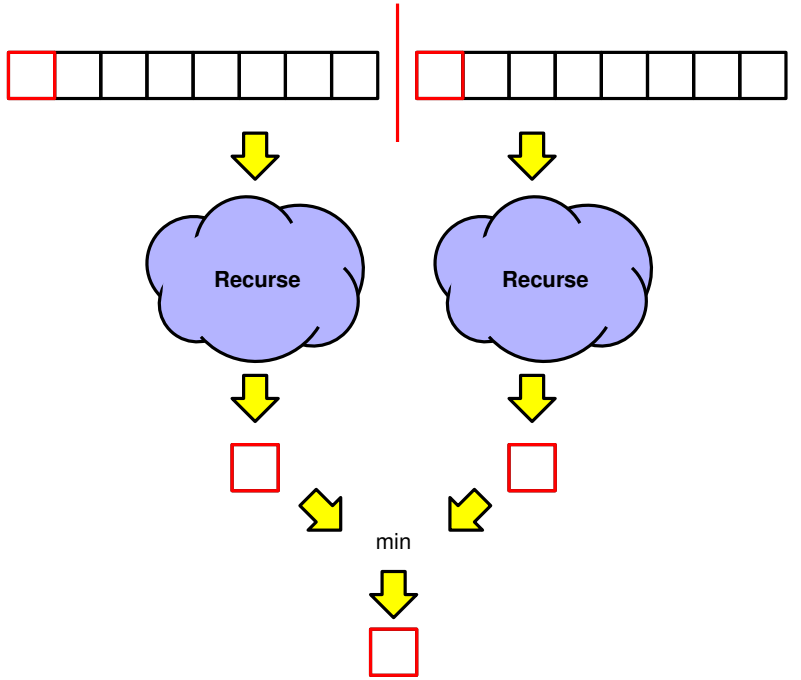
$T_P(n) = O(\frac{n}{P} + \log P)$

Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ← $P = n/2^{\ell}$

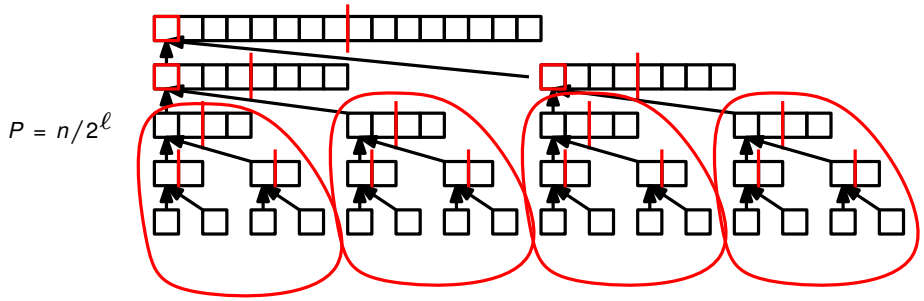


```
procedure MIN(a[i..j])
  if i = j then
      return a[i]
  else
      mid = ⌊(i+j)/2⌋, rmid = mid + 1
      in parallel do
          a[i] = MIN(a[i..mid])
          a[rmid] = MIN(a[rmid..j])
      return min(a[i], a[rmid])
```

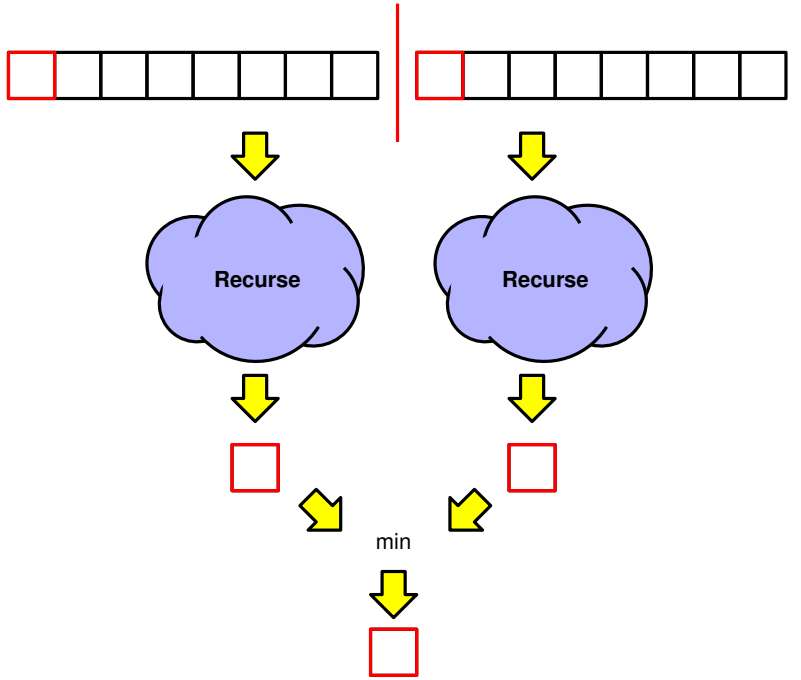


Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$

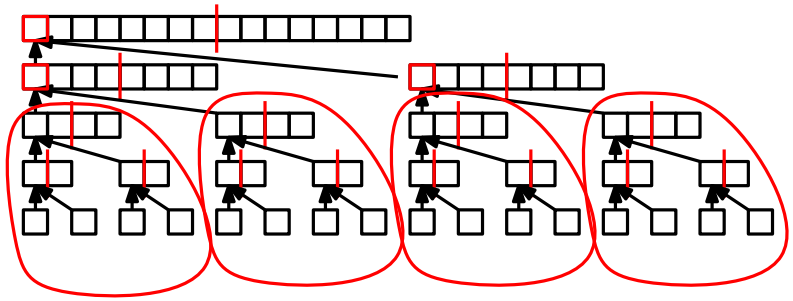



```
procedure MIN(a[i..j])
  if j − i + 1 = 1 then
      return a[i]
  else
      mid = ⌊(i+j)/2⌋, rmid = mid + 1
      in parallel do
          a[i] = MIN(a[i..mid])
          a[rmid] = MIN(a[rmid..j])
      return min(a[i], a[rmid])
```

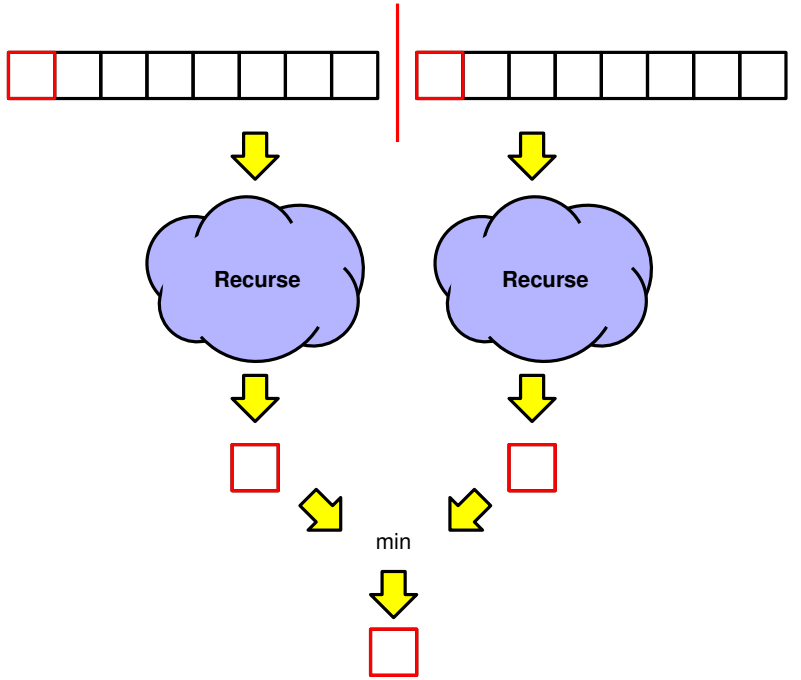


Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$

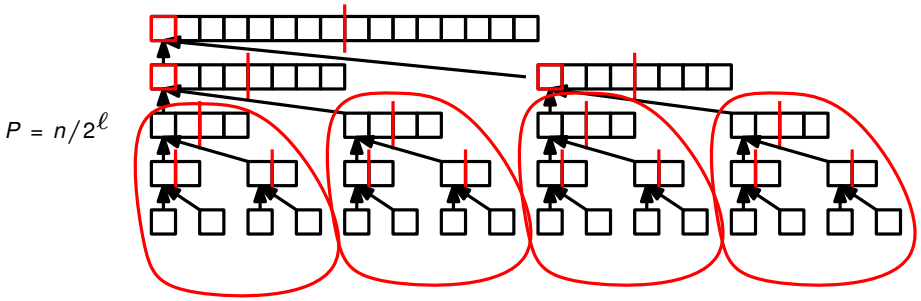


procedure MIN($a[i..j], n, P$)
  if $j - i + 1 \leq n/P$ **then**
    return SEQMIN($a[i..j]$)
  else
    $mid = \left\lfloor \frac{i+j}{2} \right\rfloor$, $rmid = mid + 1$
    in parallel do
      $a[i]$ = MIN($a[i..mid], n, P$)
      $a[rmid]$ = MIN($a[rmid..j], n, P$)
    return min($a[i], a[rmid]$)



Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ←

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  if j − i + 1 ≤ n/P then
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  else
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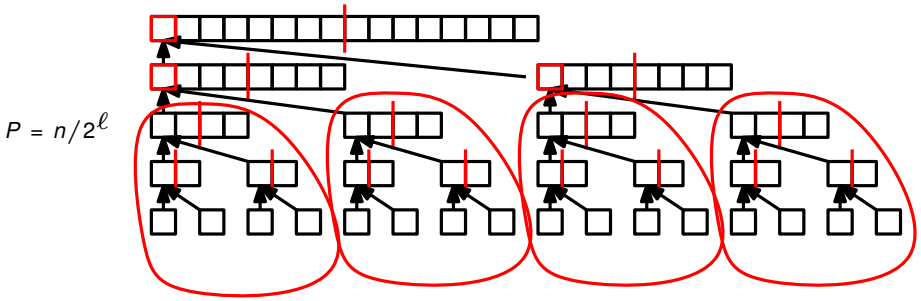
      in parallel do
          a[i] = MIN(a[i..mid], n, P)
          a[rmid] = MIN(a[rmid..j], n, P)
      return min(a[i], a[rmid])
```

```
procedure SEQMIN(a[i..j])
    for k = i + 1 to j do
        if a[k] < a[i] then
            a[i] = a[k]
    return a[i]
```

Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ←




```
procedure MIN(a[i..j], n, P)
  if j − i + 1 < 2n/P then
      return SEQMIN(a[i..j])
  else
      mid = ⌊(i+j)/2⌋, rmid = mid + 1
      in parallel do
          a[i] = MIN(a[i..mid], n, P)
          a[rmid] = MIN(a[rmid..j], n, P)
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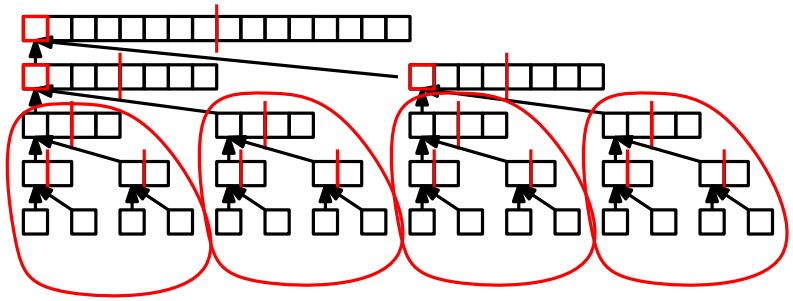
Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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  else
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    in parallel do
      $a[i]$ = MIN($a[i..mid]$, n, P)
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  for $k = i + 1$ to j **do**
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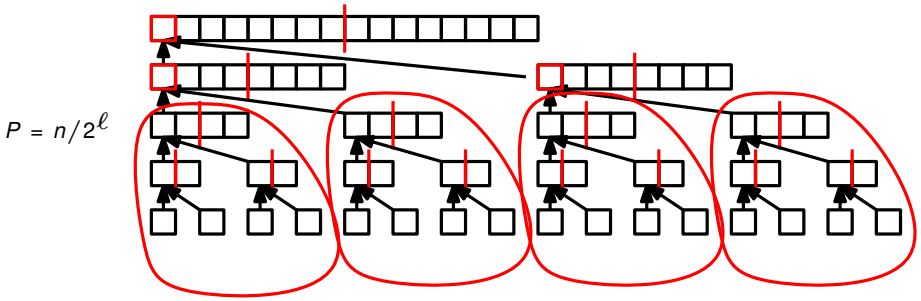
Runtime of MIN($a[1..n]$)?

Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵

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```
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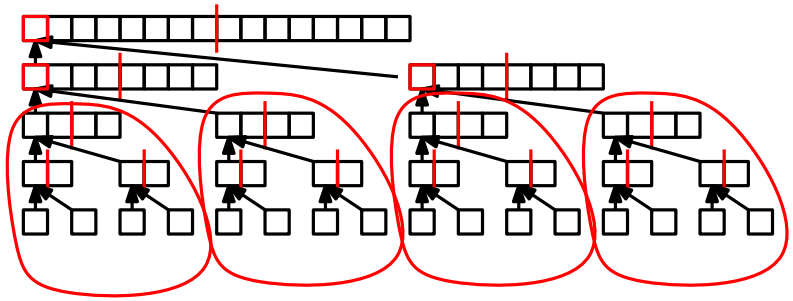
Runtime of MIN($a[1..n]$)?

$$T_P(n) = \begin{cases} T_P(n/2) + O(1) & \text{if } n \geq \frac{2n}{P} \\ O(n) & \text{if } n < \frac{2n}{P} \end{cases}$$

Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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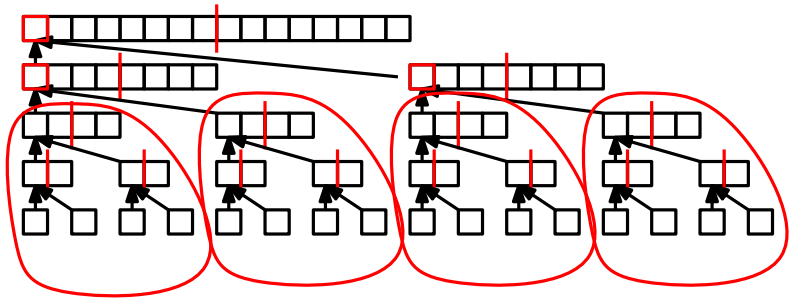
Runtime of MIN($a[1..n]$)?

same n?!

$$T_P(n) = \begin{cases} T_P(n/2) + O(1) & \text{if } n \geq \frac{2n}{P} \\ O(n) & \text{if } n < \frac{2n}{P} \end{cases}$$

Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ⟵ $P = n/2^{\ell}$



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    $mid = \left\lfloor \frac{i+j}{2} \right\rfloor$, $rmid = mid + 1$
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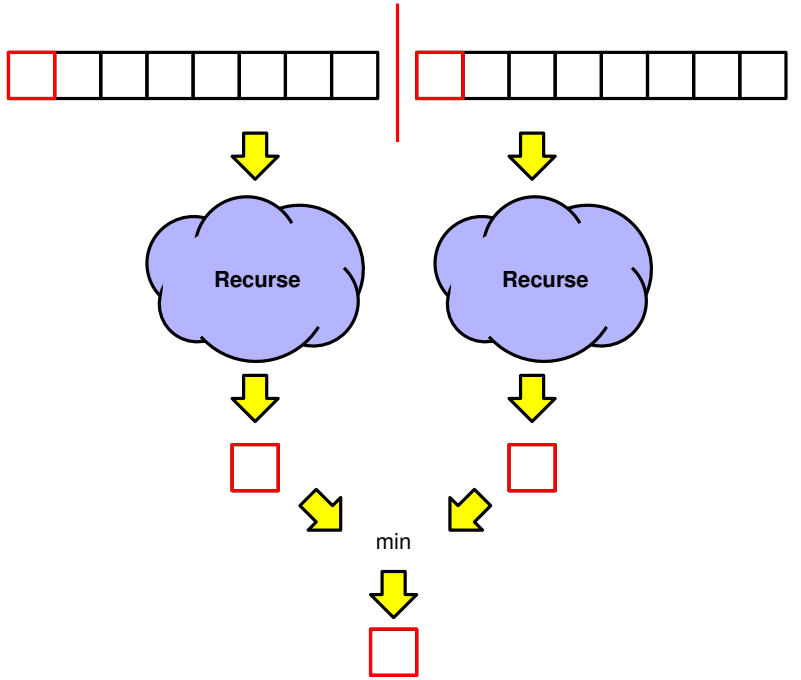
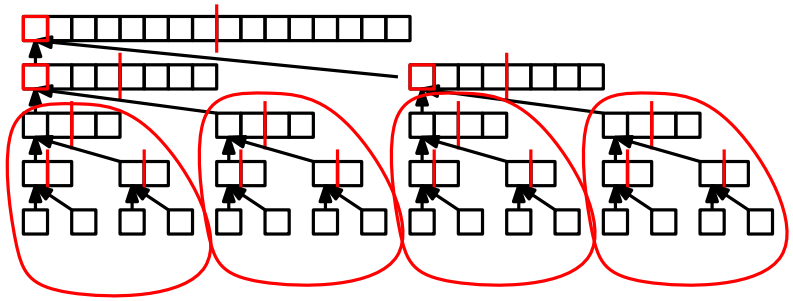


Recursion with fewer than n processors?

$\ell = \log \frac{n}{P}$ ← $P = n/2^{\ell}$



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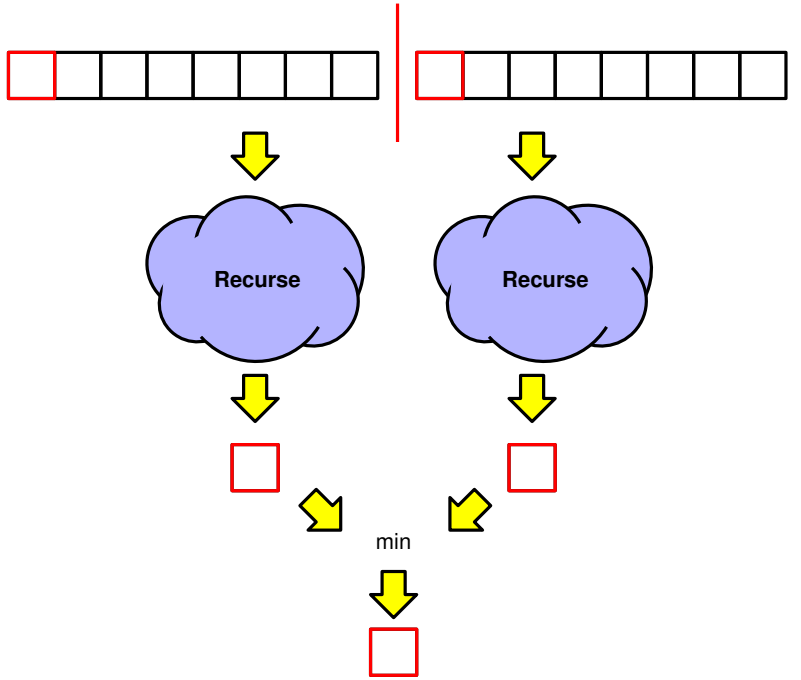

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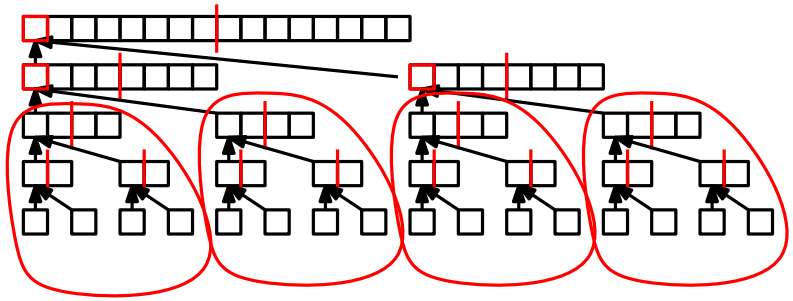

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Runtime of MIN(a[1..n])?

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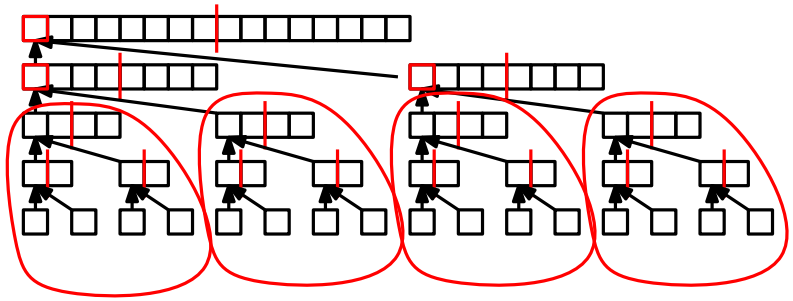

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Runtime of MIN($a[1..n]$)?

$$T_P(n) = \begin{cases} T_{P/2}(n/2) + O(1) & \text{if } P > 1 \\ O(n) & \text{if } P = 1 \end{cases}$$

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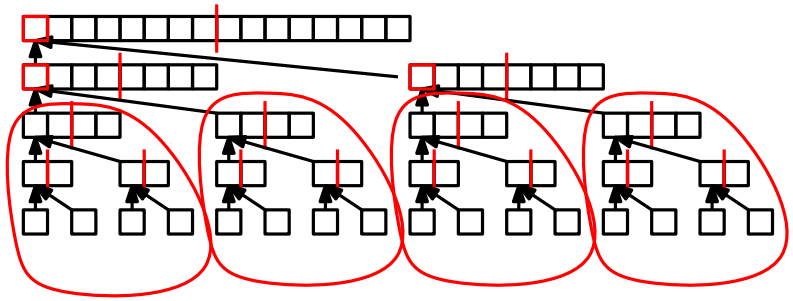

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Work-Depth Analysis

We use two metrics to analyze parallel algorithms

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Work: $W(n)$

(Parallel) Time: $T(n)$

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Also denoted as:

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- $S(n)$: span
- $T_\infty(n)$ $(P = \infty)$

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Work-efficient parallel algorithm:

$W(n) = O(\text{time of the fastest sequential algorithm})$

Analysis Example

```
procedure MIN(a[i..j])
  if i = j then
      return a[i]
  else
      mid = ⌊(i+j)/2⌋; rmid = mid + 1
      in parallel do
          left = MIN(a[i..mid])
          right = MIN(a[rmid..j])
      return min(left, right)
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procedure MIN(a[i..j])
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Theorem. *Any PRAM algorithm with work $W(n)$ and span $T(n)$, can be implemented on a P-processor PRAM to run in time*

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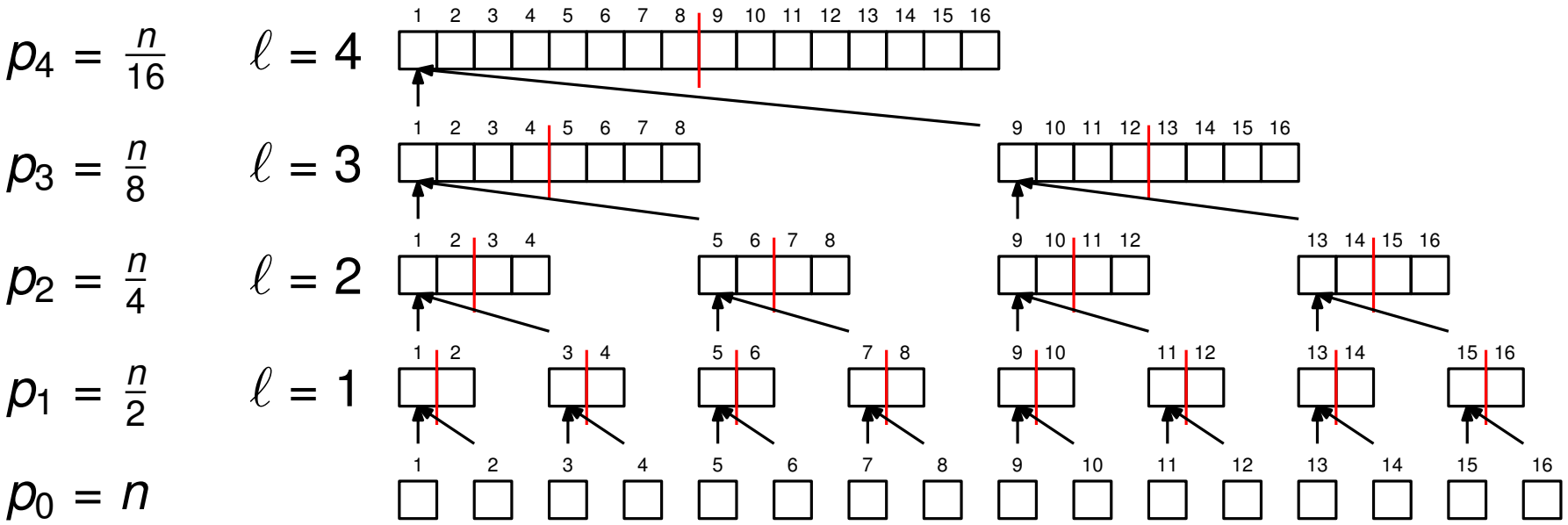

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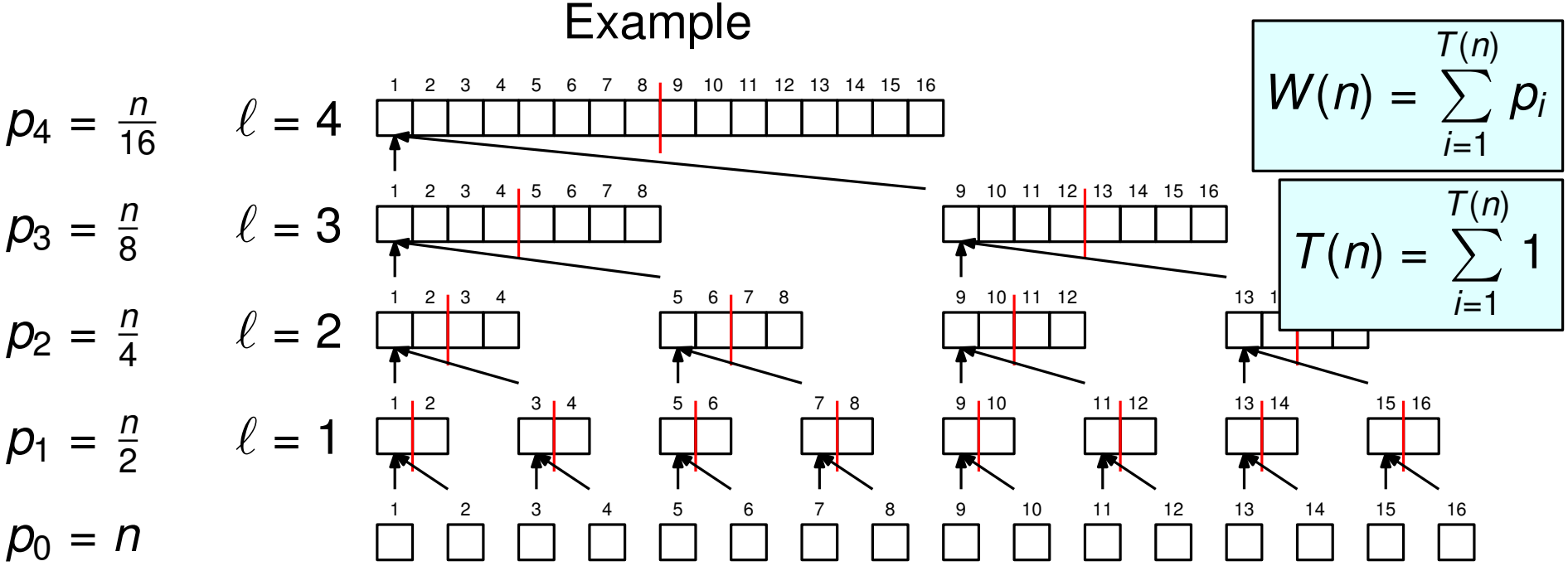

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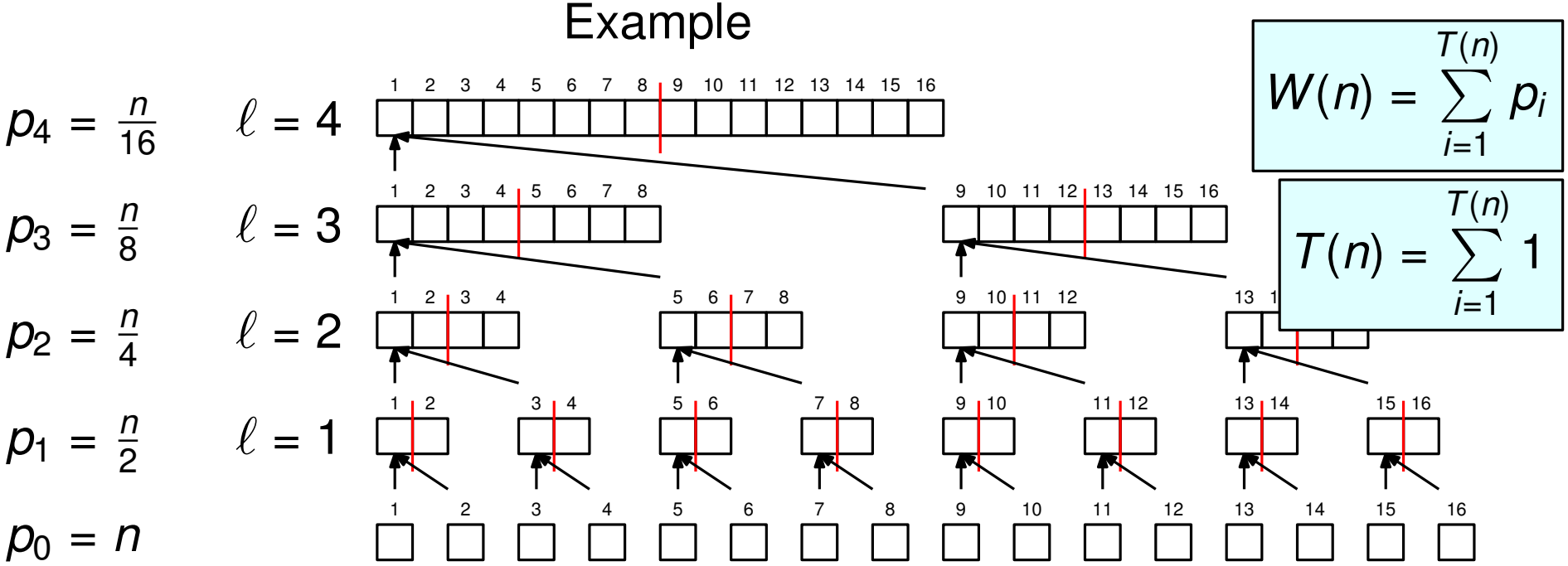

We design an algorithm A' that will run on P (physical) processors, executing the operations of $\left\lceil \frac{p_i}{P} \right\rceil$ "virtual" processors of A of step i.

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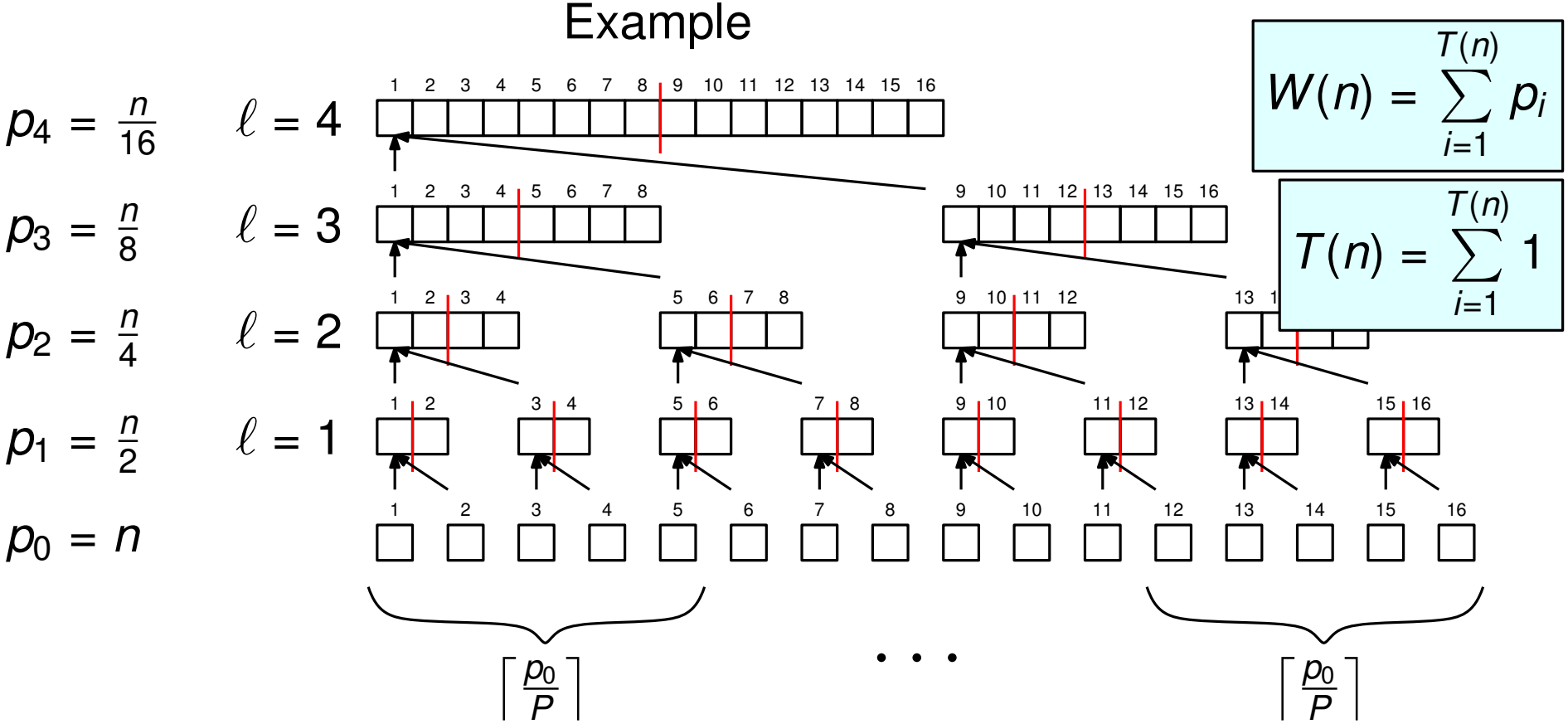

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Thus, each step of A takes $\left\lceil \frac{p_i}{P} \right\rceil$ time on P (physical) processors, for a total of

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$$= O\left(\frac{W(n)}{P} + T(n) \right)$$

Q.E.D.

$$W(n) = \sum_{i=1}^{T(n)} p_i$$

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