Height of a Treap Analysis

(Last time)

<u>Lemma</u>: Expected height of any node is $O(\log n)$

<u>Reminder of proof</u>:

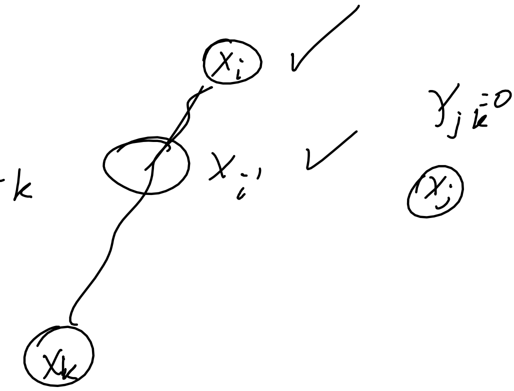
$$\underline{\underline{Y_{ij}}} = \begin{cases} 1 & \text{if } x_i \text{ is a proper ancestor of } x_j \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x_i \text{ has the smallest priority among } \underline{\underline{(x_i, \dots, x_j)}} \\ 0 & \text{o.w.} \end{cases}$$

$$\Pr\left[Y_{ij} = 1\right] = \frac{1}{\underline{\underline{|i-j|+1}}}, \qquad \underline{Y_{ii} = 0}$$

Y_k = Height of x_k

$$Y_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} Y_{ik} = \sum_{i=1}^{k-1} Y_{ik} + \sum_{i=k+1}^{n} Y_{ik}$$



$$E[Y_k] = E\left[\sum_{i=1}^{k-1} Y_{ik}\right] + E\left[\sum_{i=k+1}^{n} Y_{ik}\right]$$

$$= \sum_{i=1}^{k-1} E[Y_{ik}] + \sum_{i=k+1}^{n} E[Y_{ik}]$$

$$= \sum_{i=1}^{k-1} \Pr[Y_{ik}=1] + \sum_{i=k+1}^{n} \Pr[Y_{ik}=1]$$

$$= \sum_{t=2}^{k} \frac{1}{t} + \sum_{t=2}^{n-k} \frac{1}{t}$$

$$\leq 2 \cdot \sum_{t=2}^{n} \frac{1}{t} \leq 2 \cdot \ln n$$

$$Y_k = \underbrace{\sum_{i=1}^{k-1} Y_{ik}}_{} + \underbrace{\sum_{i=k+1}^{n} Y_{ik}}_{}$$

Lemma: $Y_{1,k}, Y_{2,k}, \dots, Y_{k-1,k}$ are independent &

$Y_{k+1,k}, Y_{k+2,k}, \dots, Y_{n,k}$ are independent.

Proof for $Y_{k+1,k}, \dots Y_{n,k}$ are independent.

By induction on # of variables

A & B are independent if $P_r[A \wedge B] = P_r[A] \cdot P_r[B]$

if $P_r[A|B] = P_r[A]$

$\Leftrightarrow P_r[A \wedge B] = P_r[A|B] \cdot P_r[B]$

$P_r[A] \cdot P_r[B]$

I.H.: For all $k+1 \leq n' < n$

$Y_{k+1,k}, Y_{k+2,k}, \dots, Y_{n',k}$

are independent

Let's show that $Y_{k+1,k}, \dots Y_{n,k}$ are independent

$$P_r[(Y_{k+1,k} = 1 \wedge Y_{k+2,k} = 1 \wedge \dots) \wedge Y_{n,k} = 1] =$$

$$= \Pr\left[\bigwedge_{i=k+1}^{n} Y_{i,k} = 1\right]$$

$$= \Pr\left[\bigwedge_{i=k+1}^{n-1} Y_{i,k}=1 \,\middle|\, Y_{n,k}=1\right] \cdot \Pr\left[Y_{n,k}=1\right]$$

x_n is proper ancestor of x_k

$\iff$ x_n is has highest priority among $x_k, x_{k+1}, \dots x_n$

$Y_{i,k} = 1$

$\iff$ x_i has the highest priority among $x_i \dots x_{n-1}$

$k+1 \le i \le n-1$

$$= \Pr\left[\bigwedge_{i=k+1}^{n-1} Y_{i,k}=1\right] \cdot \Pr\left[Y_{n,k}=1\right]$$

$$= \Pr\left[Y_{k+1,k}=1\right] \cdot \Pr\left[Y_{k+2,k}=1\right] \cdot \ldots \cdot \Pr\left[Y_{n-1,k}=1\right] \cdot \Pr\left[Y_{n,k}=1\right]$$

$$= \prod_{i=k+1}^{n} \Pr\left[Y_{i,k}=1\right]$$

$$n-1 \geq k+1$$

$$n \geq k+2$$

Base case:

$$\boxed{Y_{k+1,k} \text{ is independent}}$$

Vacuously true. Q.E.D.

Let $Y_{<k} = \sum_{i=1}^{k-1} Y_{ik}$ & $Y_{>k} = \sum_{i=k+1}^{n} Y_{ik}$

Y_k = Height of a node x_k

$$\boxed{Y_k = Y_{<k} + Y_{>k}}$$

$$= \sum_{i=1}^{k-1} Y_{ik} + \sum_{i=k+1}^{n} Y_{ik}$$

Let $Z_i = \begin{cases} 1 & \text{w/ probability } \frac{1}{i} \\ 0 & \text{o.w} \end{cases}$

$$Z^{(n)} = \sum_{i=2}^{n} Z_i$$

$$\boxed{\text{In homework: } \Pr\left[Z^{(n)} \geq 4 \cdot \ln n\right] \leq \frac{1}{n^2}}$$

$$Z^{(k)} = \sum_{i=2}^{k} Z_i = Y_{\ell k} = \sum_{i=1}^{k-1} Y_{ik}$$

$$\Pr[Z_i = 1] = \frac{1}{i}$$

$$\Pr[Y_{k-i+1,k} = 1] = \frac{1}{|k-(k-i+1)|+1} = \frac{1}{i}$$

$$\boxed{\Pr[Y_{ij} = 1] = \frac{1}{|i-j|+1}}$$

$\forall i \quad Z_i = Y_{k-i+1,k}$

$$Z^{(k)} = \sum_{i=2}^{k} Z_i = Z_2 + Z_3 + \dots + Z_k$$

$$= Y_{k-2+1,k} + Y_{k-3+1,k} + \dots + Y_{k-k+1,k}$$

$$= Y_{k-1,k} + Y_{k-2,k} + \dots + Y_{1,k}$$

$$= \sum_{i=1}^{k-1} Y_{i,k} = Y_{<k}$$

$n = k$

From HW:

$$\Pr\left[Y_{<k} \geq 4 \ln k\right] \leq \frac{1}{k^2}$$

Similarly, one can show that

$$Y_{>k} = \sum_{i=k+1}^{n} Y_{ik} = Z^{(n-k+1)}$$

$$\Pr\left[Y_{>k} \geq \ln(n-k+1)\right] \leq \frac{1}{(n-k+1)^2}$$

For node x_k:

$$\Pr\left[\text{height}(x_k) \geq r\right]$$

$$= \Pr\left[Y_{<k} + Y_{>k} \geq r\right]$$

$$= \Pr\left[Z^{(k)} + Z^{(n-k+1)} \geq r\right]$$

if $a+b \geq x$ then $\left(a \geq \frac{x}{2} \text{ or } b \geq \frac{x}{2}\right)$

$$\leq \Pr\left[Z^{(k)} \geq \frac{r}{2} \vee Z^{(n-k+1)} \geq \frac{r}{2}\right]$$

by union bound:

$$\leq \Pr\left[Z^{(k)} \geq \frac{r}{2}\right] + \Pr\left[Z^{(n-k+1)} \geq \frac{r}{2}\right]$$

$r = 8 \cdot \ln n$

$$\Pr\left[\text{height}(x_k) \geq 8 \ln n\right] \leq$$

$$\leq \Pr\left[Z^{(k)} \geq 4 \ln n\right] + \Pr\left[Z^{(n-k+1)} \geq 4 \ln n\right]$$

$$\leq \Pr\left[\underline{Z^{(n)} \geq 4 \ln n}\right] + \Pr\left[\underline{Z^{(n)} \geq 4 \ln n}\right]$$

$$\boxed{\begin{gathered} Z^{k} = Z_2 + Z_3 + \dots + Z_k \qquad k \leq n \\ Z^{n} = Z_2 + Z_3 + \dots + Z_n \\ \Rightarrow \underline{\Pr\left[Z^{n} \geq 4 \ln n\right]} \geq \\ \geq \underline{\Pr\left[Z^{(k)} \geq 4 \ln n\right]} \end{gathered}}$$

$$= 2 \cdot \Pr\left[Z^{n} \geq 4 \ln n\right] \leq 2 \cdot \frac{1}{n^2}$$

$$\boxed{\Pr\left[\text{height}(x_k) \geq 8 \ln n\right] \leq \underline{\frac{2}{n^2}}}$$

$$\underline{\Pr\left[\text{height of a treap} \geq 8 \ln n\right]}$$

$$= \Pr\left[\begin{aligned} &\text{height}(x_1) \geq 8 \ln n \ \vee \\ &\text{height}(x_2) \geq 8 \ln n \ \vee \\ &\quad \vdots \\ \vee\ &\text{height}(x_n) \geq 8 \ln n \end{aligned} \right]$$

by union bound

$$\leq \sum_{k=1}^{n} \Pr[\text{height}(x_k) \geq 8 \ln n]$$

$$= n \cdot \frac{2}{n^2} = \frac{2}{n}$$

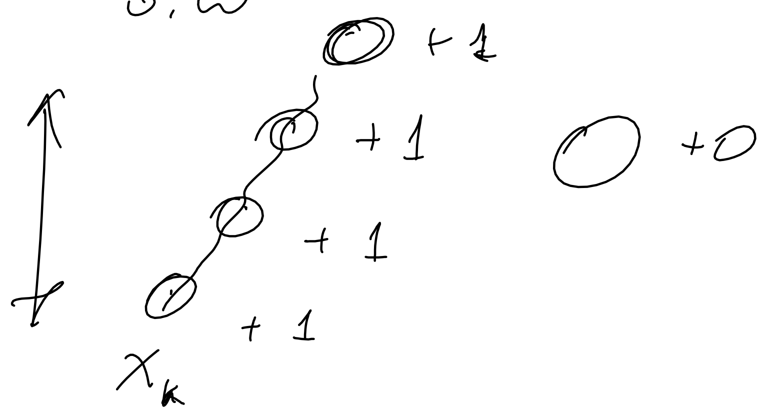
$\frac{1}{n^c}$

<u>Homework</u>: $\Pr[\text{height of a treap} \geq c \cdot \ln n] \leq \frac{2}{n^{c \ln c - c}}$

$$\boxed{\Pr[\text{Height of a treap} \geq 8 \ln n] \leq \frac{2}{n}}$$

$$\text{Height}(x_k) = \overbrace{\underset{=1}{Y_{1,k}} + \underset{1}{Y_{2,k}} + \dots \underset{1}{Y_{k-1,k}}}^{Y' = Z^{(k)}} + \underbrace{Y_{k+1,k} + \dots Y_{n,k}}_{Y'' = Z^{(n-k+1)}}$$

$$Y_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is proper ancestor of } x_j \\ 0 & \text{o.w} \end{cases}$$



$$\text{Height}(x_k) = Z^{(k)} + Z^{(n-k+1)}$$

$$\Pr\{\text{Height}(x_k) = O(\log n)\} \le$$

$Z_2 + \dots Z_k$

$\le Z_2 + \dots + Z_n$

$$\Pr[Z^{(k)} = O(\log n)] + \Pr[Z^{(n-k+1)} = O(\log n)]$$

$$\leq \Pr\left[Z^{(n)} = O(\log n)\right]$$

$$+ \Pr\left[Z^{(n)} = O(\log n)\right]$$

$$= 2 \cdot \underline{\Pr\left[Z^{(n)} = O(\log n)\right]}$$

Homework $\Downarrow$

$$\leq \boxed{2 \cdot \frac{1}{n^2}}$$

$$Z^{(k)} \quad \underbrace{Z_2 + Z_3 + Z_4 + \dots + Z_k} \leq$$

$$\underbrace{Z_2 + Z_3 + Z_4 + \quad + Z_k + \dots + Z_n} = Z^{(n)}$$

$$\Pr\left[Z^{(k)} \geq 8 \ln n\right] \leq \Pr\left[Z^{(n)} \geq 8 \ln n\right]$$