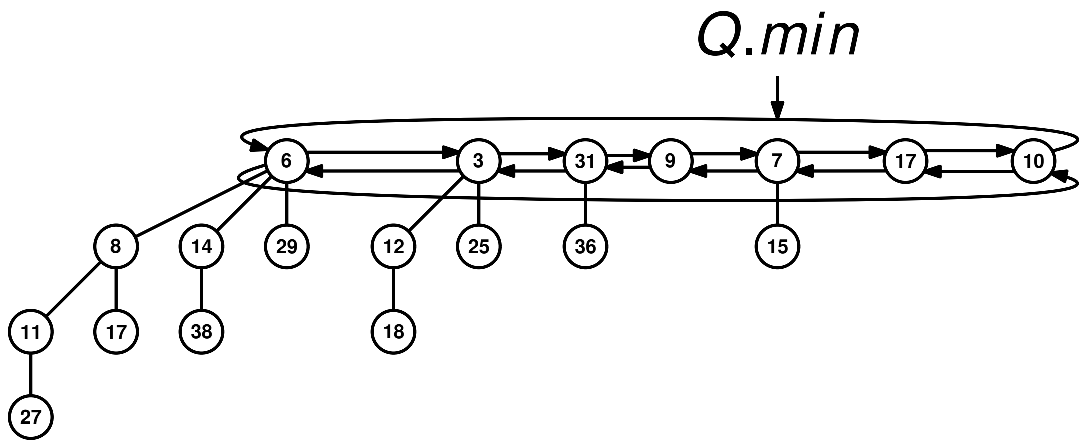
INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

AlgoPARC

# ICS 621: Analysis of Algorithms

Prof. Nodari Sitchinava




# Mergeable Priority Queues: Fibonacci Heaps

# Heaps

| | Binomial |
|---|---|
| ■ MAKE() | $O(1)$ |
| ■ INSERT($Q$, $x$) | $O(1)^*$ |
| ■ MINIMUM($Q$) | $O(1)$ |
| ■ EXTRACT-MIN($Q$) | $O(\log n)$ |
| ■ DECREASE-KEY($Q$, $x$, $k$) | $O(\log n)$ |
| ■ DELETE($Q$, $x$) | $O(\log n)$ |
| | |
| ■ UNION($Q_1$, $Q_2$) | $O(\log n)$ |

$^*$ Amortized cost

# Heaps

| | Binomial | Lazy Binomial |
|---|---|---|
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# Heaps

| | Binomial | Lazy Binomial | Fibonacci |
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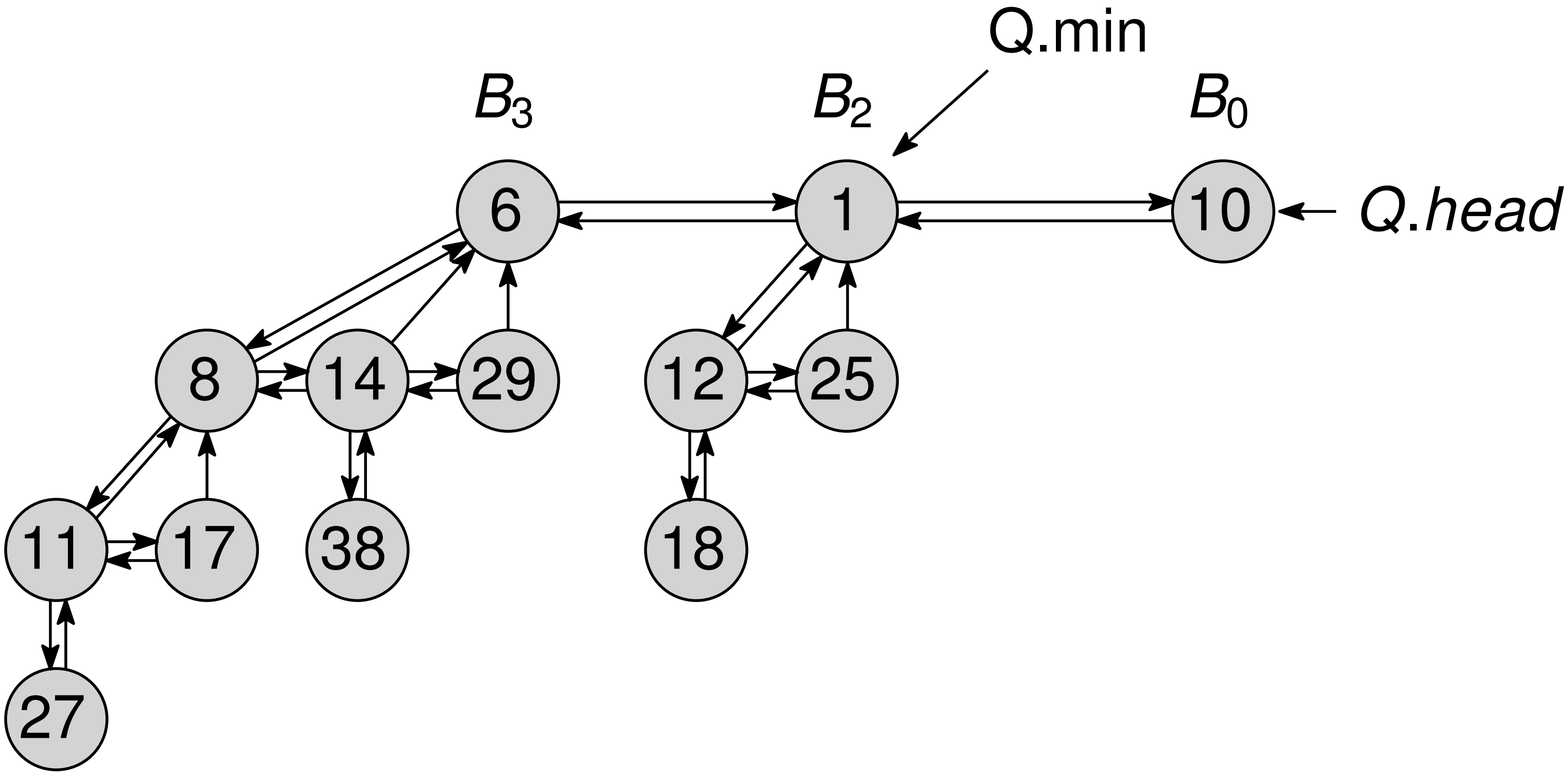
| MINIMUM($Q$) | $O(1)$ | $O(1)$ | $O(1)$ |
| EXTRACT-MIN($Q$) | $O(\log n)$ | $O(\log n)^*$ | $O(\log n)^*$ |
| DECREASE-KEY($Q$, $x$, $k$) | $O(\log n)$ | $O(\log n)$ | $O(1)^*$ |
| DELETE($Q$, $x$) | $O(\log n)$ | $O(\log n)^*$ | $O(\log n)^*$ |
| UNION($Q_1$, $Q_2$) | $O(\log n)$ | $O(1)$ | $O(1)$ |

* Amortized cost

# Reminder: Binomial Heaps

Collection of heap-ordered binomial trees:

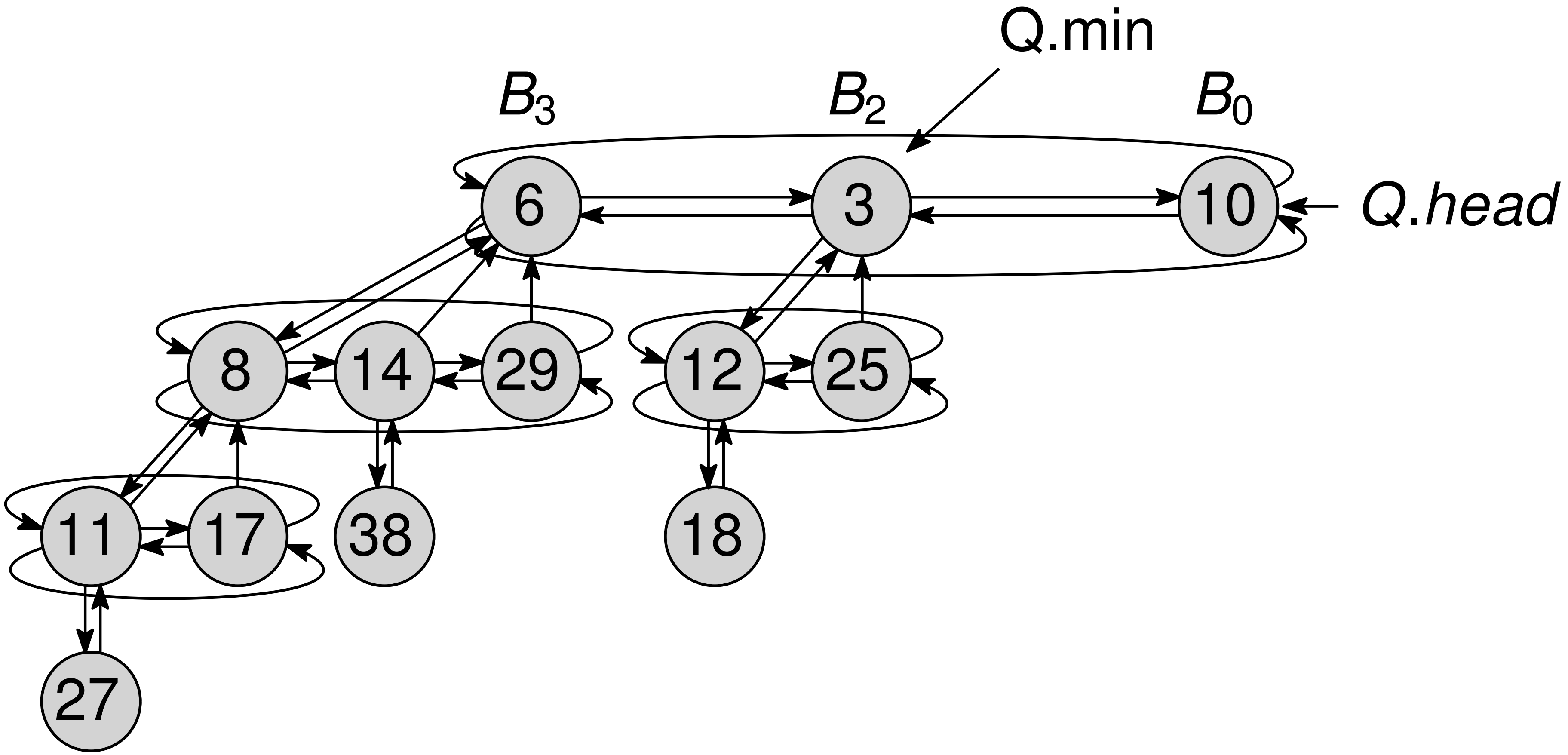
- Each tree is heap-ordered
- At most **one** tree $B_k$, for $k = 0, 1, 2, \ldots \lfloor \log n \rfloor$



# Reminder: Lazy Binomial Heaps

Collection of heap-ordered binomial trees:

- Each tree is heap-ordered
- *Arbitrary* number of trees in the root list
- Sibling lists are doubly-linked *circular* lists



# Reminder: Lazy Binomial Heaps

Collection of heap-ordered binomial trees:

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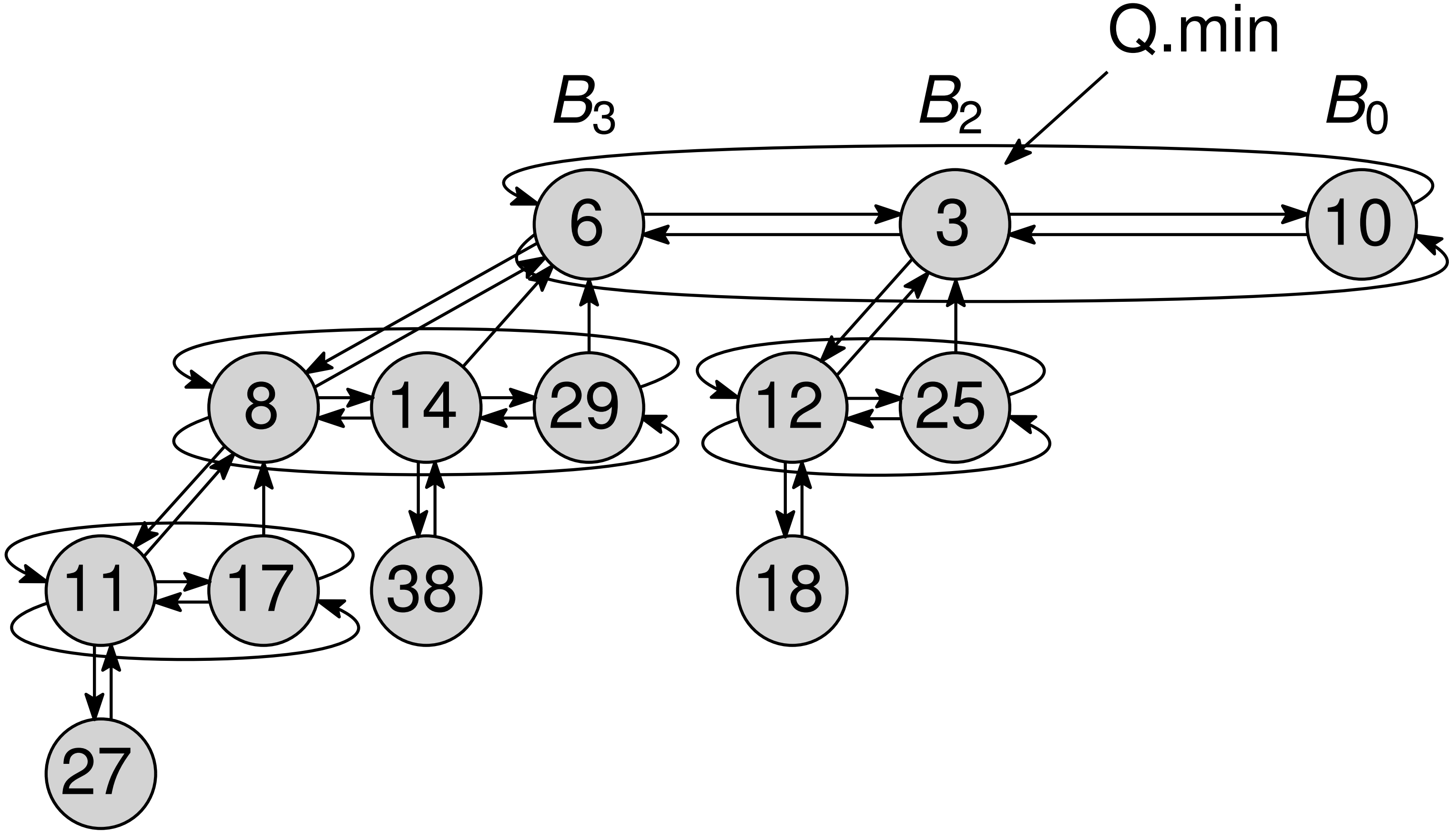
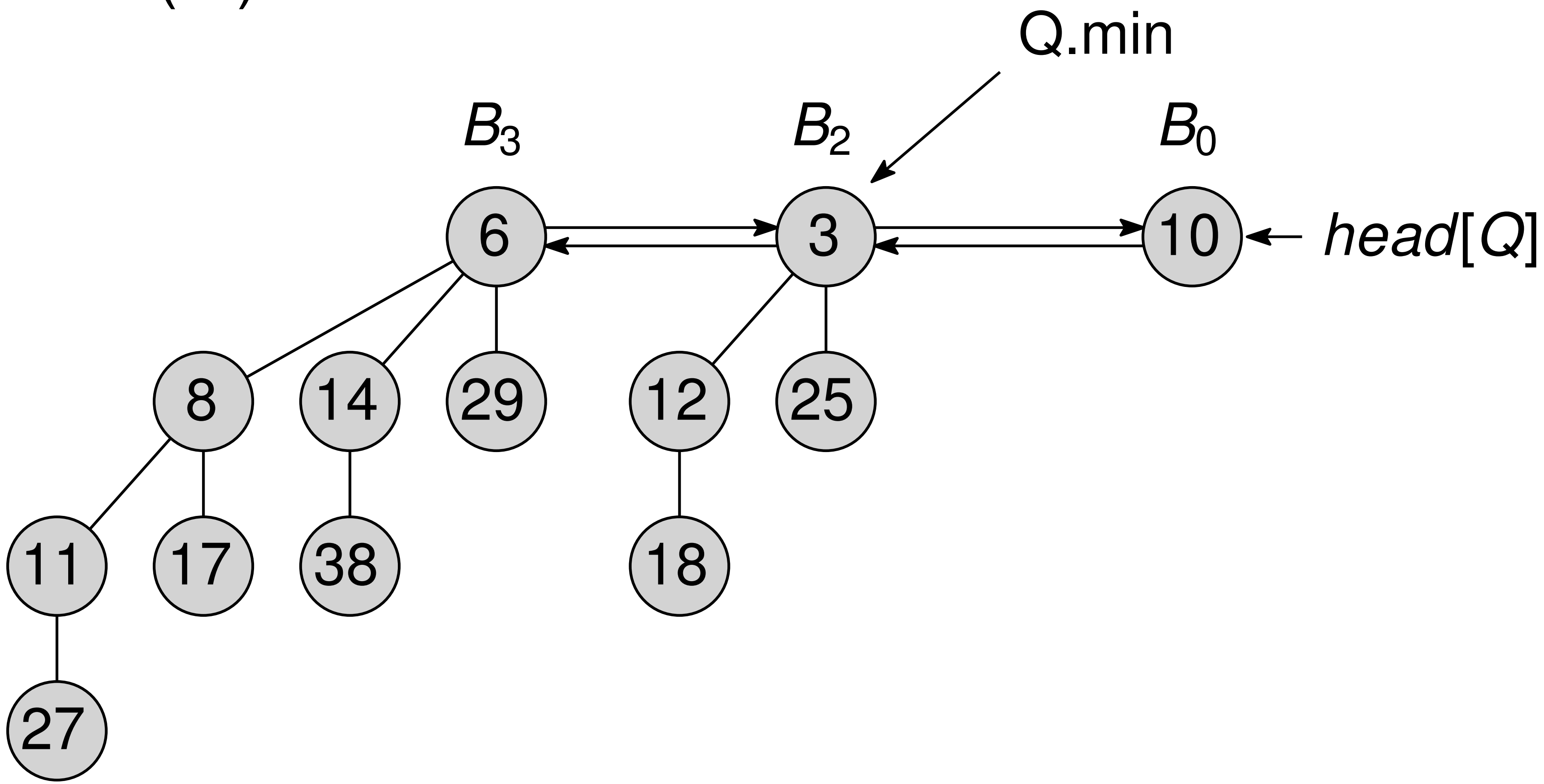
- *Arbitrary* number of trees in the root list
- Sibling lists are doubly-linked *circular* lists



# MINIMUM($Q$)



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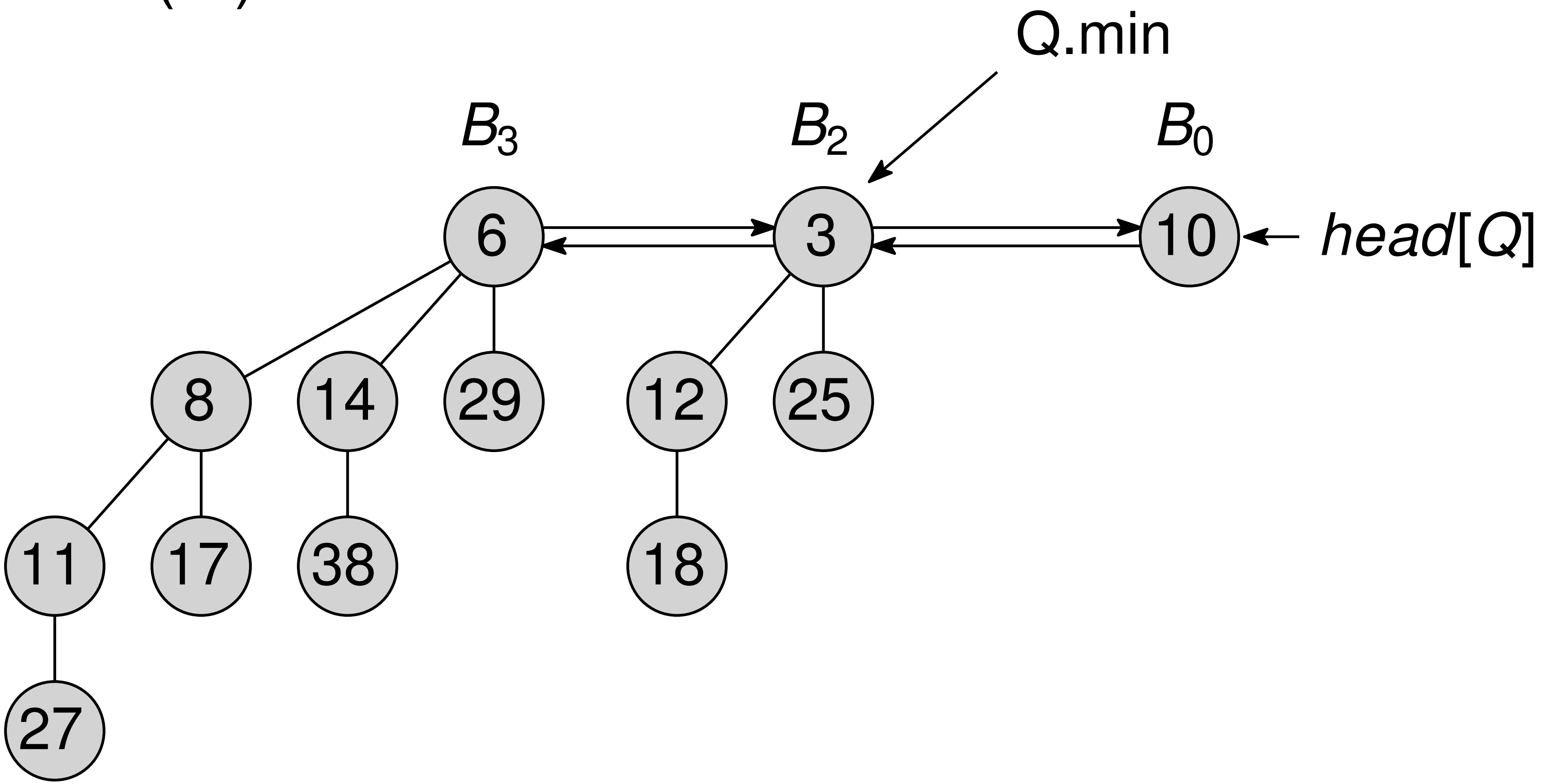
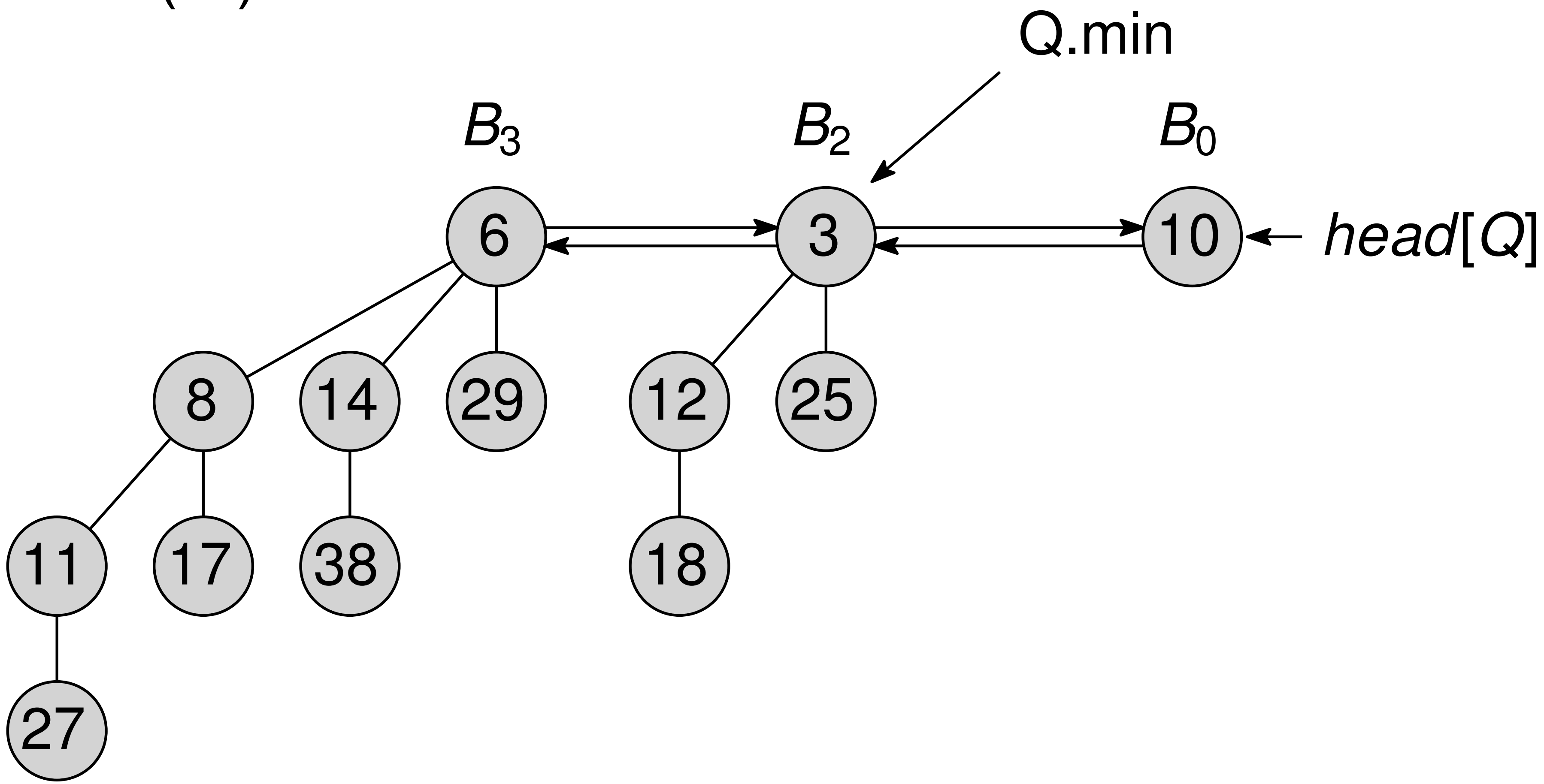



**function** MINIMUM($Q$)
**return** $Q.min$

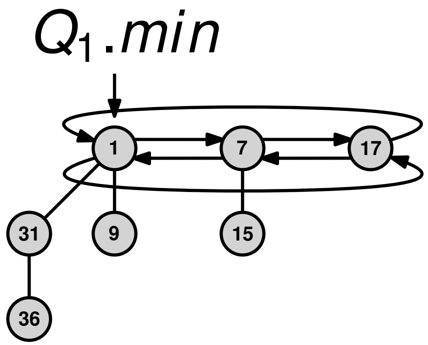
# MINIMUM($Q$)

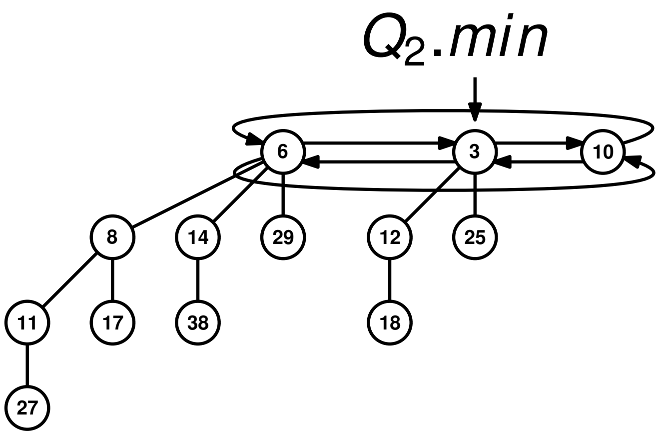



**function** MINIMUM($Q$)
**return** $Q.min$

$O(1)$ time worst-case

# Lazy UNION($Q_1$, $Q_2$)



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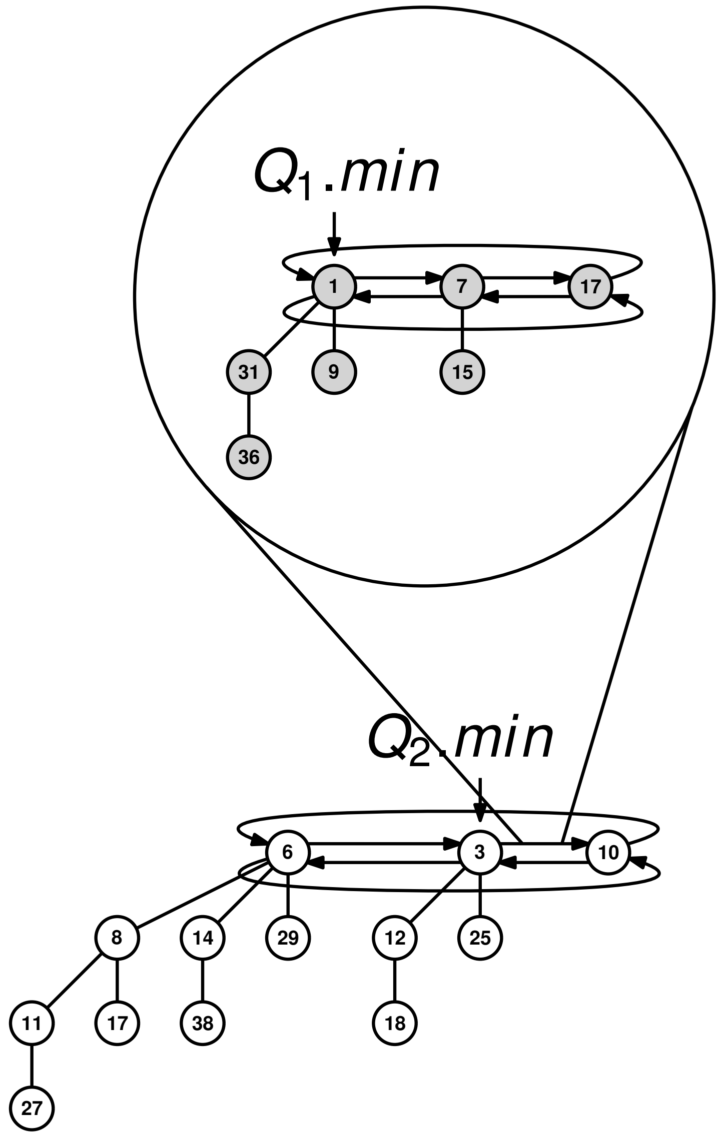

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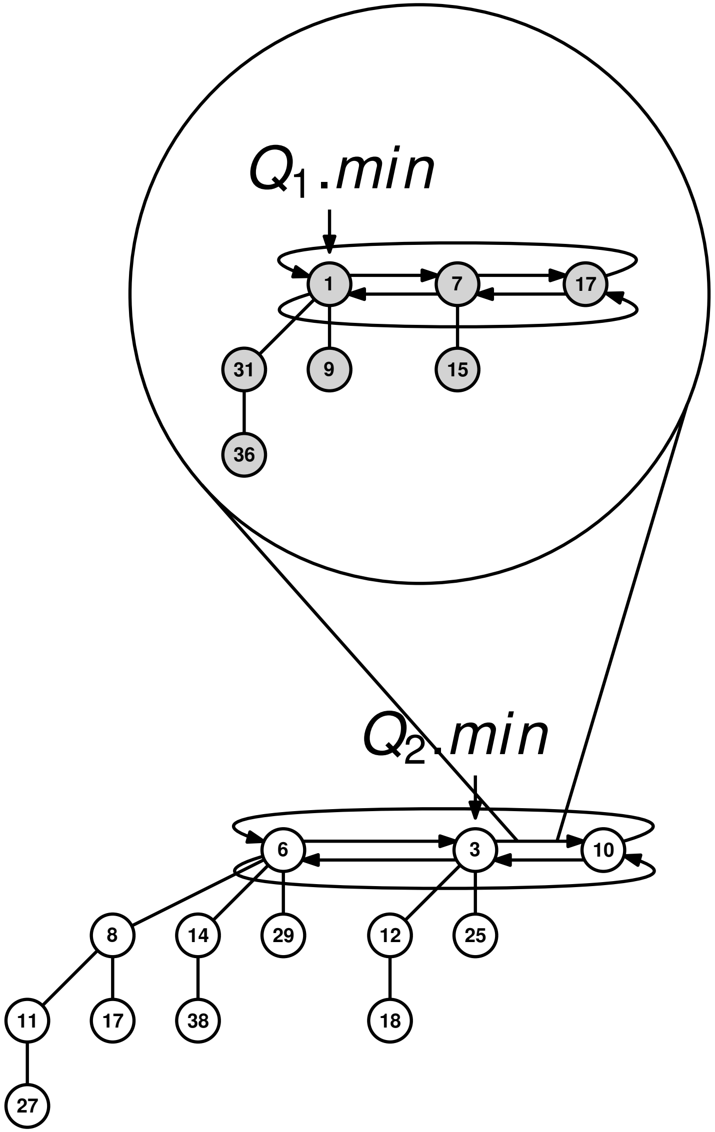


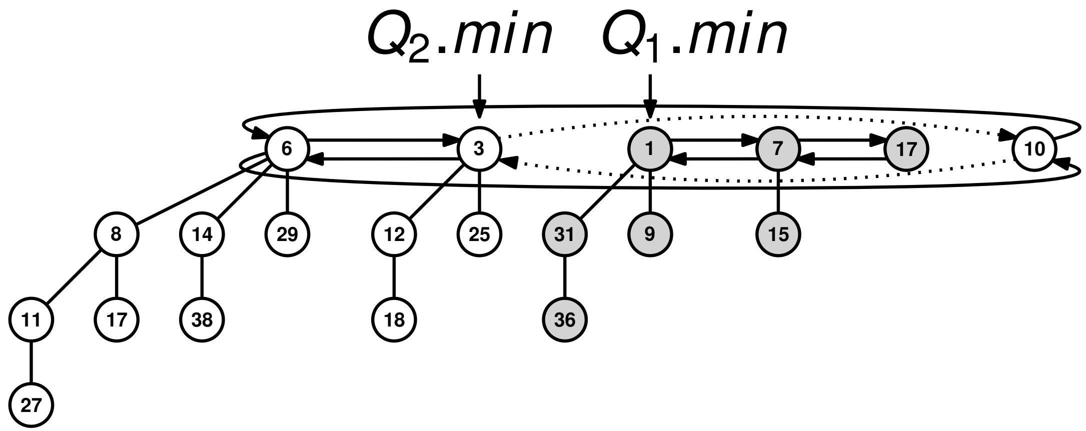

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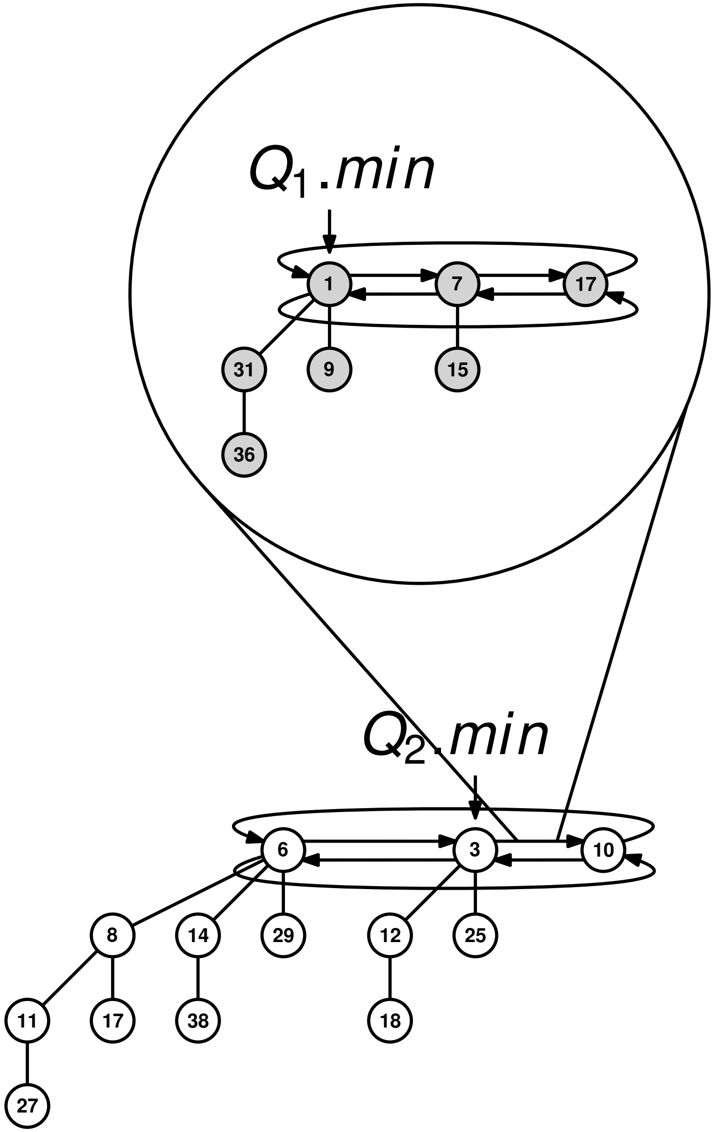


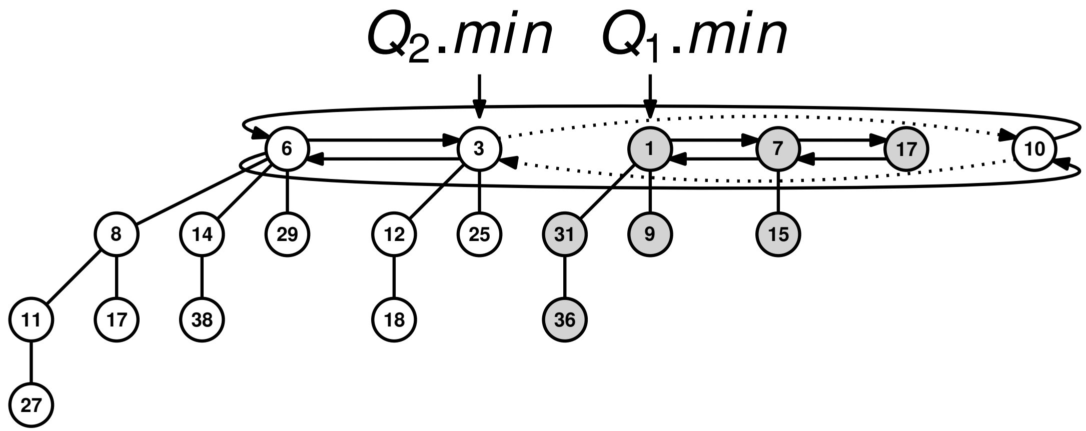


**function** UNION($Q_1$, $Q_2$)

$L_1 \leftarrow Q_1.min.left$

$R_2 \leftarrow Q_2.min.right$

$L_1.right \leftarrow R_2$

$R_2.left \leftarrow L_1$

$Q_2.min.right \leftarrow Q_1.min$

$Q_1.min.left \leftarrow Q_2.min$

**if** $Q_1.min.key < Q_2.min.key$ **then**

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**return** $Q_2$

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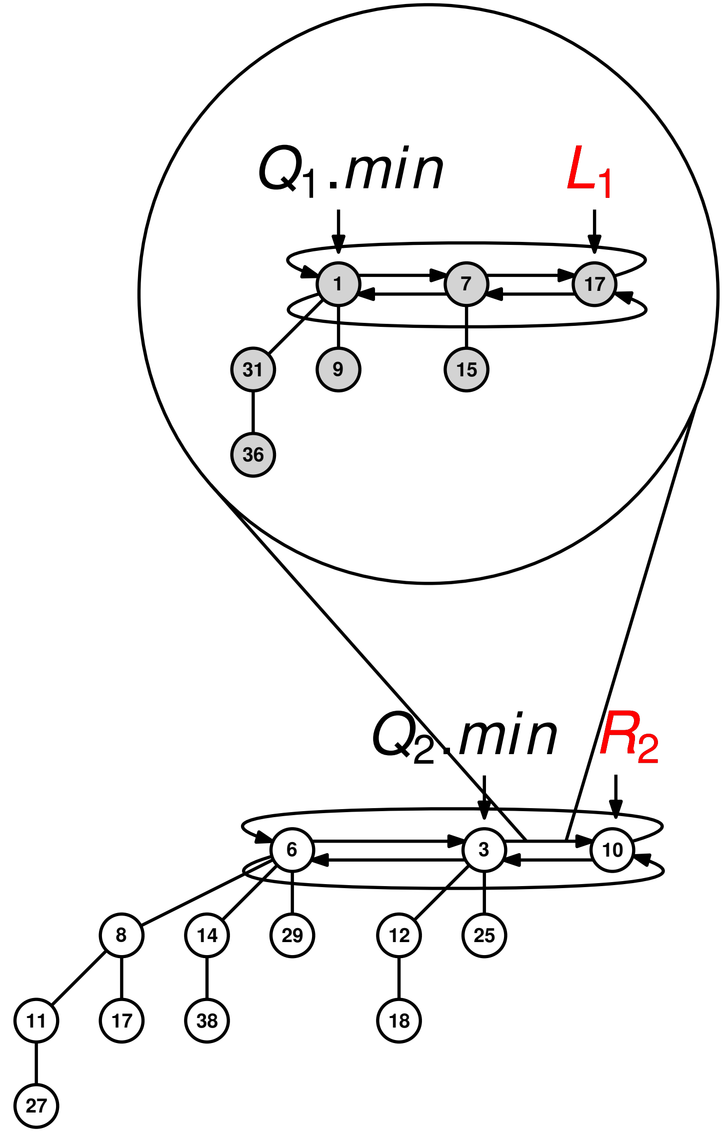


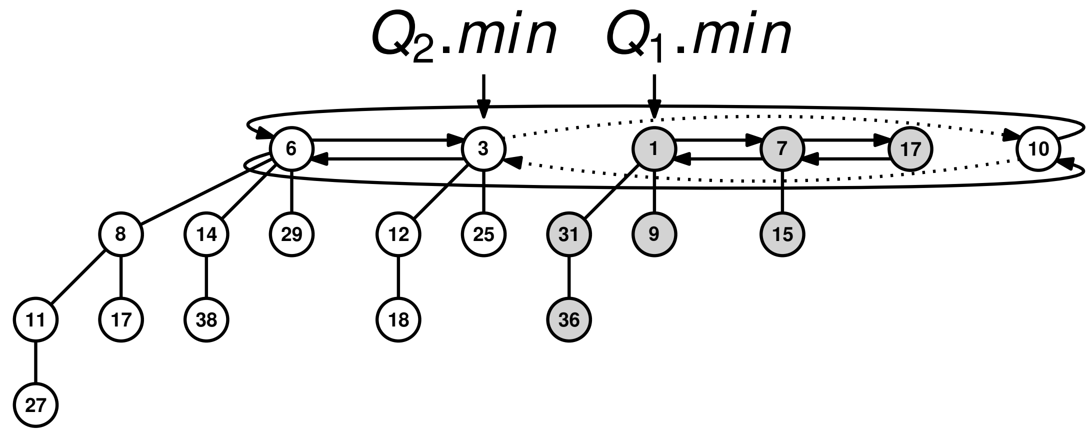


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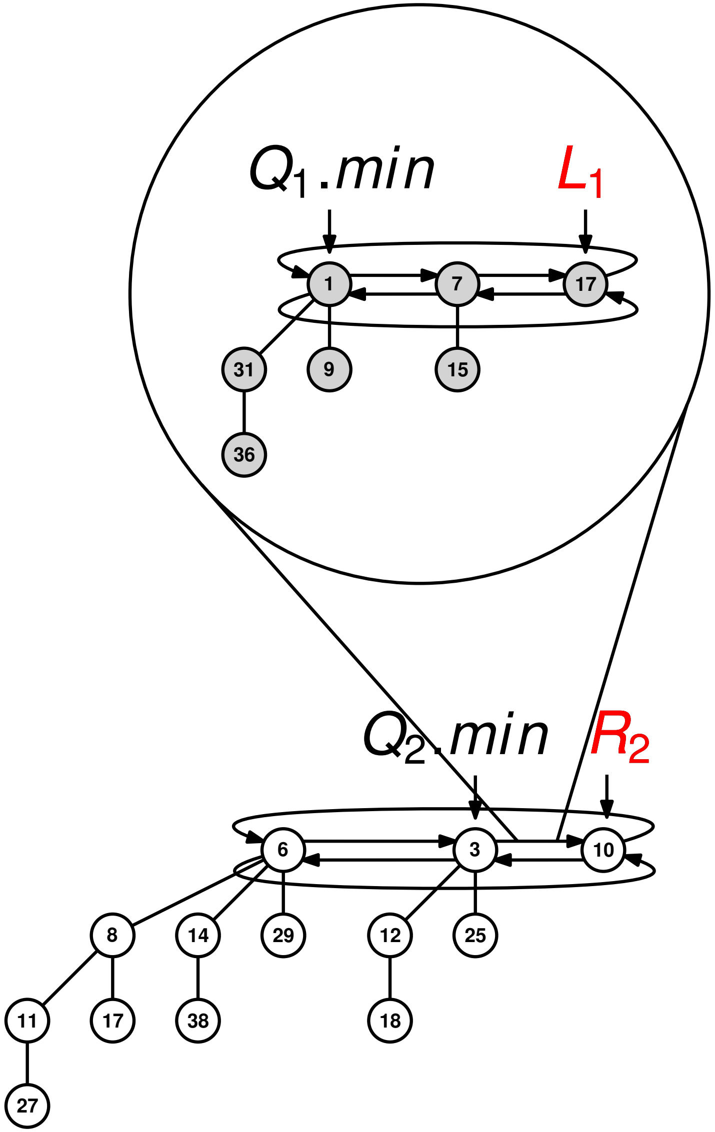


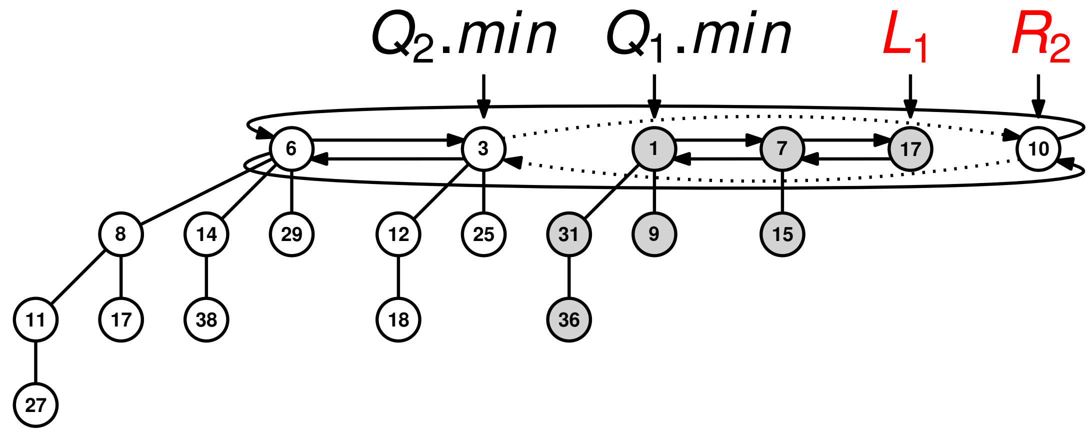


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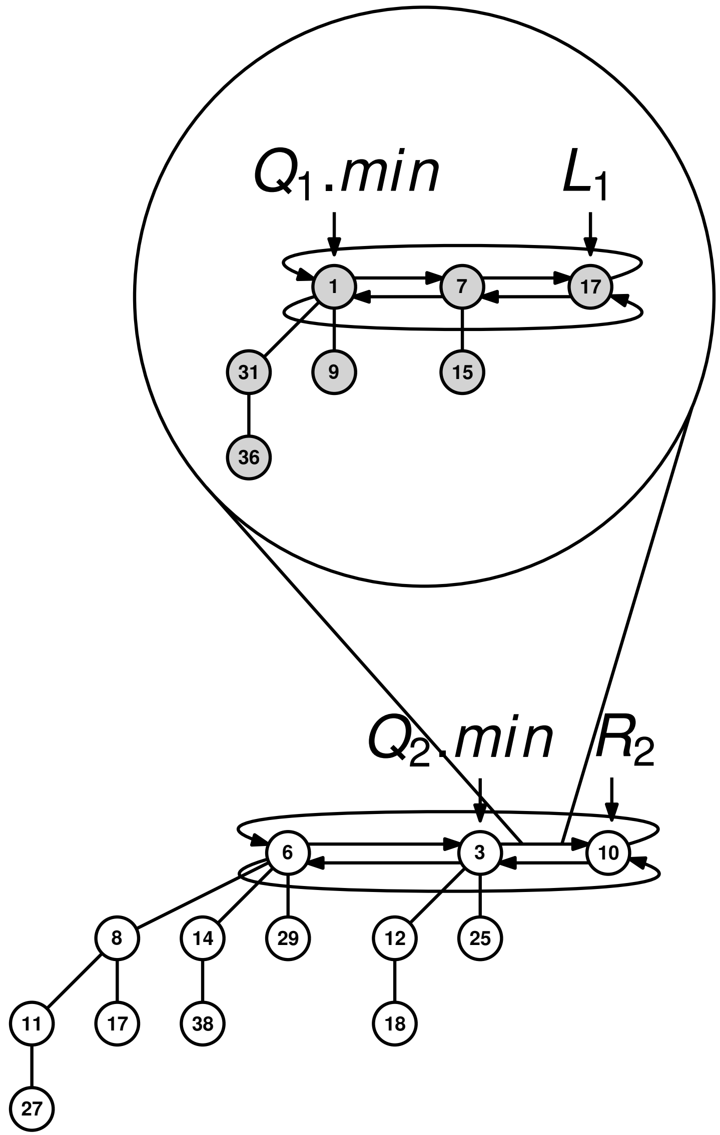


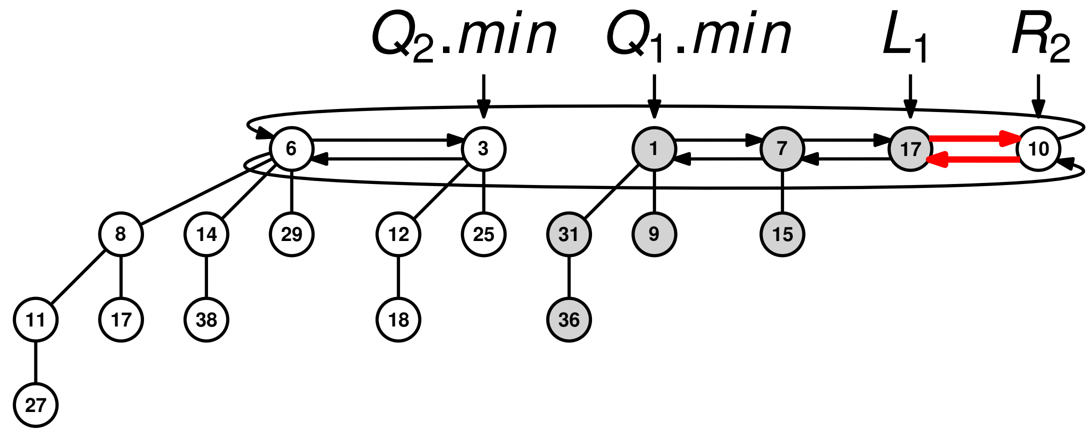


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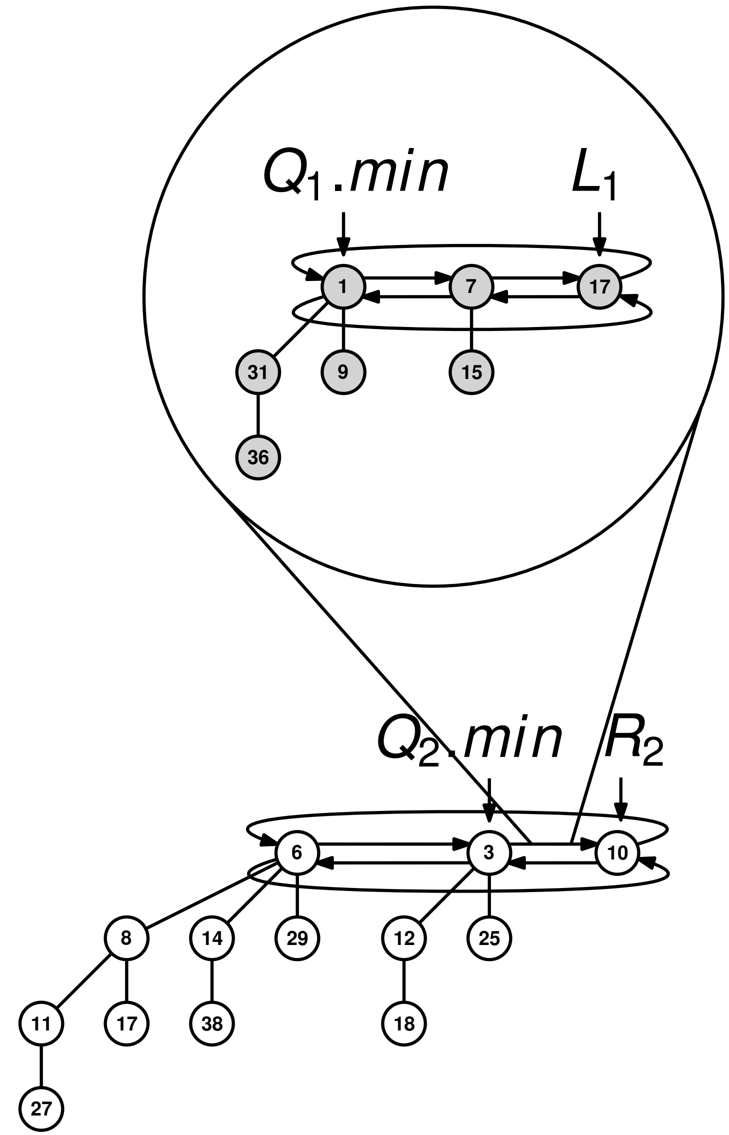


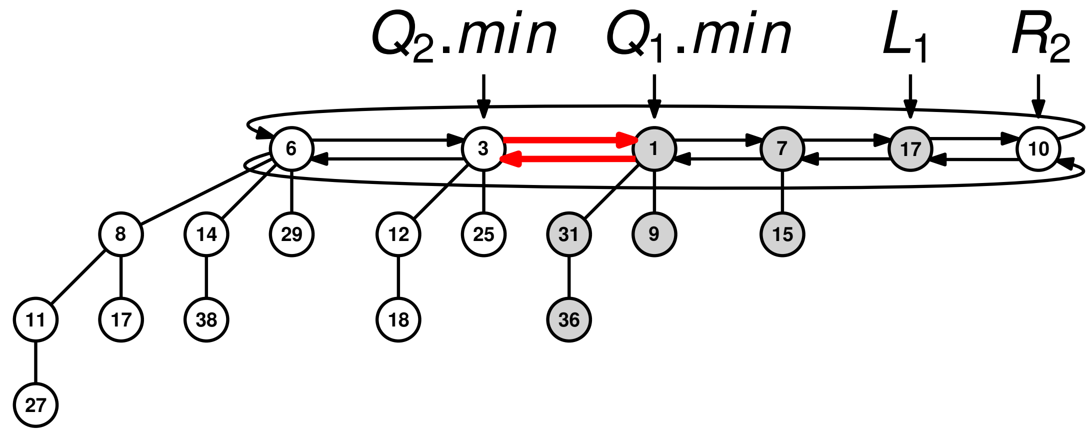


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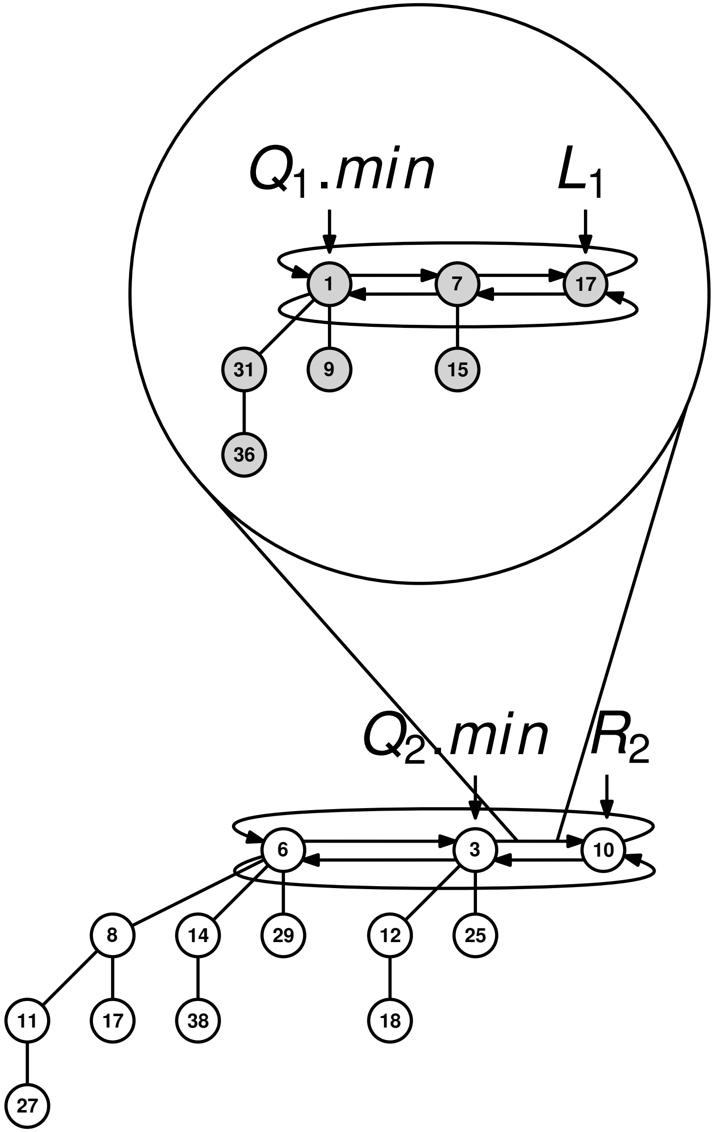


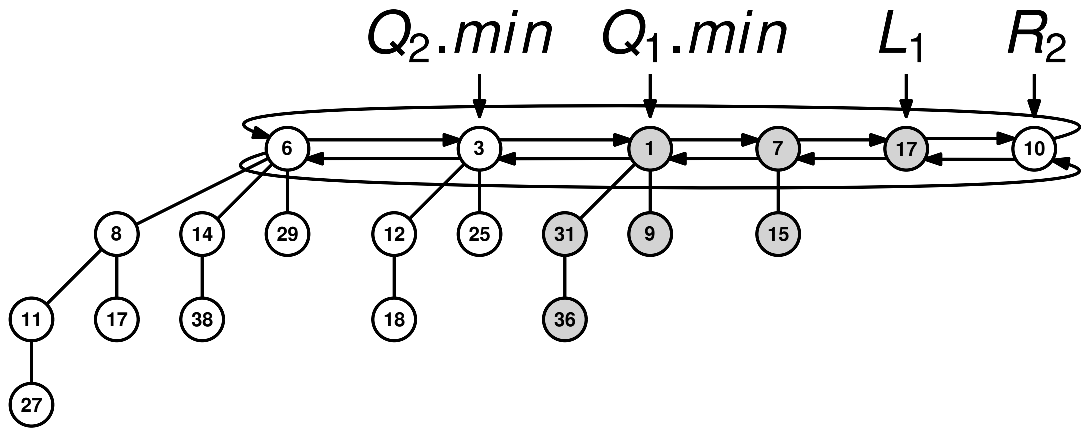


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- → **if** $Q_1.min.key < Q_2.min.key$ **then**
  - → $Q_2.min \leftarrow Q_1.min$
- **return** $Q_2$

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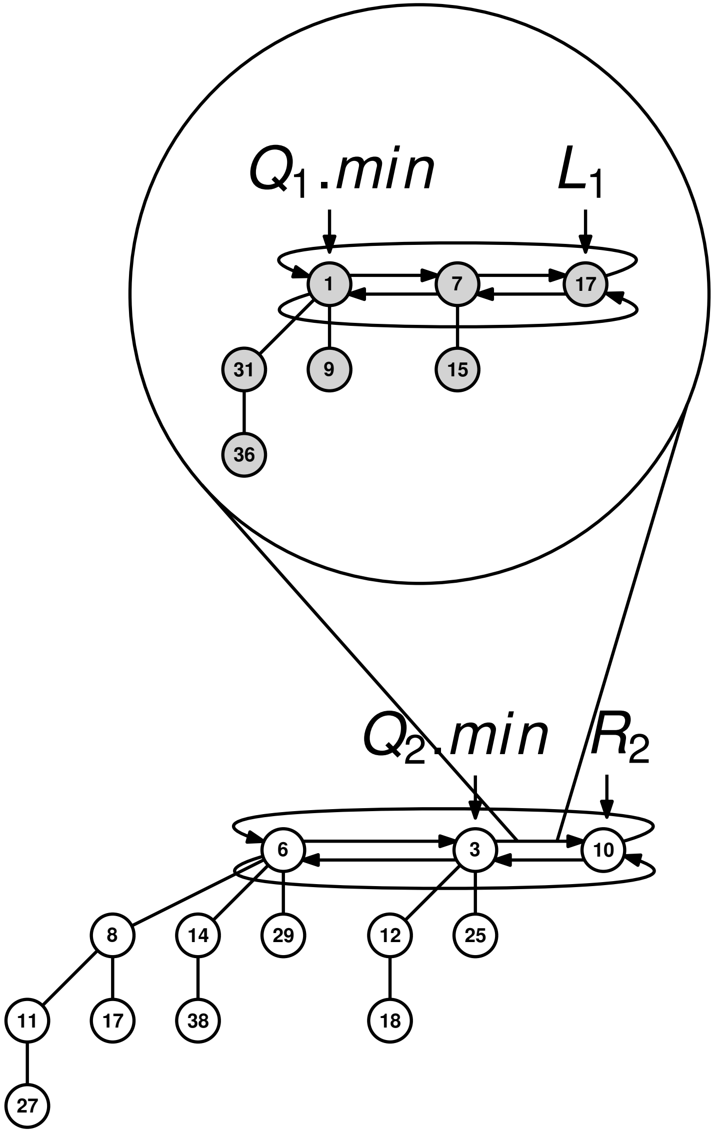
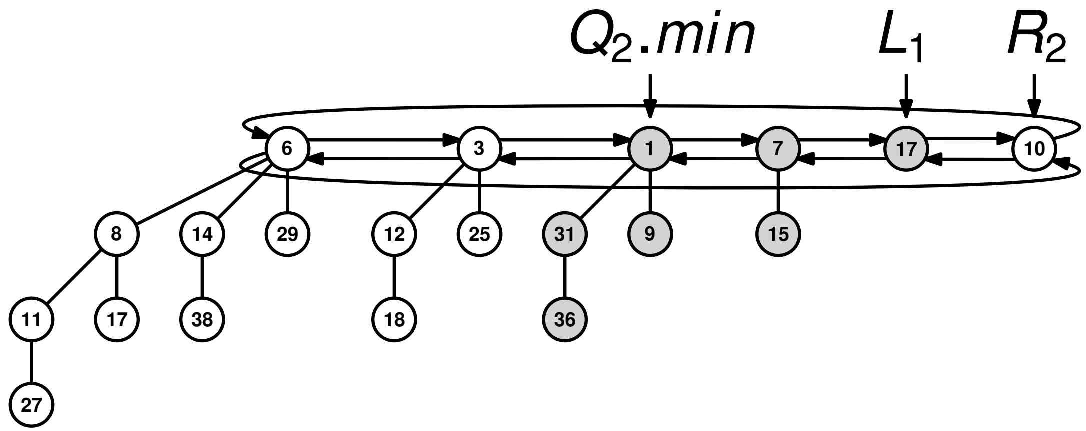
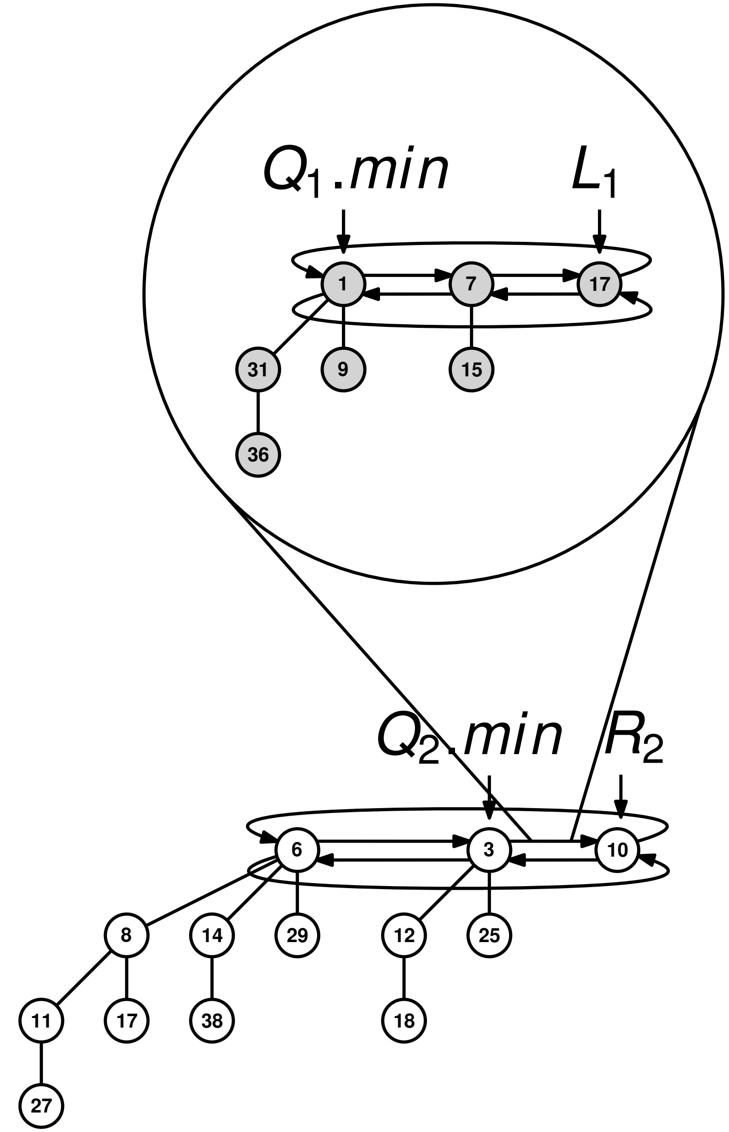
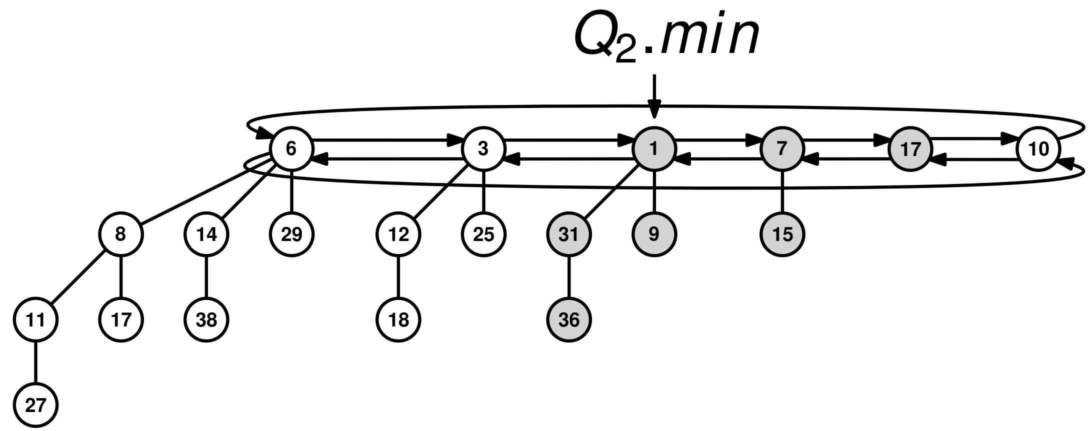



```
function Union(Q1, Q2)
    L1 ← Q1.min.left
    R2 ← Q2.min.right
    L1.right ← R2
    R2.left ← L1
    Q2.min.right ← Q1.min
    Q1.min.left ← Q2.min
→   if Q1.min.key < Q2.min.key then
→       Q2.min ← Q1.min
    return Q2
```

# Lazy UNION($Q_1$, $Q_2$)

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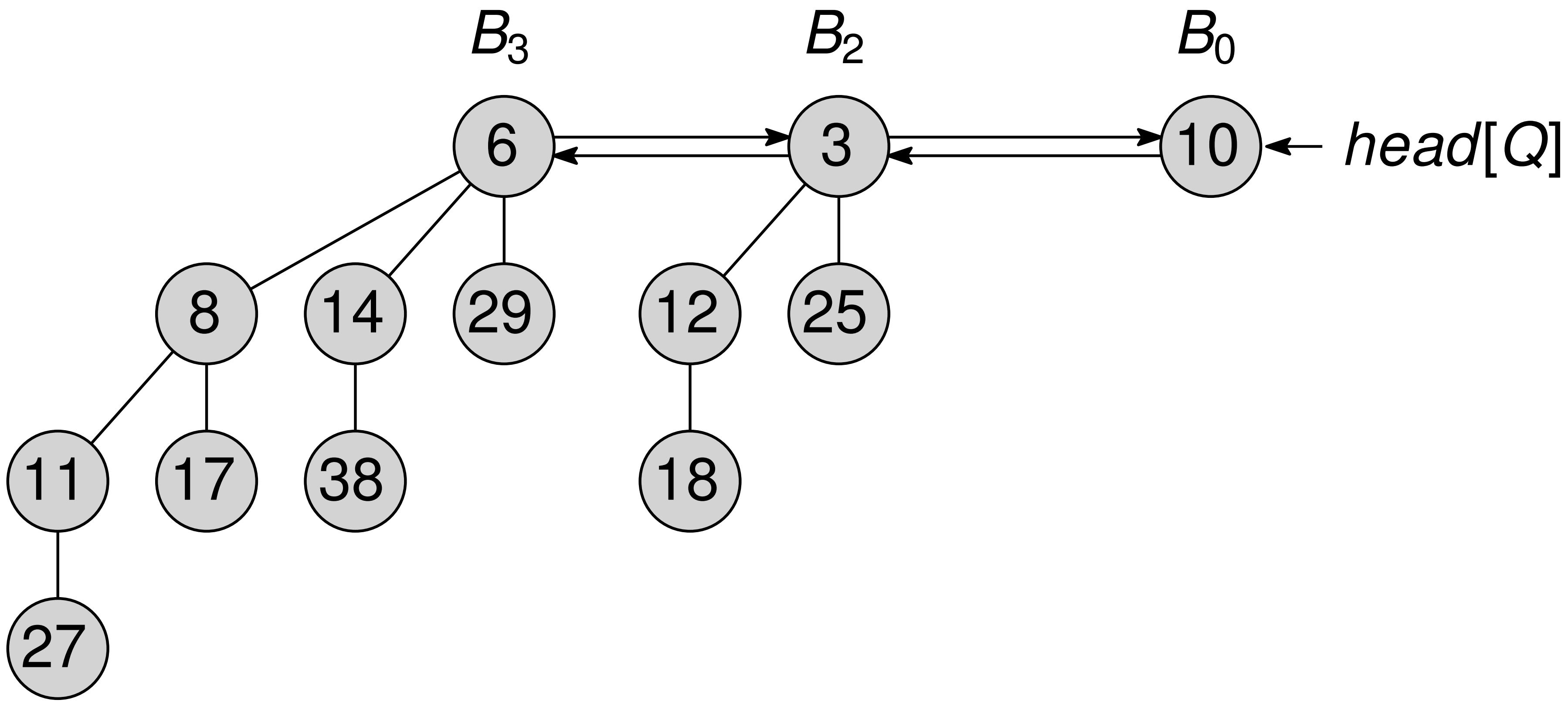
$Q_2.min \leftarrow Q_1.min$

**return** $Q_2$

$O(1)$ time worst-case

# DECREASE-KEY($Q$, $v$, $k$)



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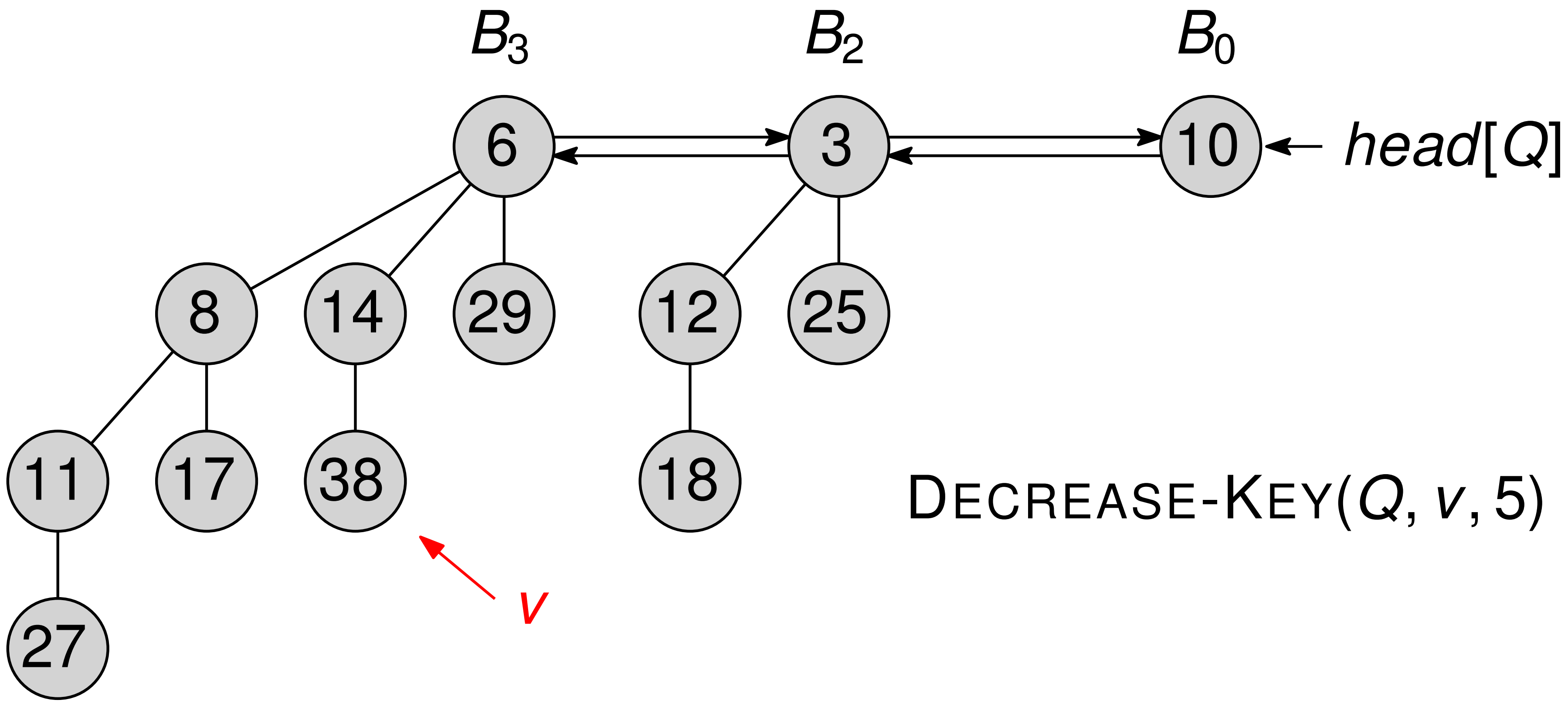
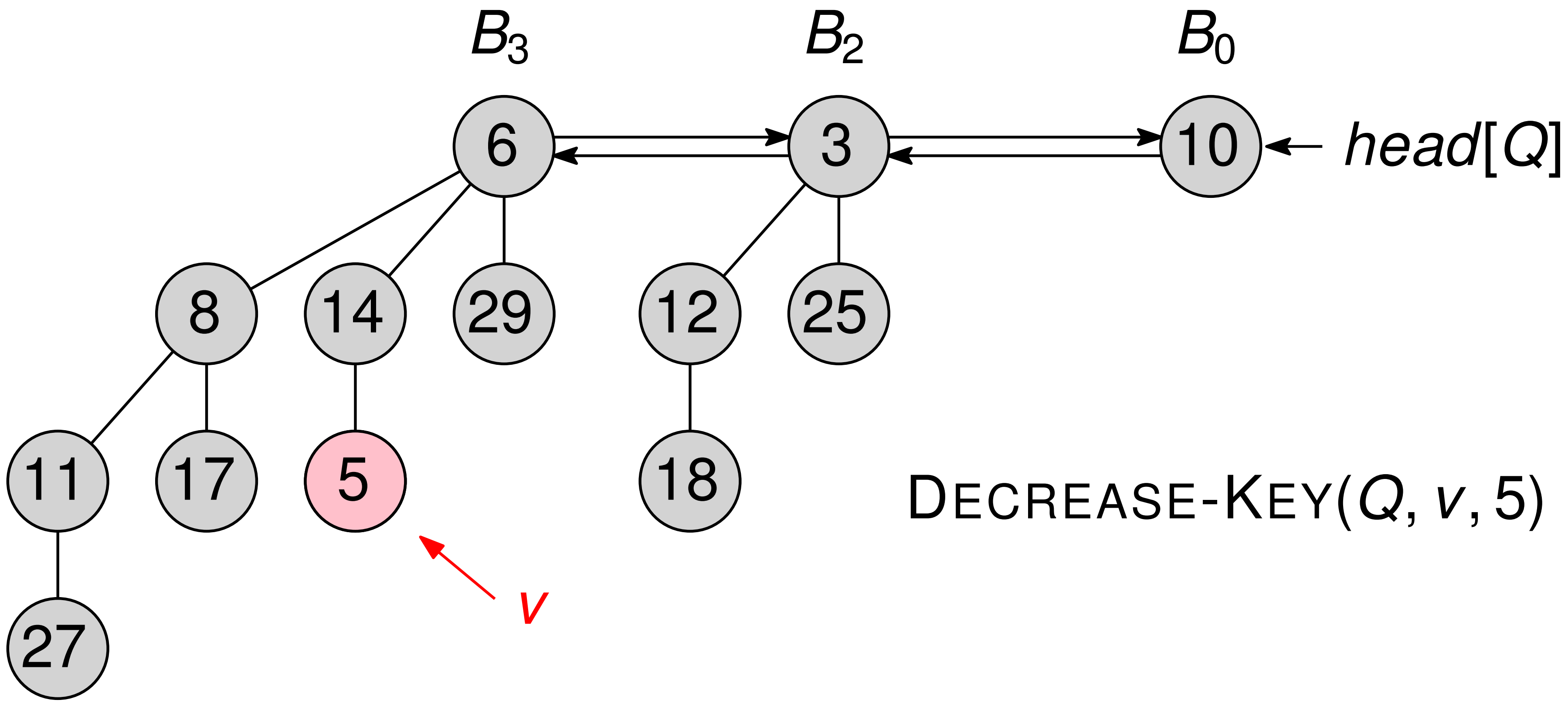



DECREASE-KEY($Q$, $v$, 5)

# DECREASE-KEY($Q$, $v$, $k$)

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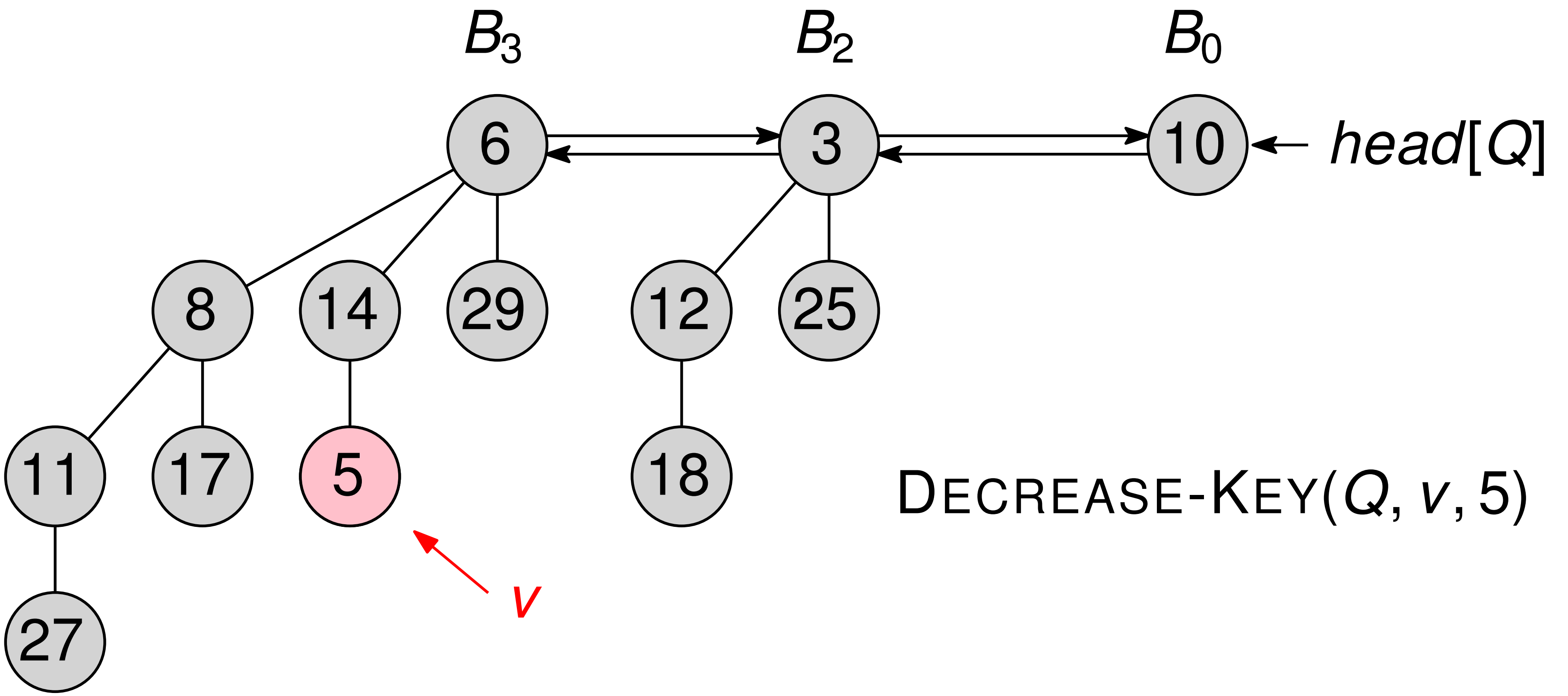
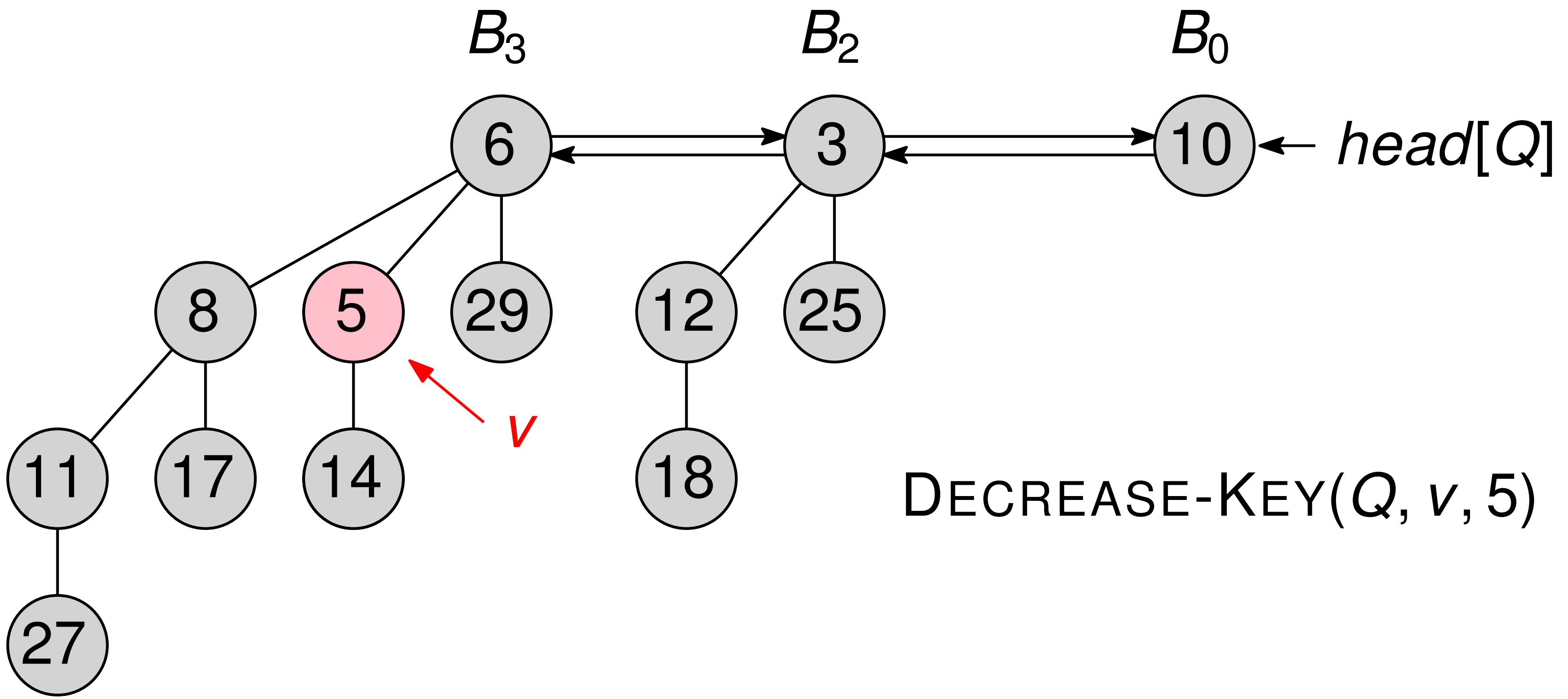



DECREASE-KEY($Q$, $v$, 5)

- Fix heap order

# DECREASE-KEY($Q$, $v$, $k$)

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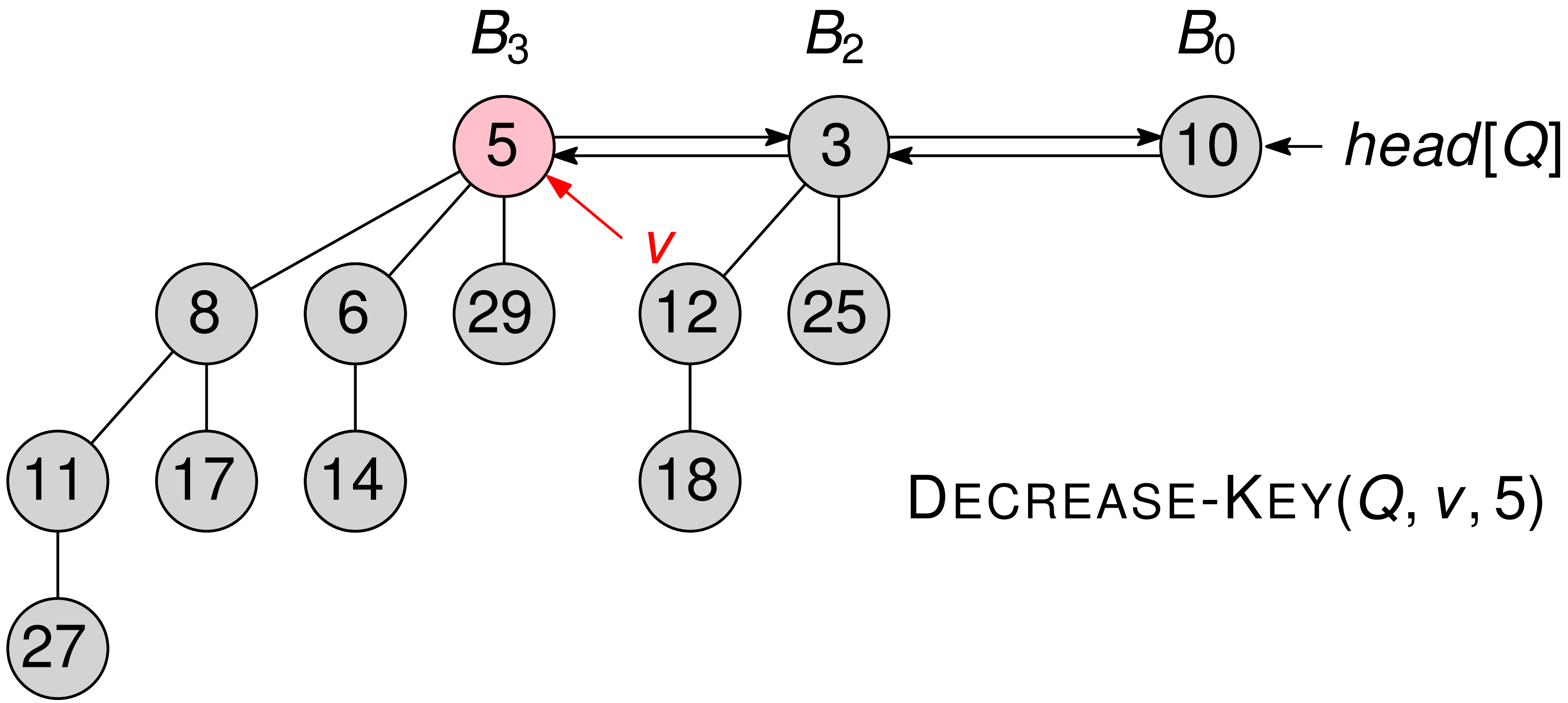
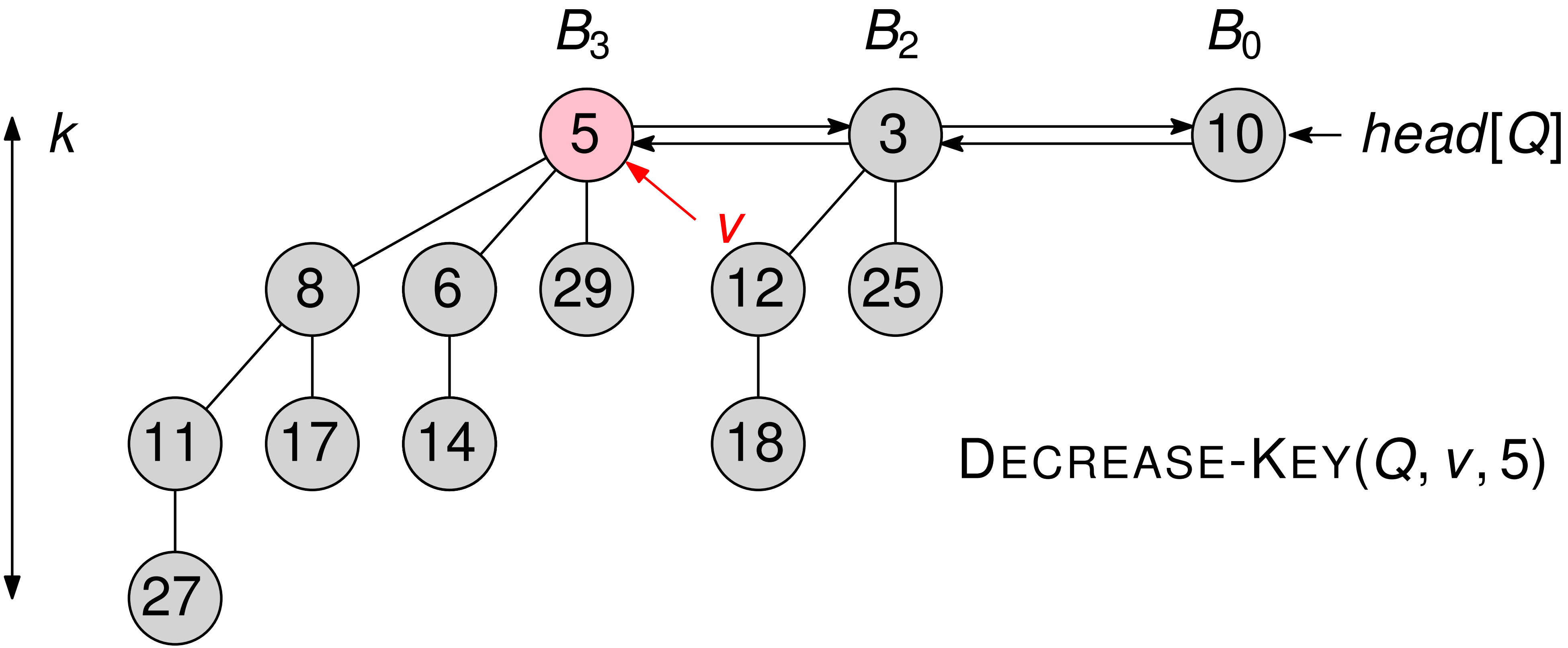



- Fix heap order

# DECREASE-KEY($Q$, $v$, $k$)

Depth of $B_k$ is $k \leq \log n$

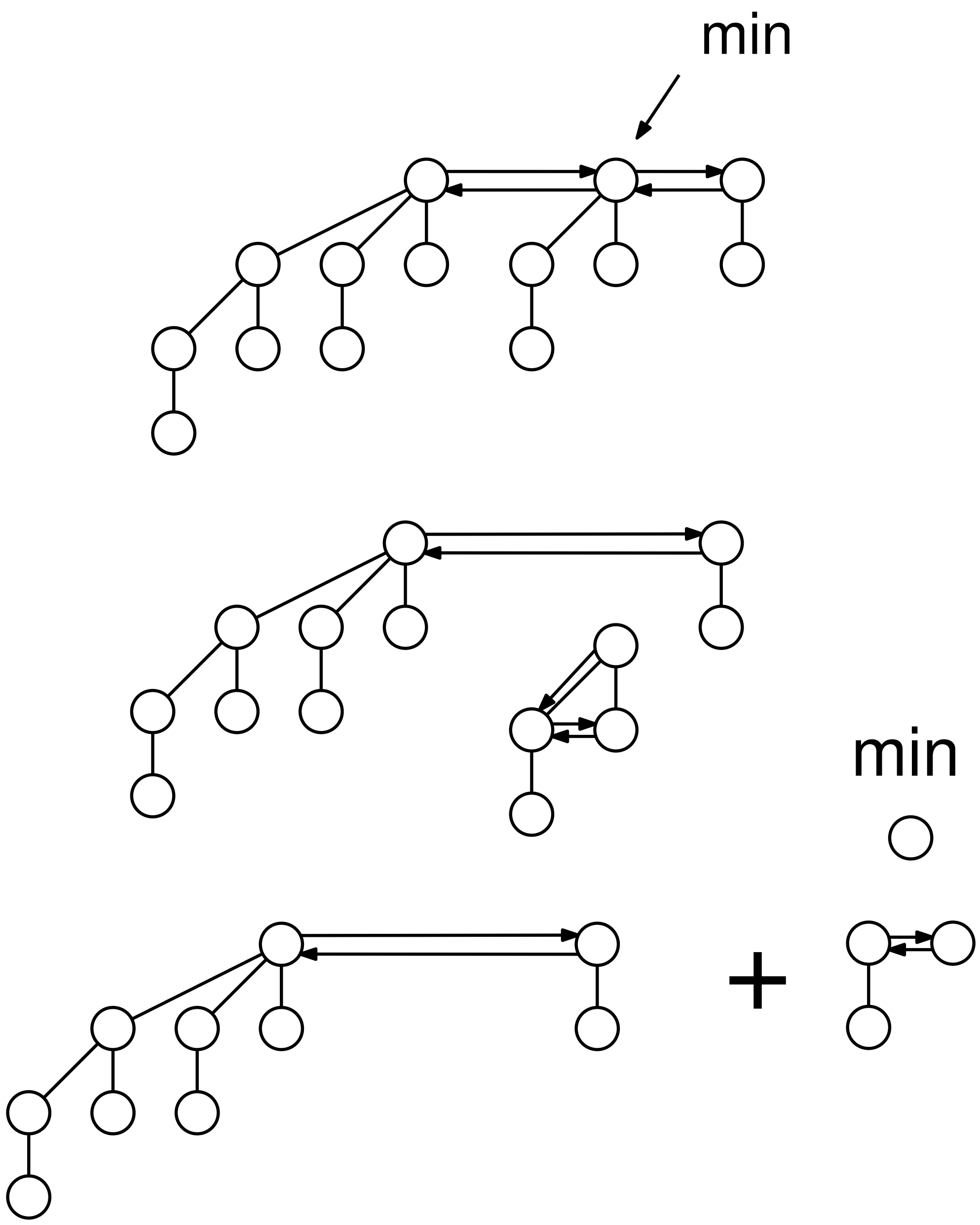



DECREASE-KEY($Q$, $v$, 5)

- Fix heap order

# Reminder: EXTRACT-MIN($Q$)

**function** EXTRACT-MIN($Q$)
    $x$ = MINIMUM($Q$)
    $Q'$ = MAKE()
    $Q'.head = x.leftchild$
    LINKEDLIST-EXTRACT(x)
    **for each** child $y$ of $x$ **do**
        $y.parent$ = NIL
    $Q$ = UNION($Q$, $Q'$)
    **return** $x$



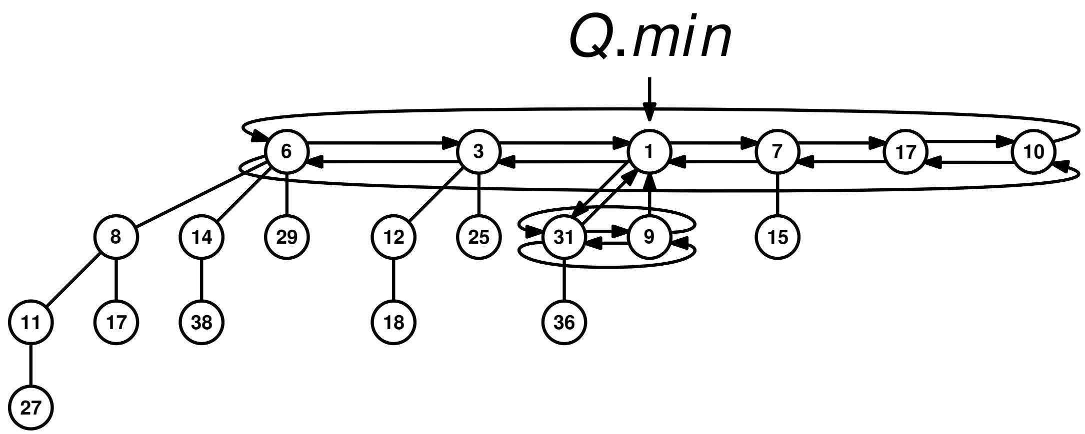

Analysis: $O(\log n)$

# Lazy EXTRACT-MIN($Q$)

```
function EXTRACT-MIN(Q)
    v = MINIMUM(Q)
    c = v.child
    L_v = v.left
    R_v = v.right
    if c ≠ NIL then
        L_c = c.left
        L_v.right = c
        c.left = L_v
        R_v.left = L_c
        L_c.right = R_v
    else
        L_v.right = R_v
        R_v.left = L_v
    Q.min = R_v    ▷ arbitrary head
    CONSOLIDATE(Q)
    return v
```

# Lazy EXTRACT-MIN($Q$)

**function** EXTRACT-MIN($Q$)
  $v$ = MINIMUM($Q$)
  $c = v.child$
  $L_v = v.left$
  $R_v = v.right$
  **if** $c \neq$ NIL **then**
    $L_c = c.left$
    $L_v.right = c$
    $c.left = L_v$
    $R_v.left = L_c$
    $L_c.right = R_v$
  **else**
    $L_v.right = R_v$
    $R_v.left = L_v$
  $Q.min = R_v$ ▷ arbitrary head
  CONSOLIDATE($Q$)
  **return** $v$



# Lazy EXTRACT-MIN($Q$)

**function** EXTRACT-MIN($Q$)

$v$ = MINIMUM($Q$)

→ $c = v.child$

→ $L_v = v.left$

→ $R_v = v.right$

**if** $c \neq$ NIL **then**

$L_c = c.left$

$L_v.right = c$

$c.left = L_v$

$R_v.left = L_c$

$L_c.right = R_v$

**else**

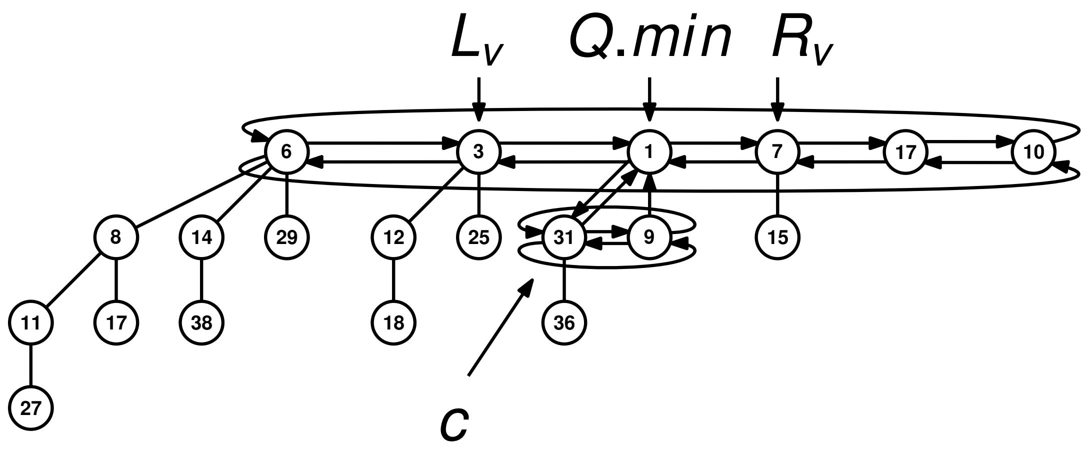
$L_v.right = R_v$

$R_v.left = L_v$

$Q.min = R_v$ ▷ arbitrary head

CONSOLIDATE($Q$)

**return** $v$



# Lazy Extract-Min($Q$)

**function** Extract-Min($Q$)

  $v$ = Minimum($Q$)

  $c = v.child$

  $L_v = v.left$

  $R_v = v.right$

  **if** $c \neq$ NIL **then**

    $L_c = c.left$

    $L_v.right = c$

    $c.left = L_v$

    $R_v.left = L_c$

    $L_c.right = R_v$

  **else**

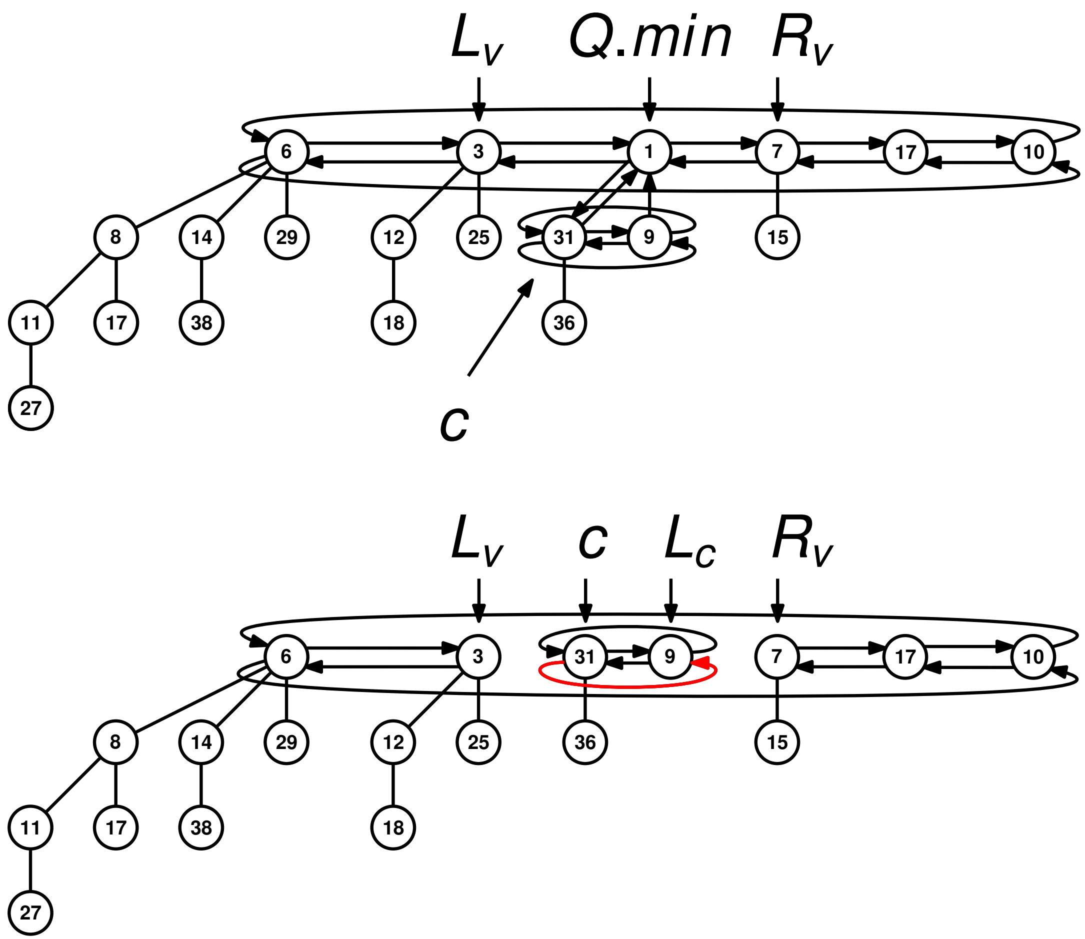
    $L_v.right = R_v$

    $R_v.left = L_v$

  $Q.min = R_v$  ▷ arbitrary head

  Consolidate($Q$)

  **return** $v$



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- **if** $c \neq$ NIL **then**
  - $L_c = c.left$
  - $L_v.right = c$
  - $c.left = L_v$
  - $R_v.left = L_c$
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- **else**
  - $L_v.right = R_v$
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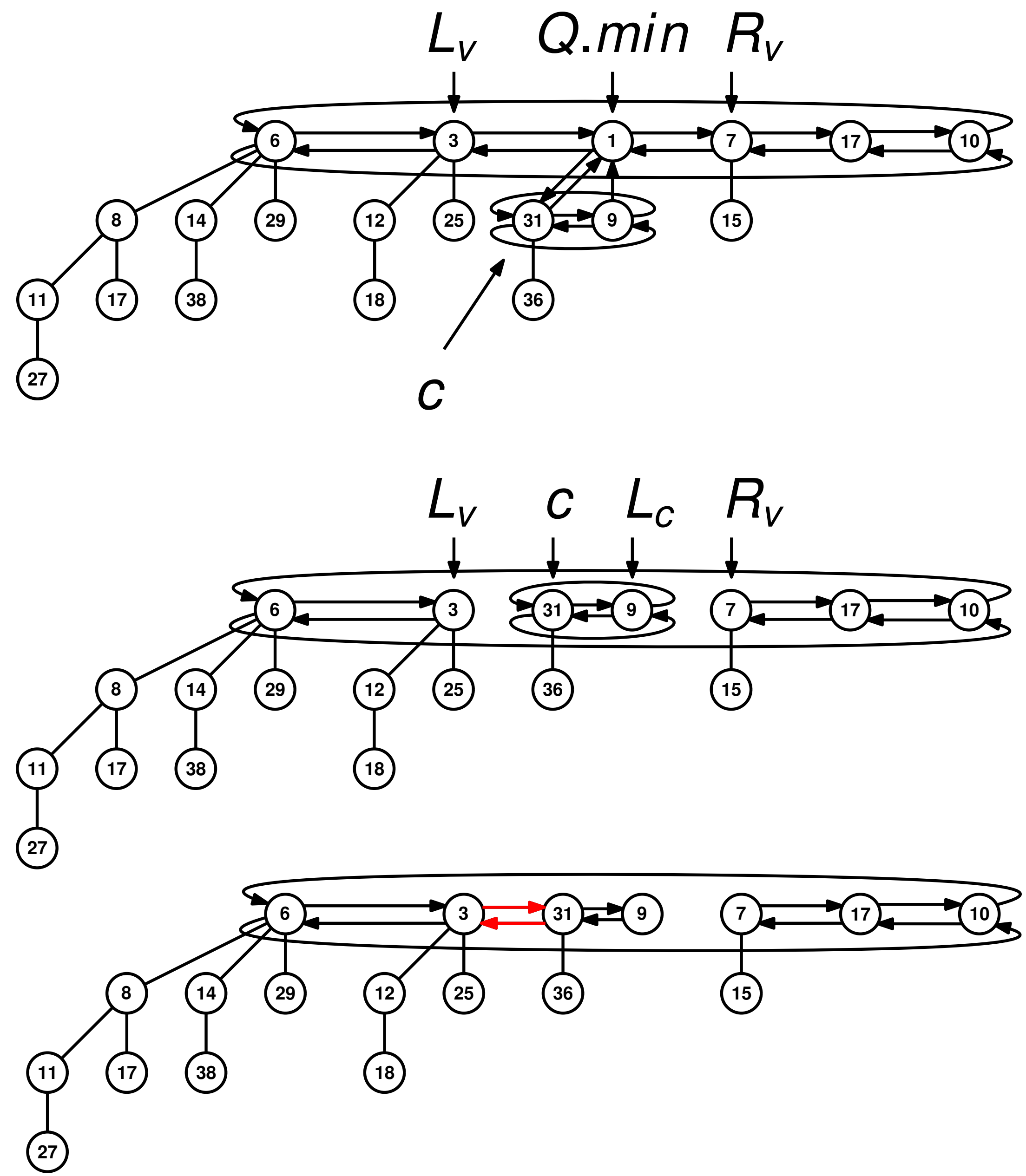
- $Q.min = R_v$ ▷ arbitrary head
- Consolidate($Q$)
- **return** $v$



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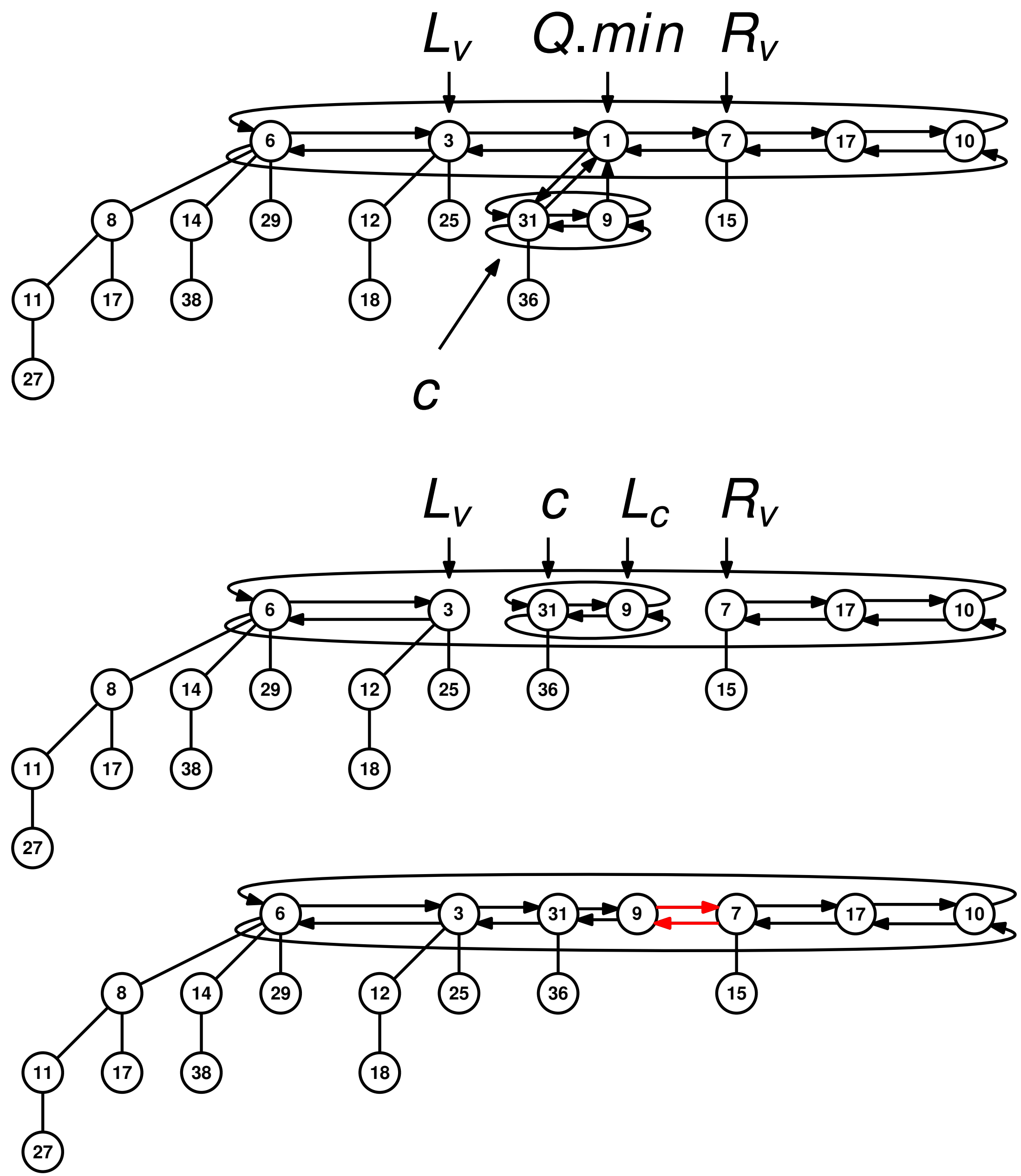
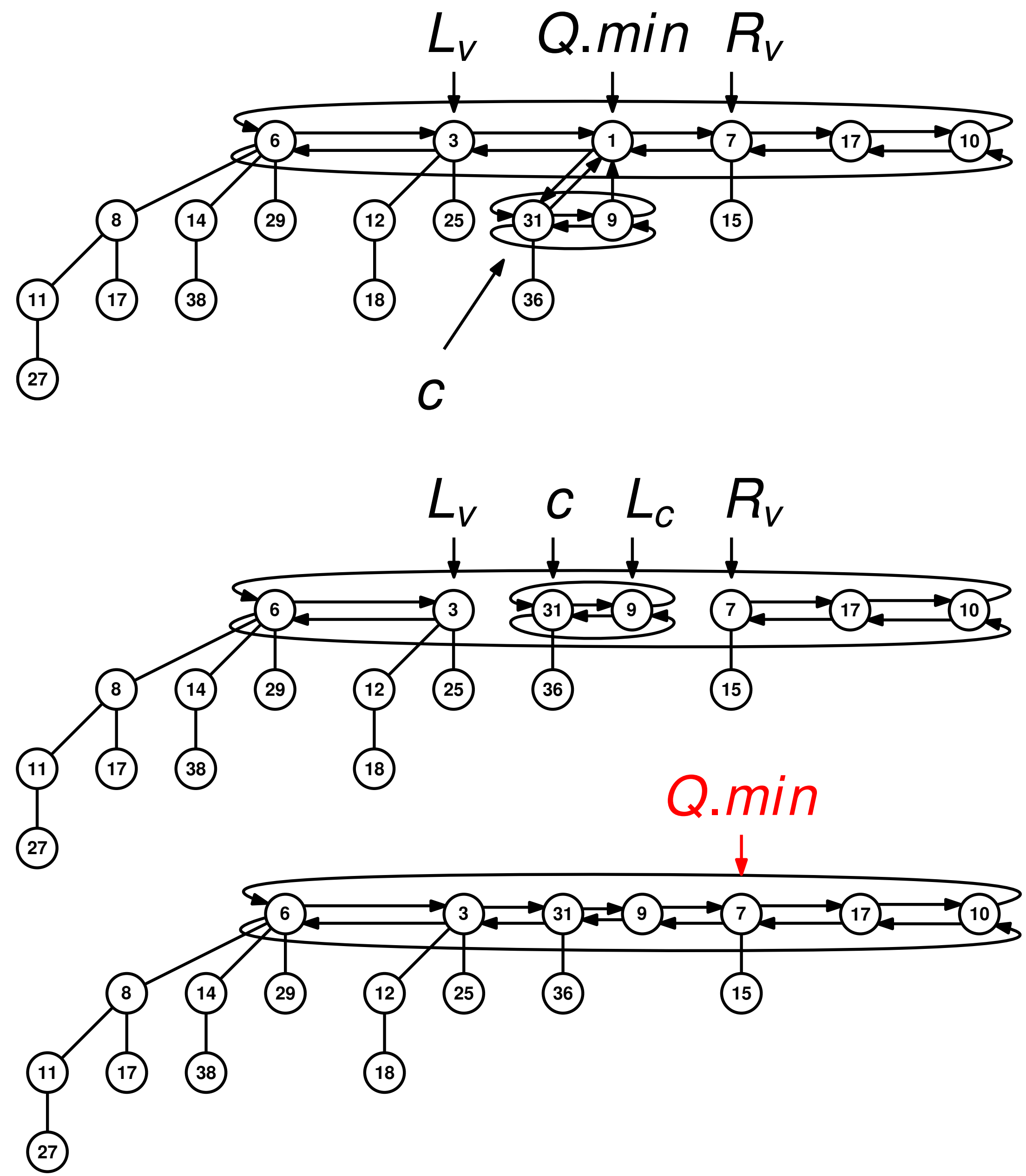
  - $L_v.right = c$
  - $c.left = L_v$
  - → $R_v.left = L_c$
  - → $L_c.right = R_v$
- **else**
  - $L_v.right = R_v$
  - $R_v.left = L_v$
- $Q.min = R_v$ ▷ arbitrary head
- CONSOLIDATE($Q$)
- **return** $v$



# Lazy EXTRACT-MIN($Q$)

**function** EXTRACT-MIN($Q$)

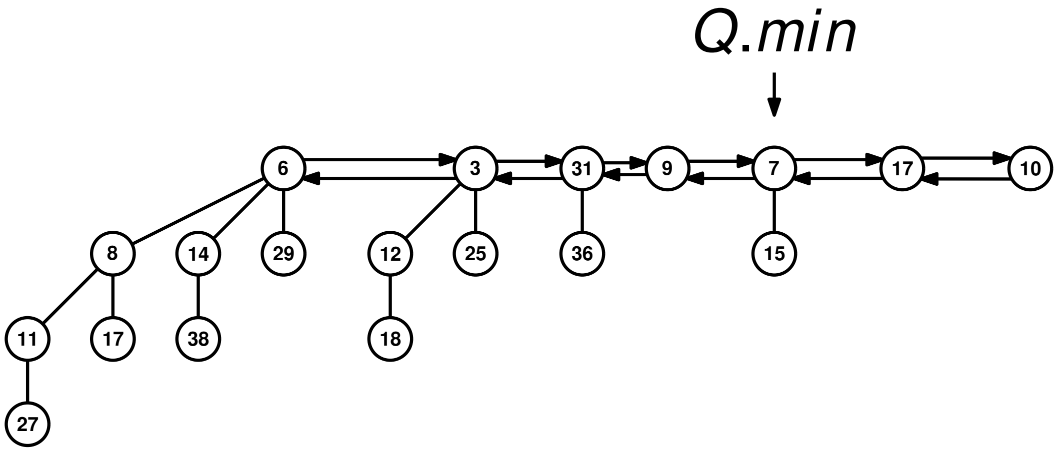
$v$ = MINIMUM($Q$)
$c = v.child$
$L_v = v.left$
$R_v = v.right$
**if** $c \neq$ NIL **then**
$L_c = c.left$
$L_v.right = c$
$c.left = L_v$
$R_v.left = L_c$
$L_c.right = R_v$
**else**
$L_v.right = R_v$
$R_v.left = L_v$
→ $Q.min = R_v$ ▷ arbitrary head
CONSOLIDATE($Q$)
**return** $v$



# CONSOLIDATE($Q$)

Root list:

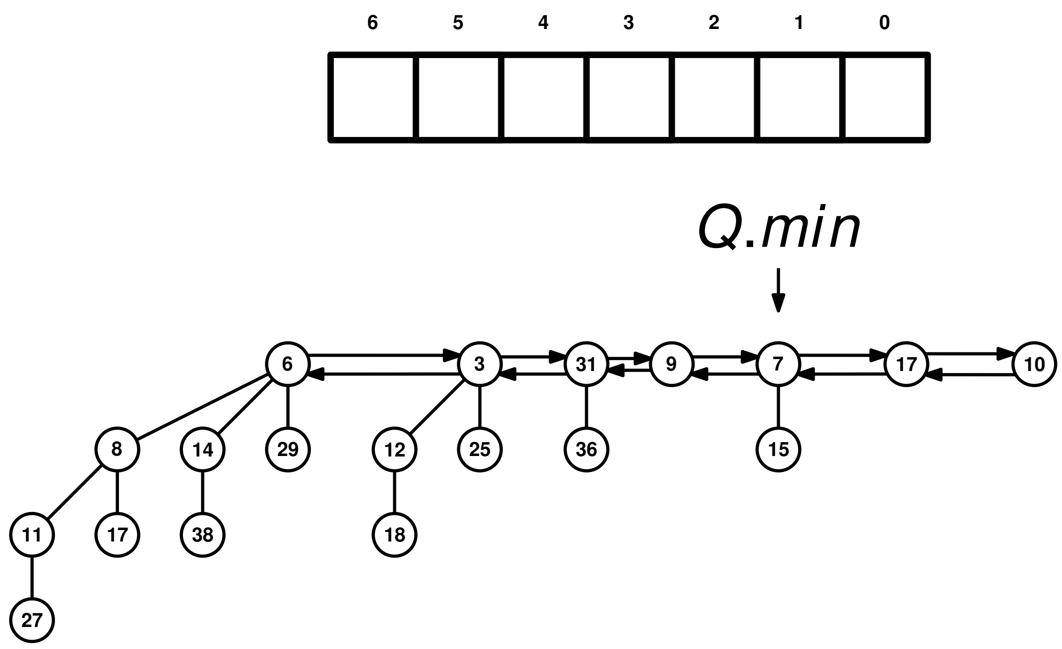
- At most $\log n$ distinct tree orders
- Use $\log n$-sized array with pointers to each tree order



# CONSOLIDATE($Q$)

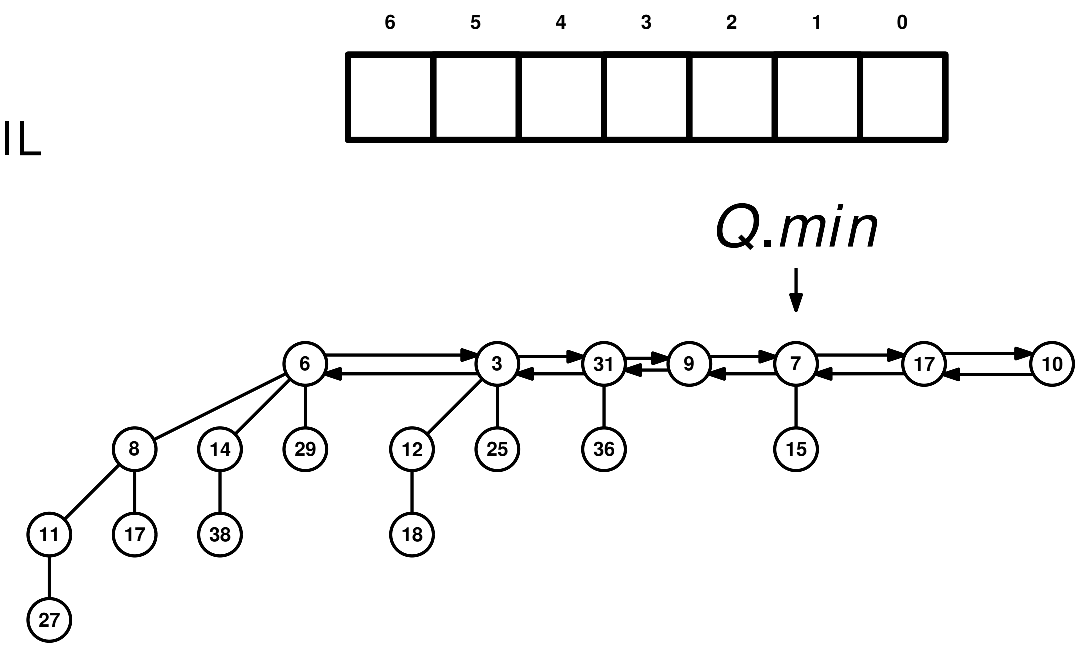
Root list:

- At most log $n$ distinct tree orders
- Use log $n$-sized array with pointers to each tree order



# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
  Initialize log $n$-sized array $A$ to NIL
  **for each** $v$ in root list **do**
    $d = v.degree$
    **while** $A[d] \neq$ NIL **do**
      $v$ = LINK($v$, $A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL



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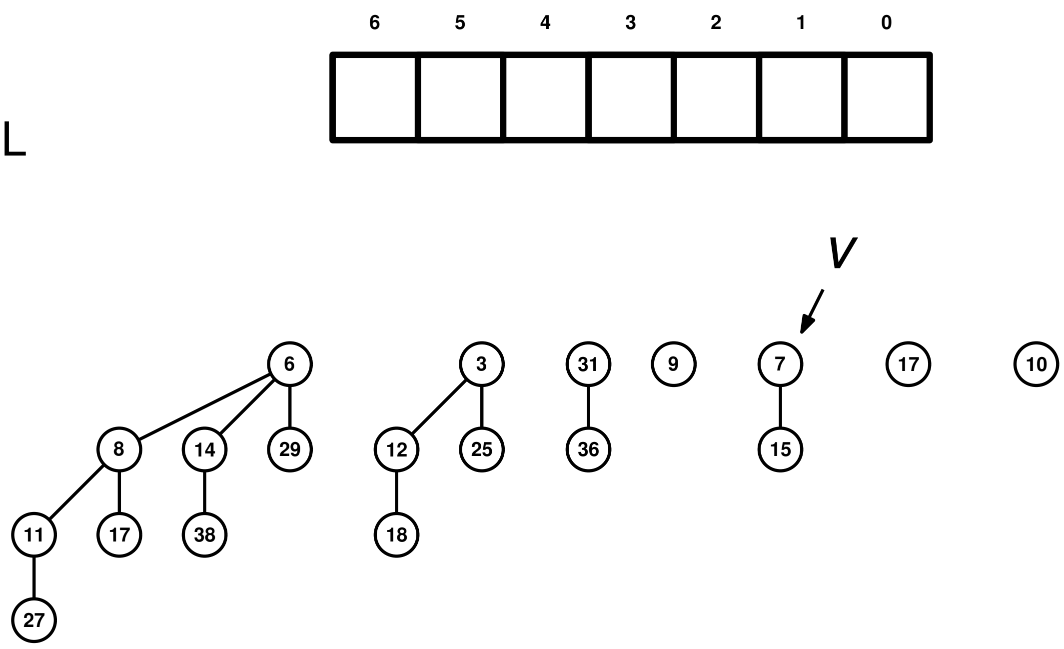

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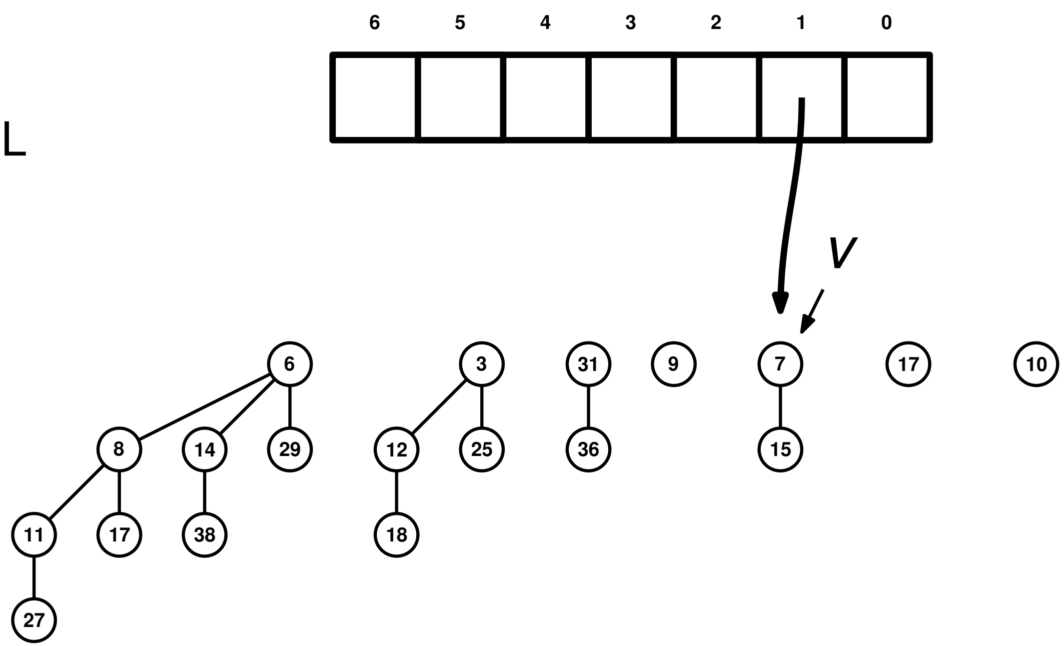

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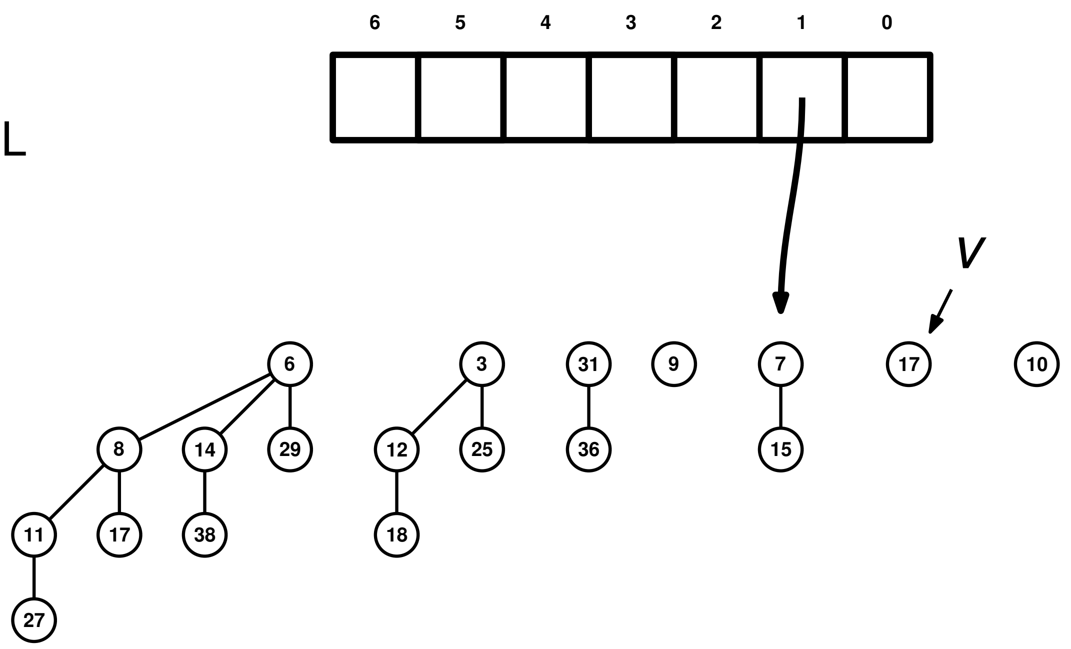

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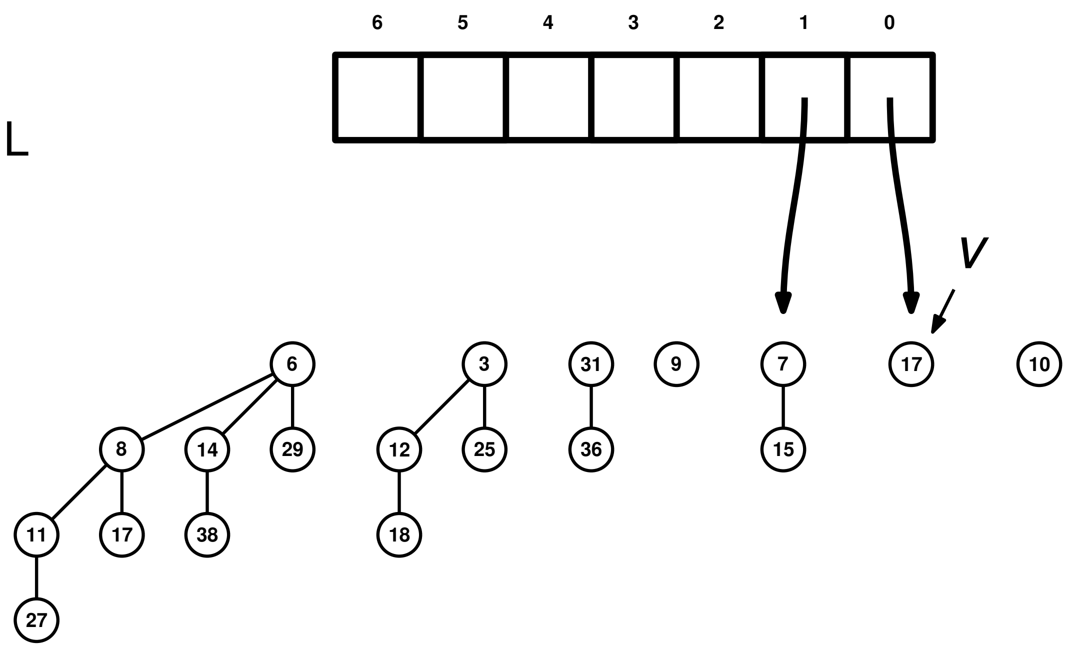

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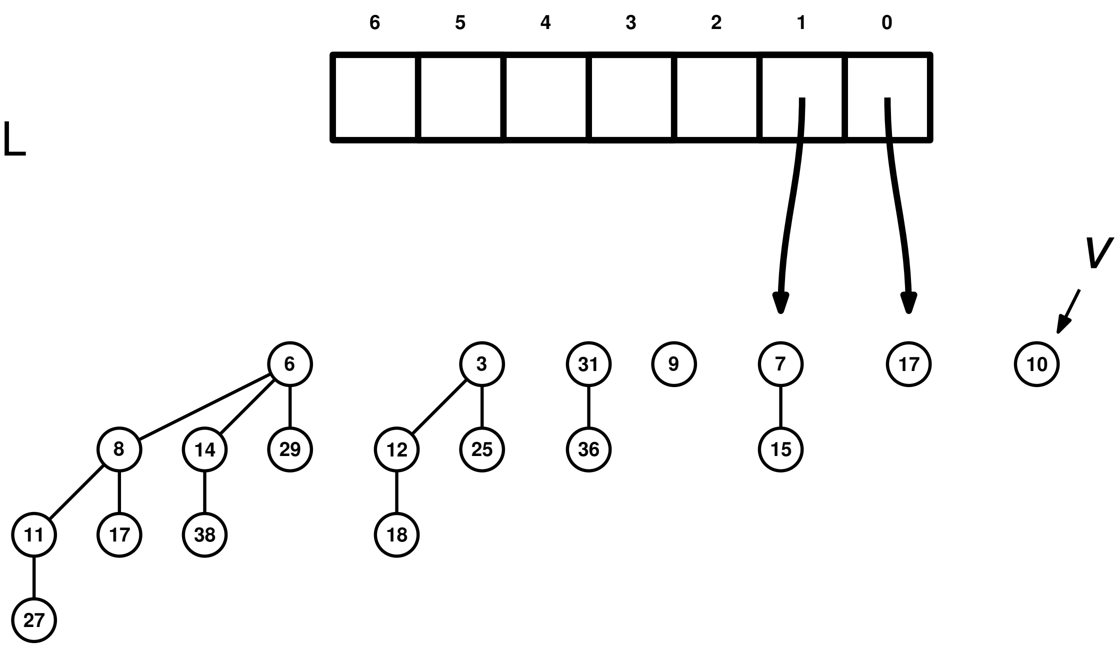
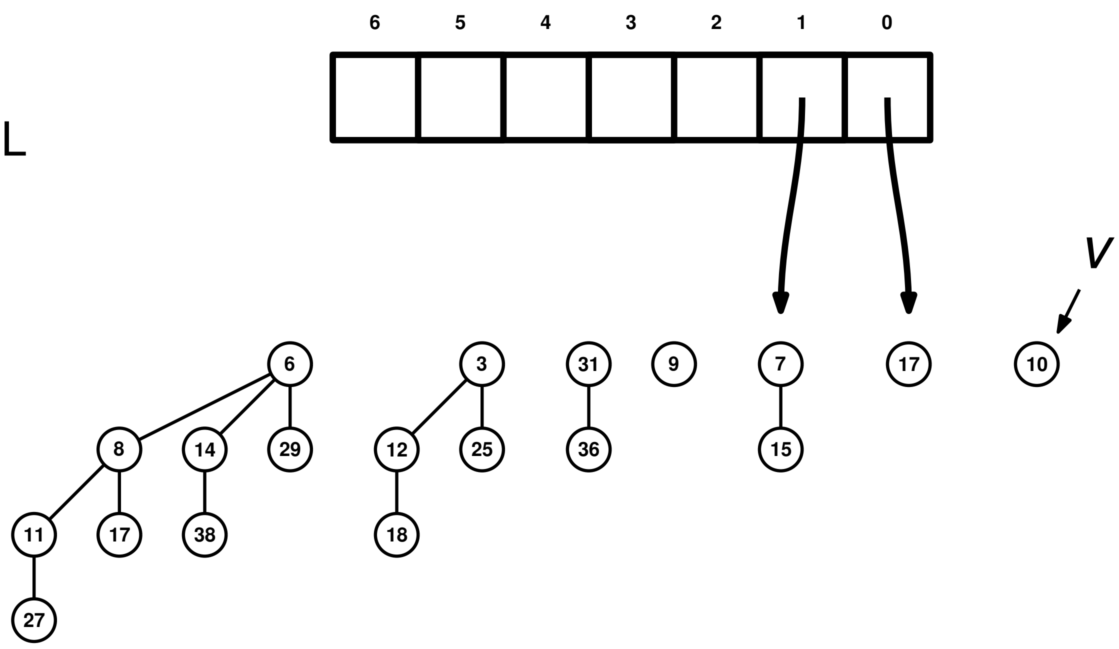


# CONSOLIDATE(*Q*)

**function** CONSOLIDATE(Q)
    Initialize log *n*-sized array *A* to NIL
    **for each** *v* in root list **do**
        *d* = *v*.*degree*
        **while** *A*[*d*] ≠ NIL **do**
            *v* = LINK(*v*, *A*[*d*])
            *A*[*d*] = NIL
            *d* = *d* + 1
        *A*[*d*] = *v*; *v*.*parent* = NIL



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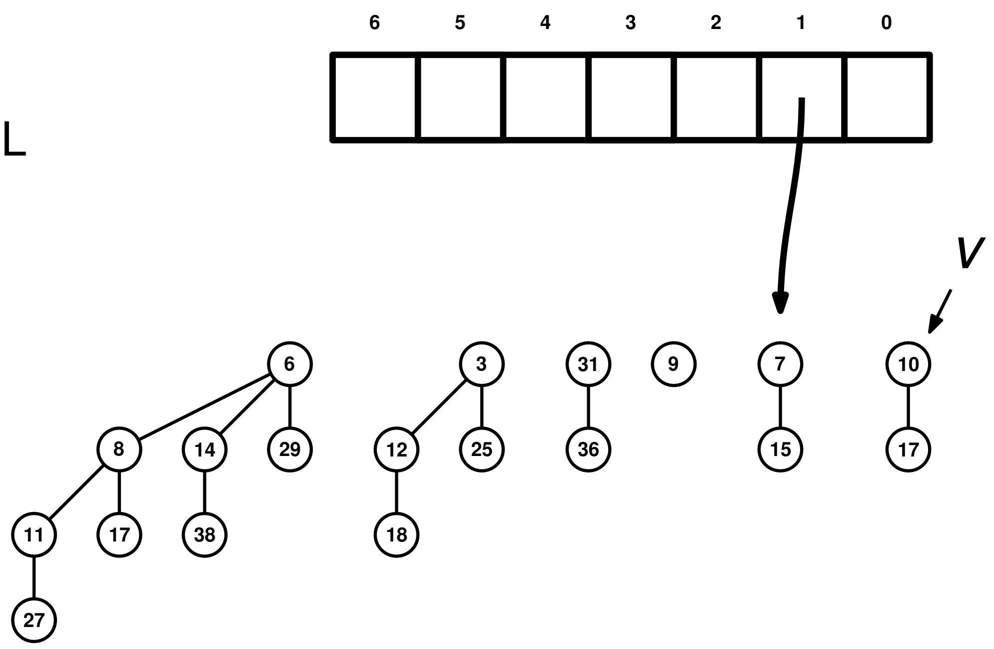

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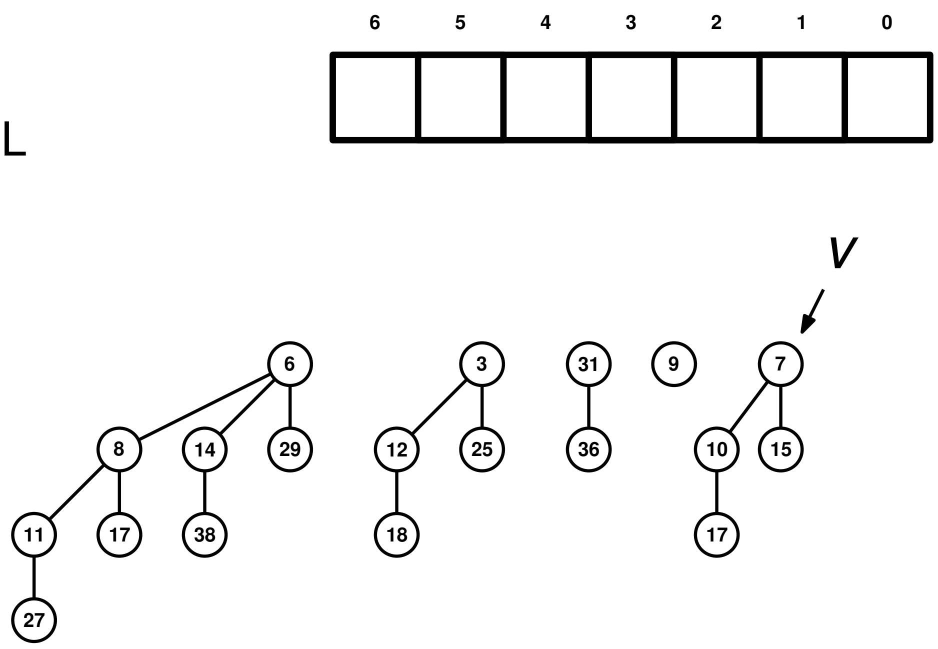

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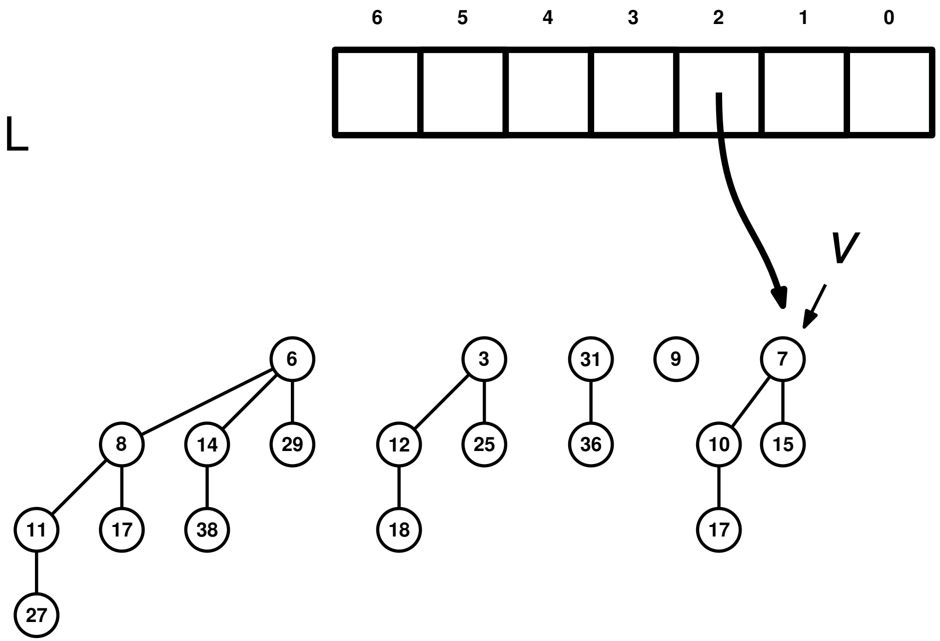

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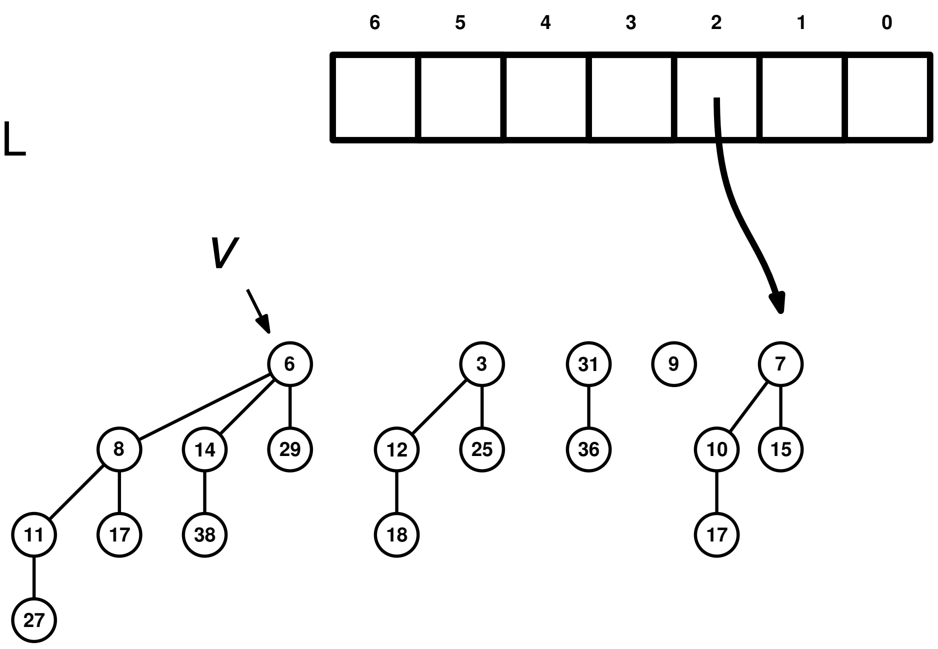

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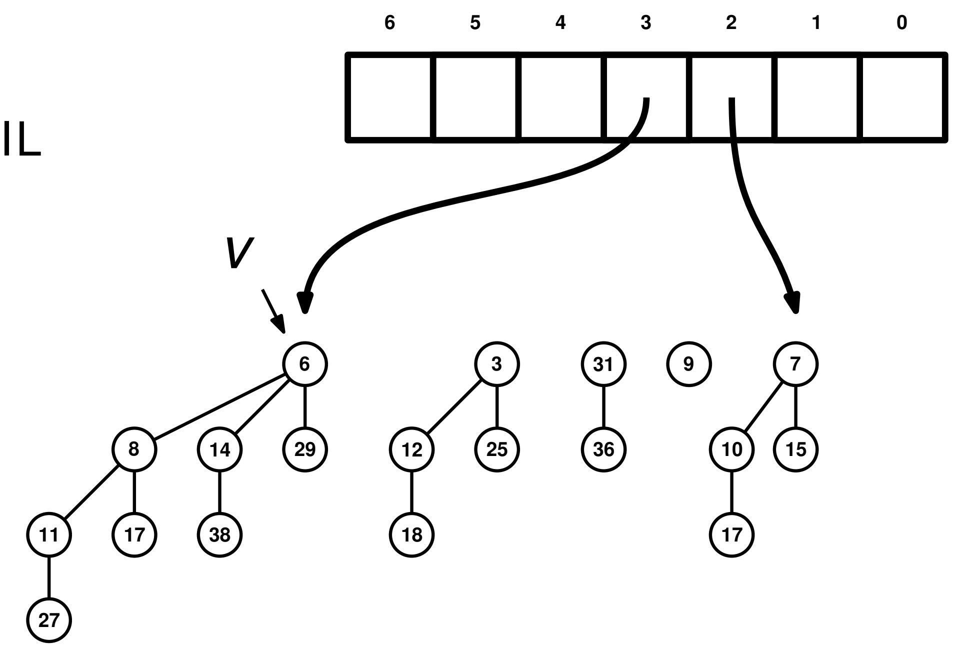

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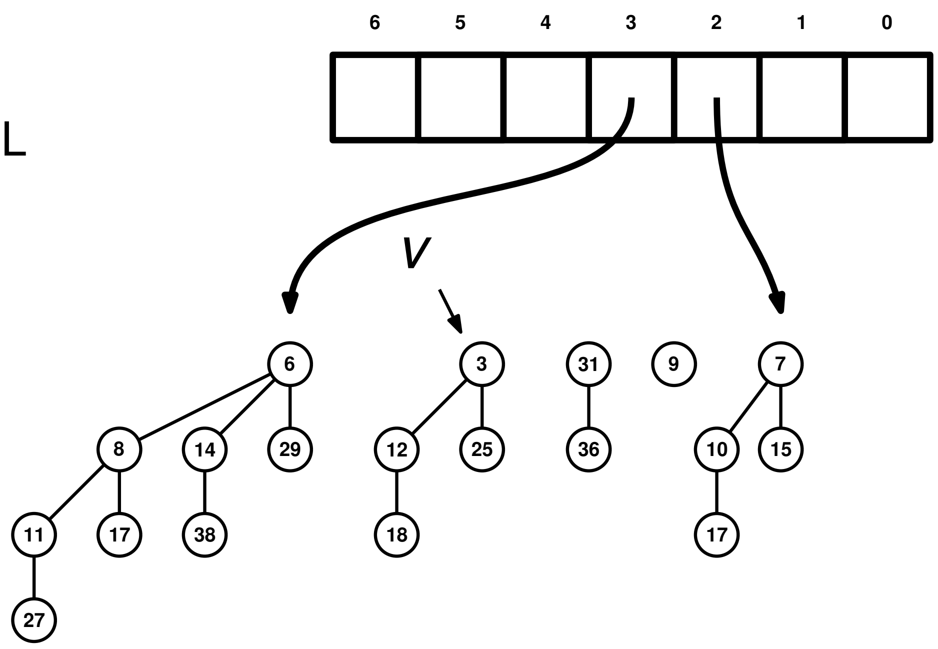

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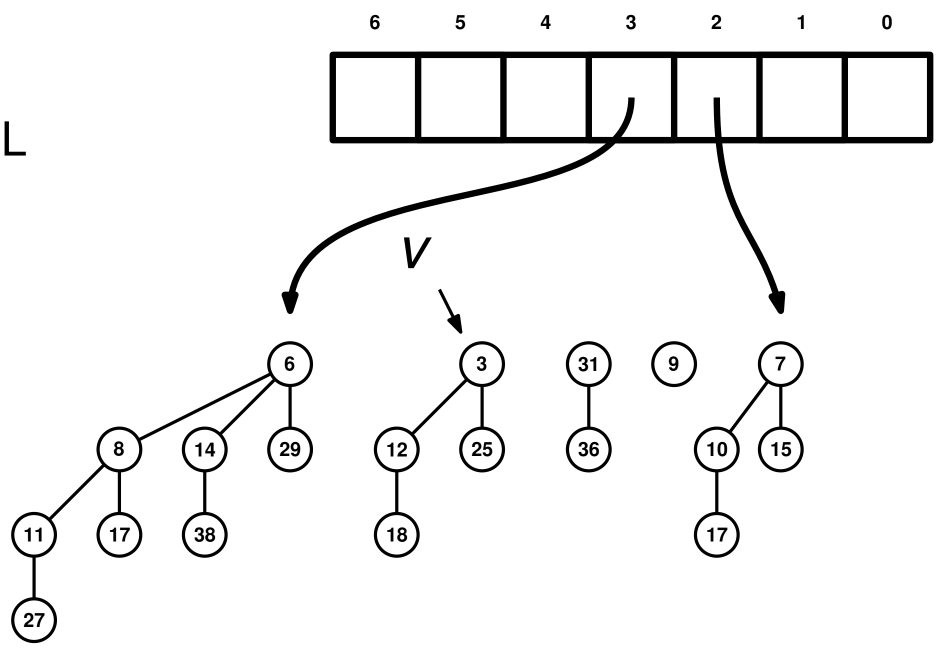

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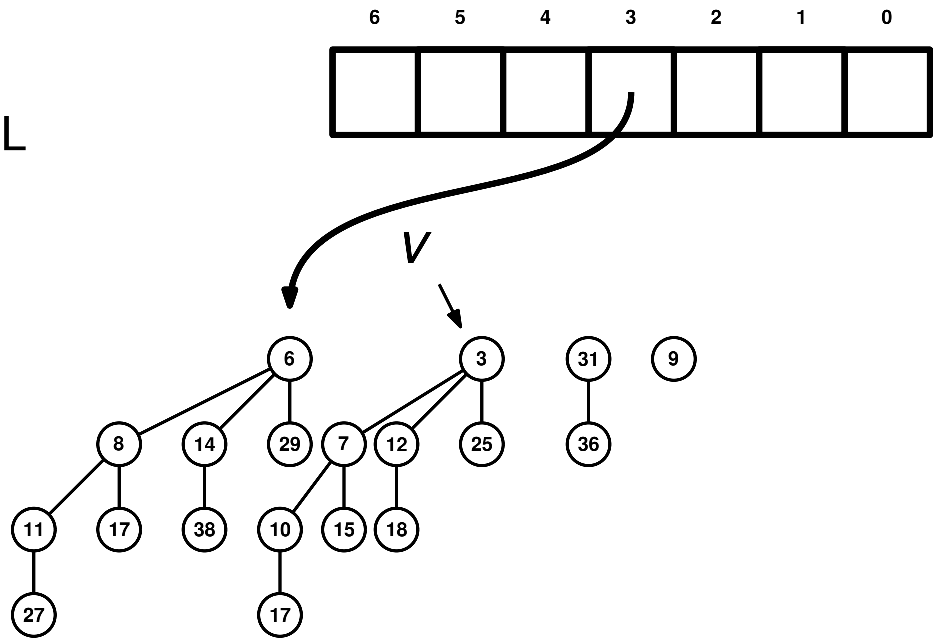

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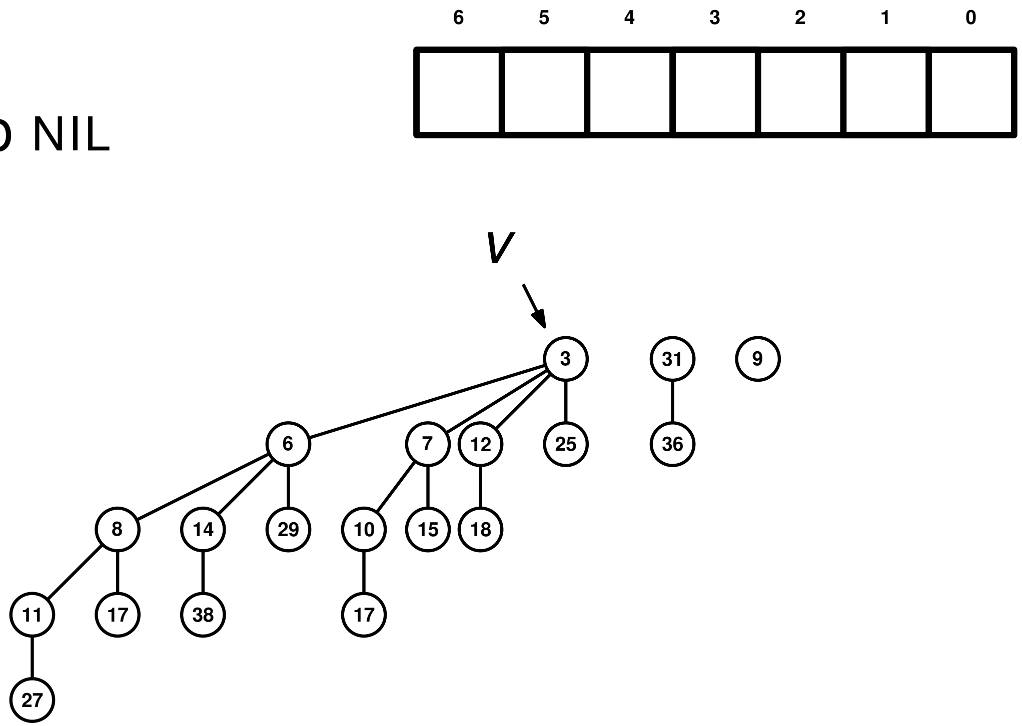

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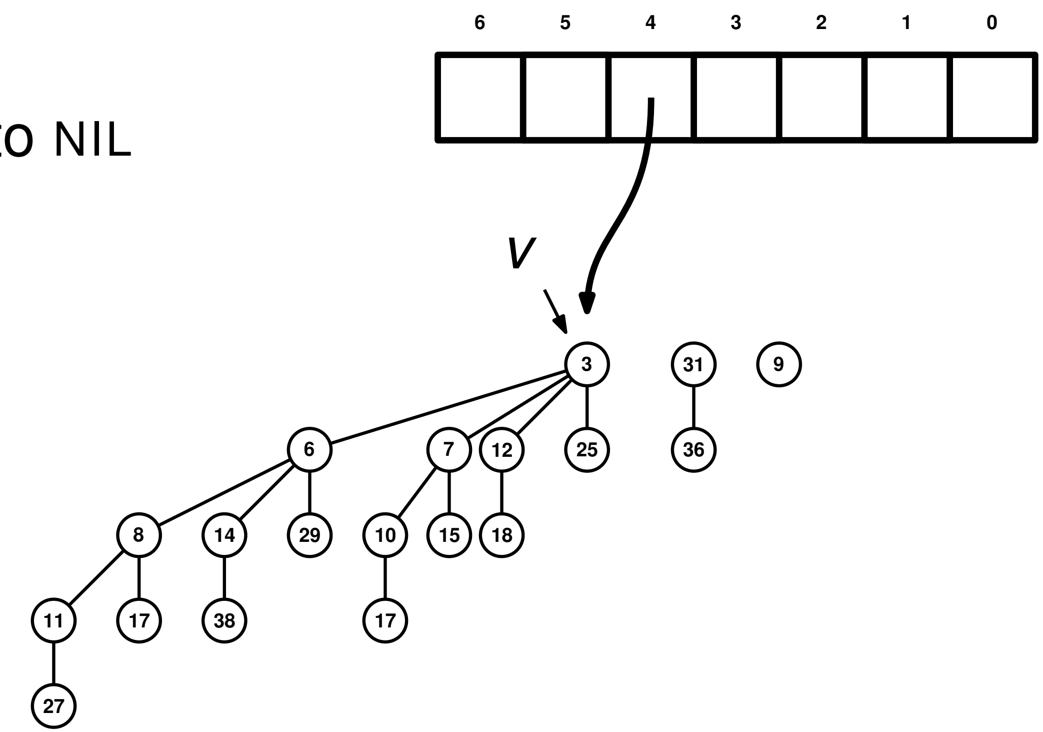

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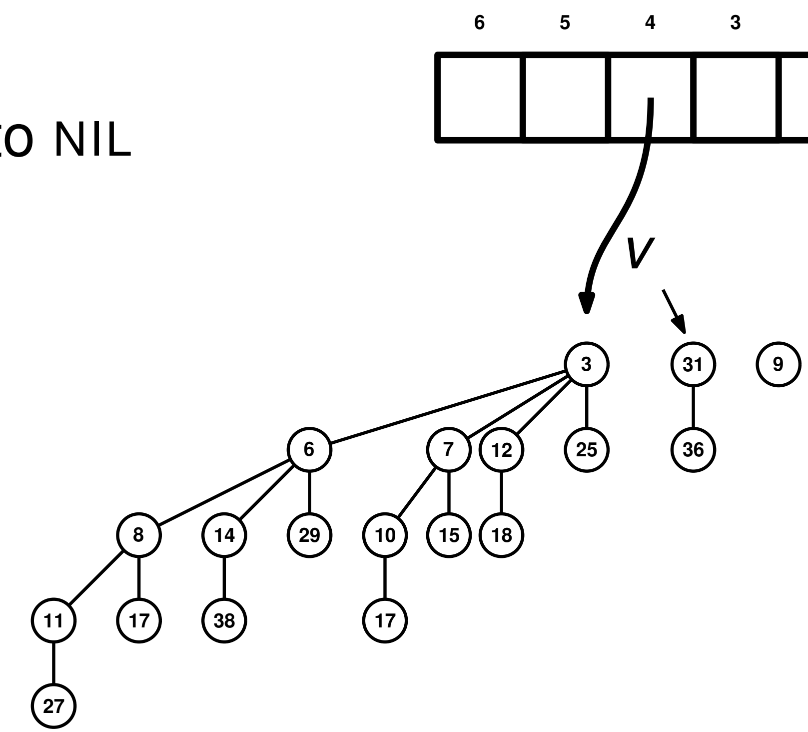

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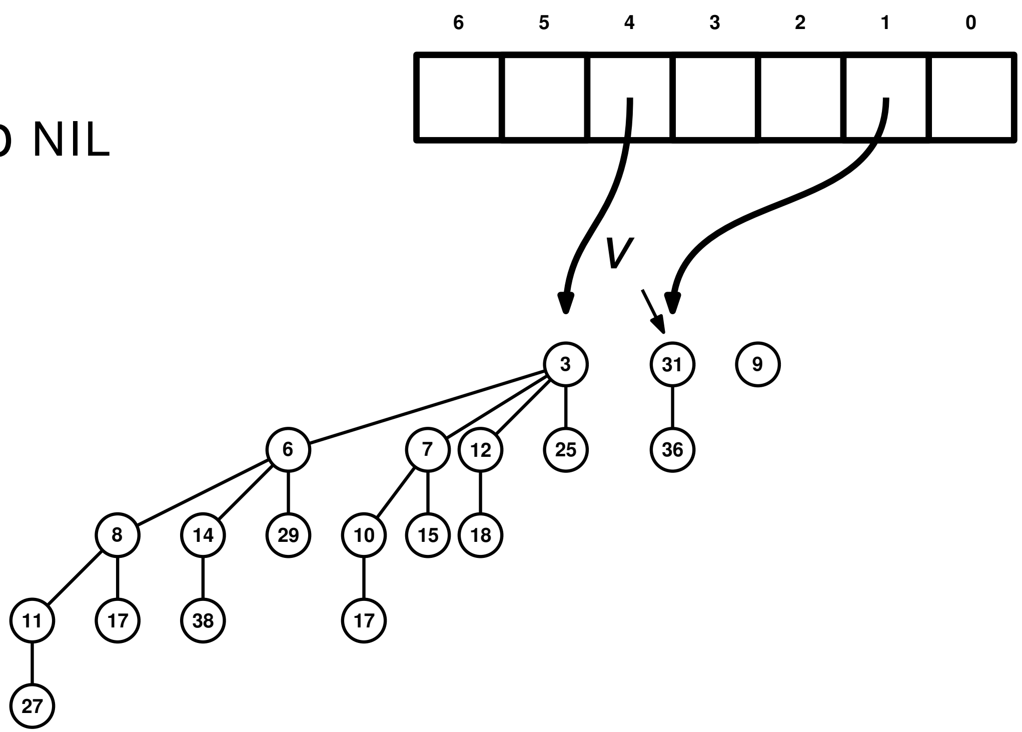

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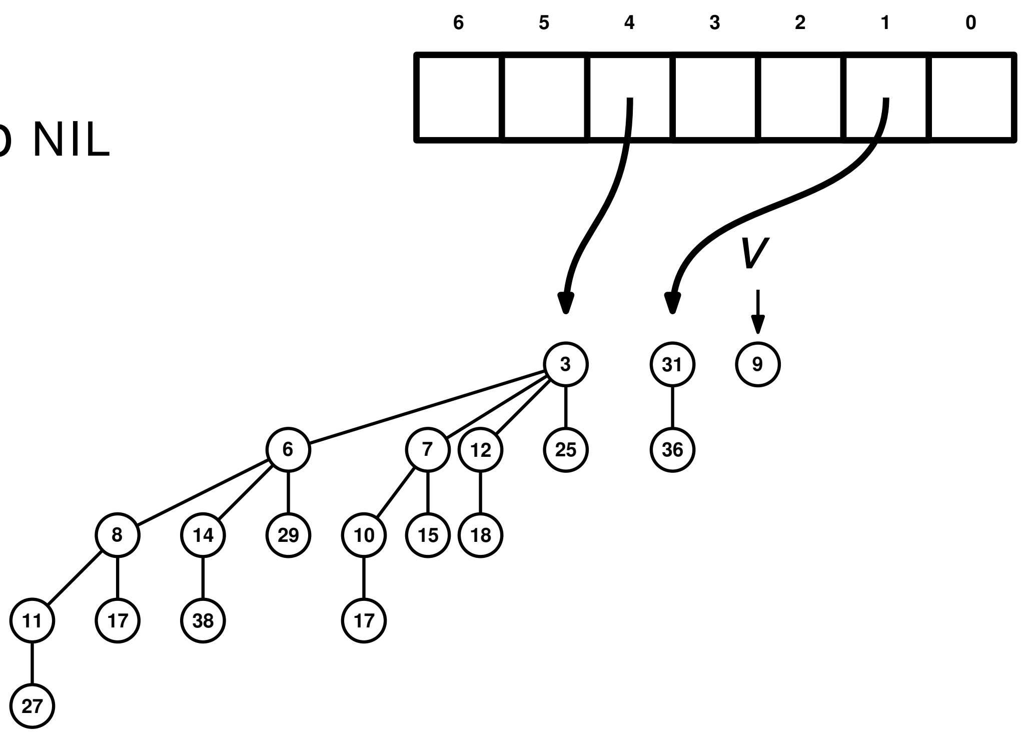

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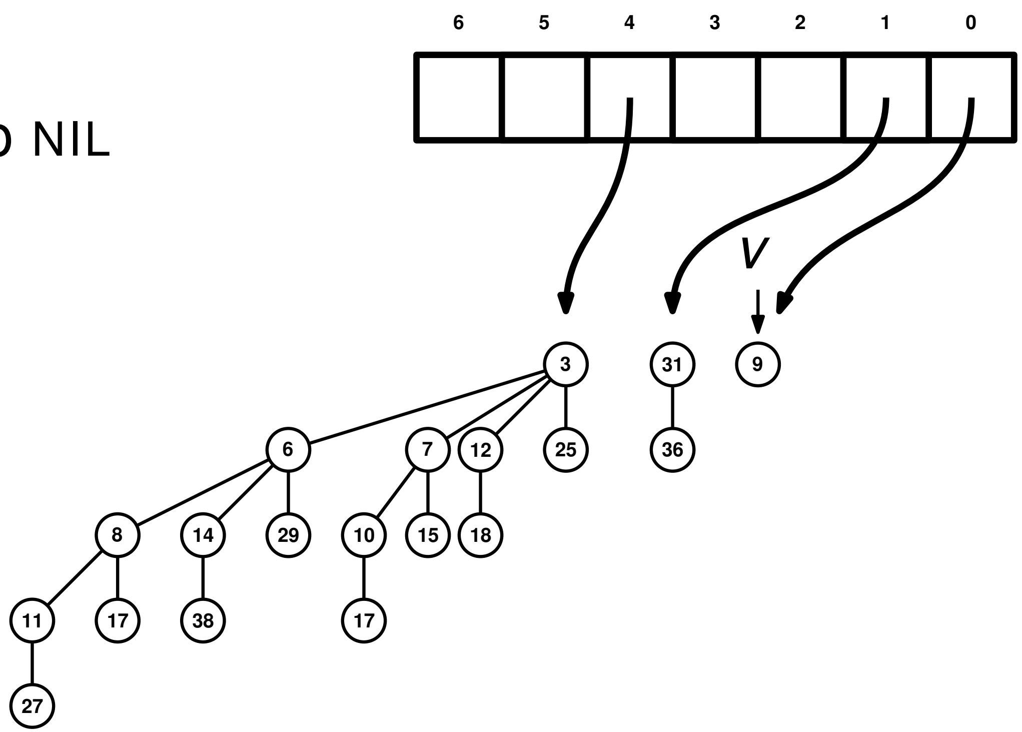

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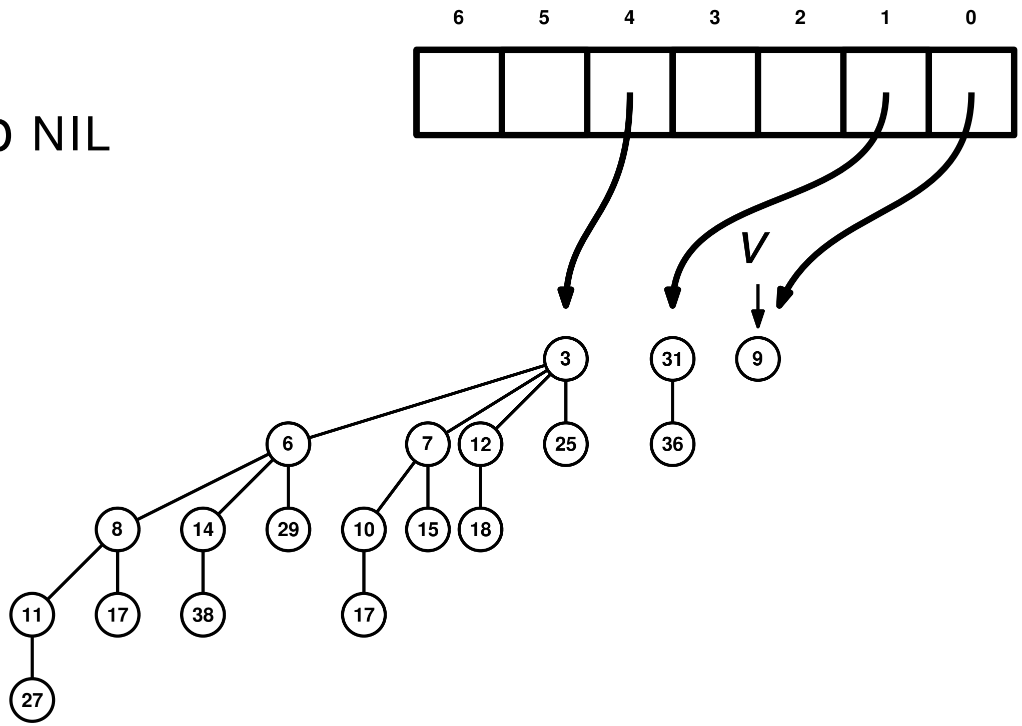
    $A[d] = v$; $v.parent$ = NIL
  $min = +\infty$
  **for** $i = 0$ **to** $\log n - 1$ **do**
    **if** $A[i] \neq$ NIL **then**
      Add $A[i]$ to the root list
      **if** $A[i].key < min$ **then**
        $Q.min \leftarrow A[i]$; $min = A[i].key$



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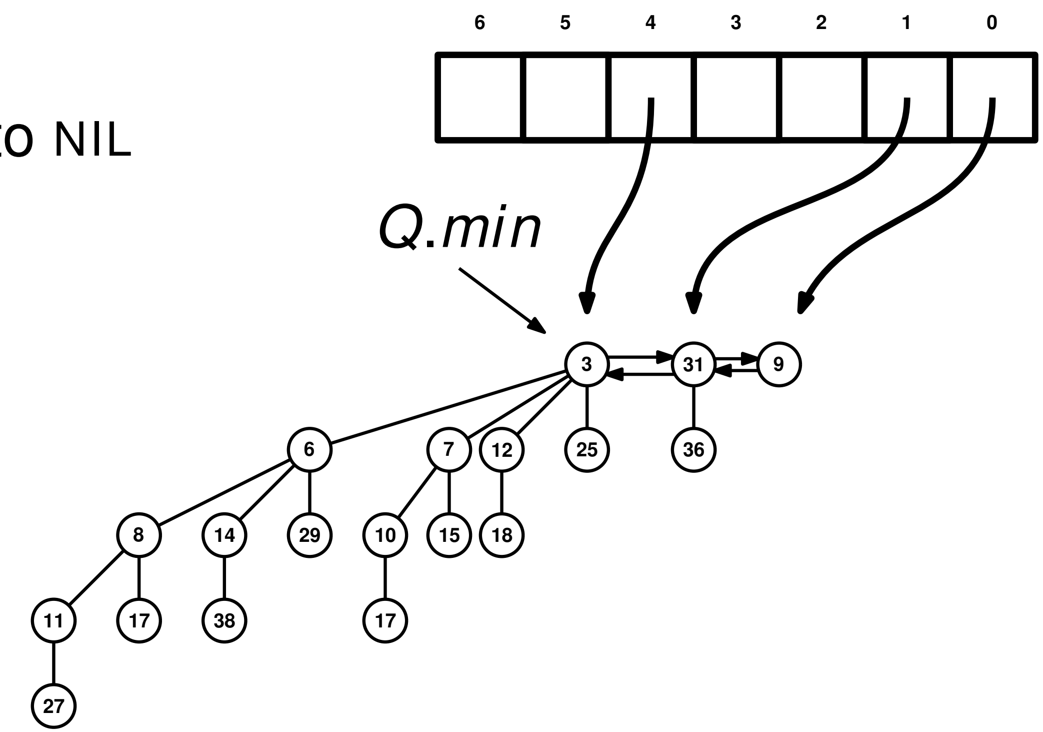
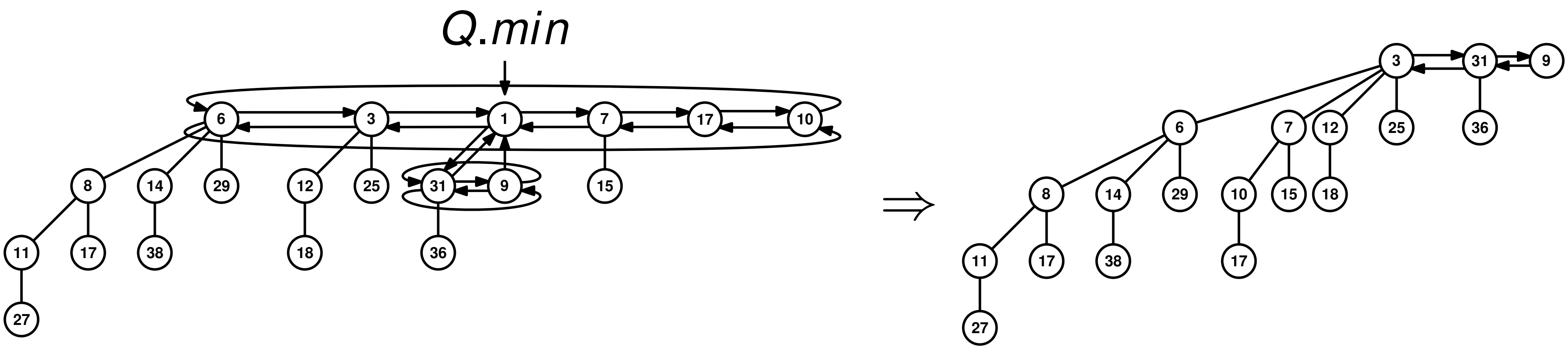


# Lazy EXTRACT-MIN($Q$) Analysis

We will use the Potential Method with $\Phi_i = t_i$ = number of trees in the root list after the $i$-th operation.

Let $d$ be the number of children of the $Q.min$ ($d \leq \log n$)




```
function CONSOLIDATE(Q)
    Initialize log n-sized array A to NIL
    for each v in root list do
        d = v.degree
        while A[d] ≠ NIL do
            v = LINK(v, A[d])
            A[d] = NIL
            d = d + 1
        A[d] = v; v.parent = NIL
    min = +∞
    for i = 0 to log n − 1 do
        if A[i] ≠ NIL then
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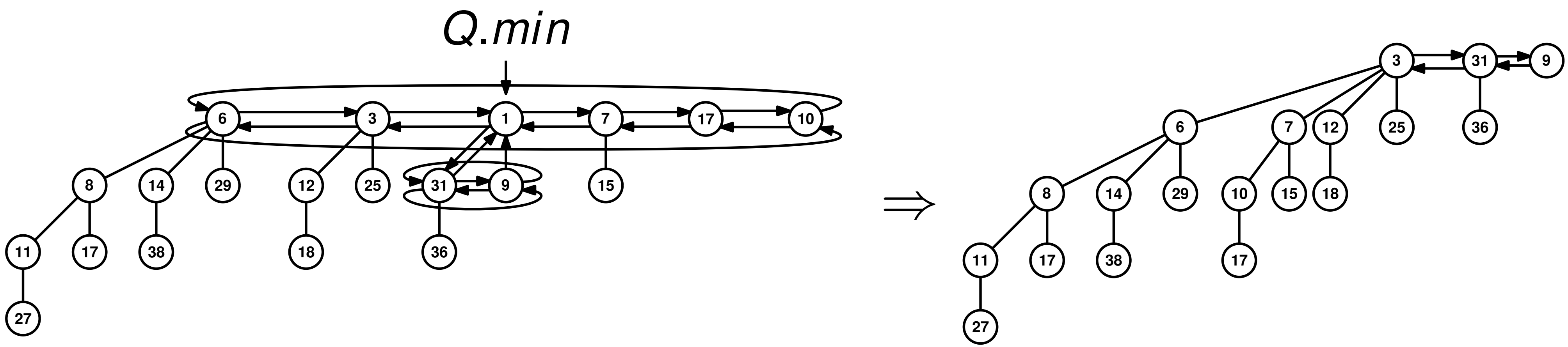


- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log n \leq O(1) + t_{i-1} + 2 \log n$
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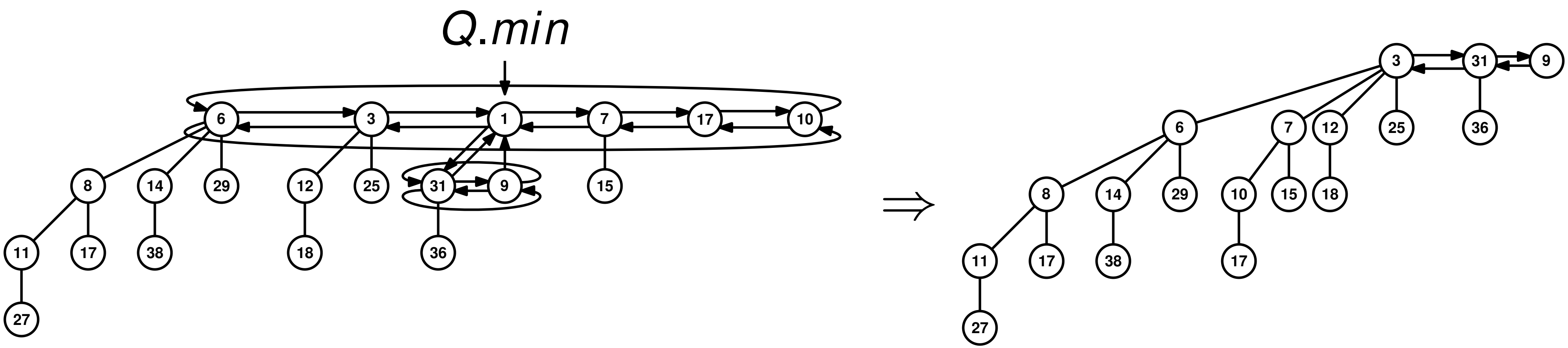


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$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2\log n + t_i$$

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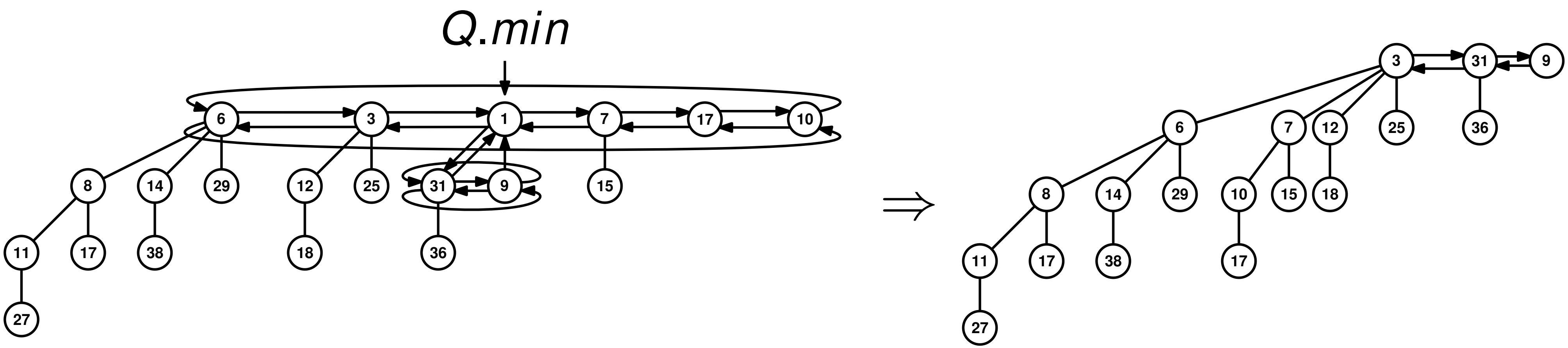



- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log n \leq O(1) + t_{i-1} + 2 \log n$
- Change in potential: $\Delta\Phi_i = t_i - t_{i-1}$

$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2 \log n + t_i$$

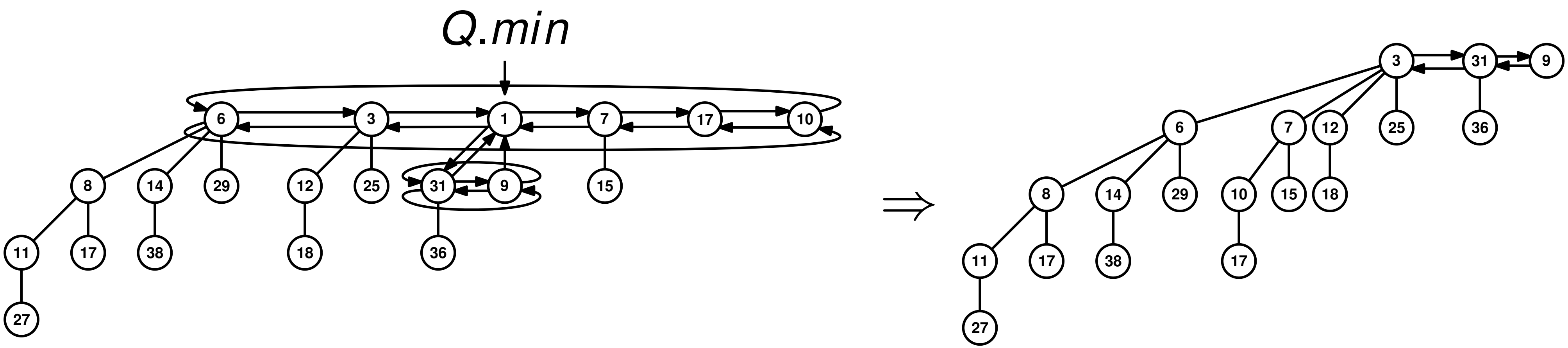
there are $t_i \leq \log n$ trees in the root list after consolidation

```
function CONSOLIDATE(Q)
    Initialize log n-sized array A to NIL
    for each v in root list do
        d = v.degree
        while A[d] ≠ NIL do
            v = LINK(v, A[d])
            A[d] = NIL
            d = d + 1
        A[d] = v; v.parent = NIL
    min = +∞
    for i = 0 to log n − 1 do
        if A[i] ≠ NIL then
            Add A[i] to the root list
            if A[i].key < min then
                Q.min ← A[i]; min = A[i].key
```

# Lazy EXTRACT-MIN($Q$) Analysis

We will use the Potential Method with $\Phi_i = t_i$ = number of trees in the root list after the $i$-th operation.

Let $d$ be the number of children of the $Q.min$ ($d \leq \log n$)




- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log n \leq O(1) + t_{i-1} + 2 \log n$
- Change in potential: $\Delta\Phi_i = t_i - t_{i-1}$

$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2 \log n + t_i \leq O(1) + 3 \log n = O(\log n)$$

there are $t_i \leq \log n$ trees in the root list after consolidation

```
function CONSOLIDATE(Q)
   Initialize log n-sized array A to NIL
   for each v in root list do
      d = v.degree
      while A[d] ≠ NIL do
         v = LINK(v, A[d])
         A[d] = NIL
         d = d + 1
      A[d] = v; v.parent = NIL
   min = +∞
   for i = 0 to log n − 1 do
      if A[i] ≠ NIL then
         Add A[i] to the root list
         if A[i].key < min then
            Q.min ← A[i]; min = A[i].key
```

# Heaps

| | Binomial | Lazy Binomial | Fibonacci |
|---|---|---|---|
| MAKE() | $O(1)$ | $O(1)$ | $O(1)$ |
| INSERT($Q$, $x$) | $O(1)^*$ | $O(1)$ | $O(1)$ |
| MINIMUM($Q$) | $O(1)$ | $O(1)$ | $O(1)$ |
| EXTRACT-MIN($Q$) | $O(\log n)$ | $O(\log n)^*$ | $O(\log n)^*$ |
| DECREASE-KEY($Q$, $x$, $k$) | $O(\log n)$ | $O(\log n)$ | $O(1)^*$ |
| DELETE($Q$, $x$) | $O(\log n)$ | $O(\log n)^*$ | $O(\log n)^*$ |
| UNION($Q_1$, $Q_2$) | $O(\log n)$ | $O(1)$ | $O(1)$ |

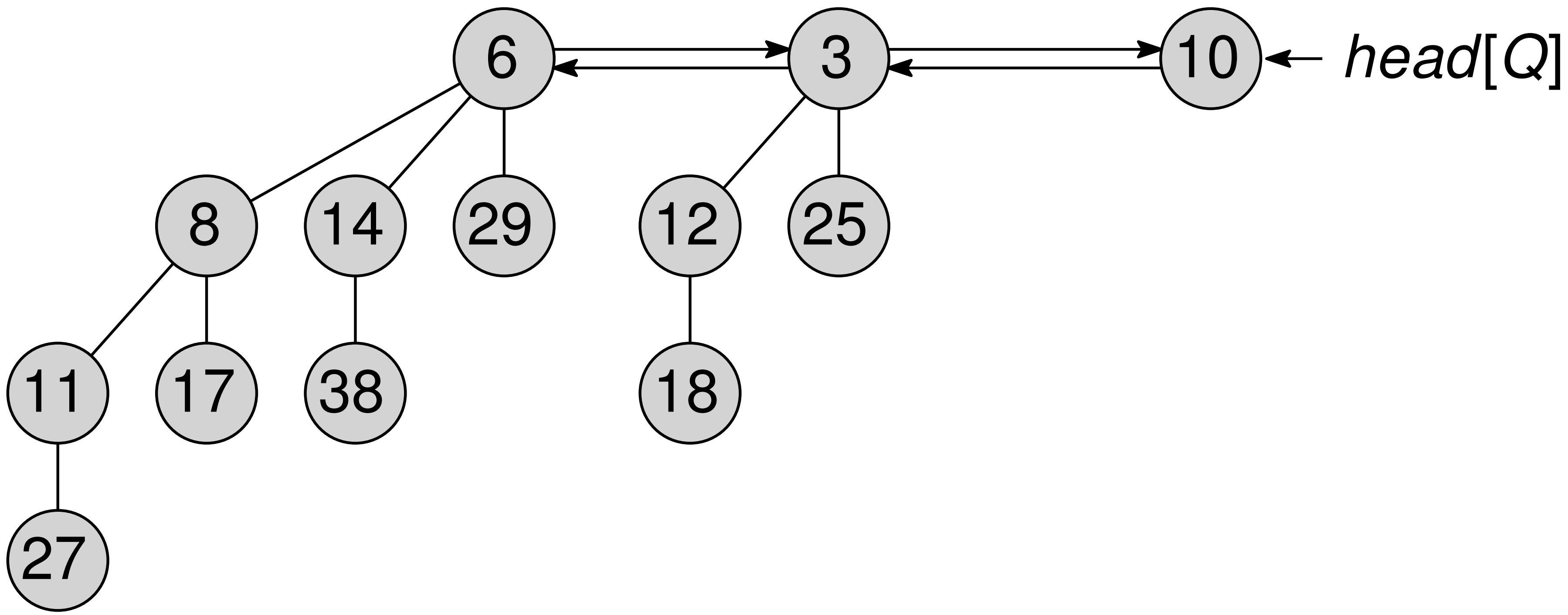
* Amortized cost

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time

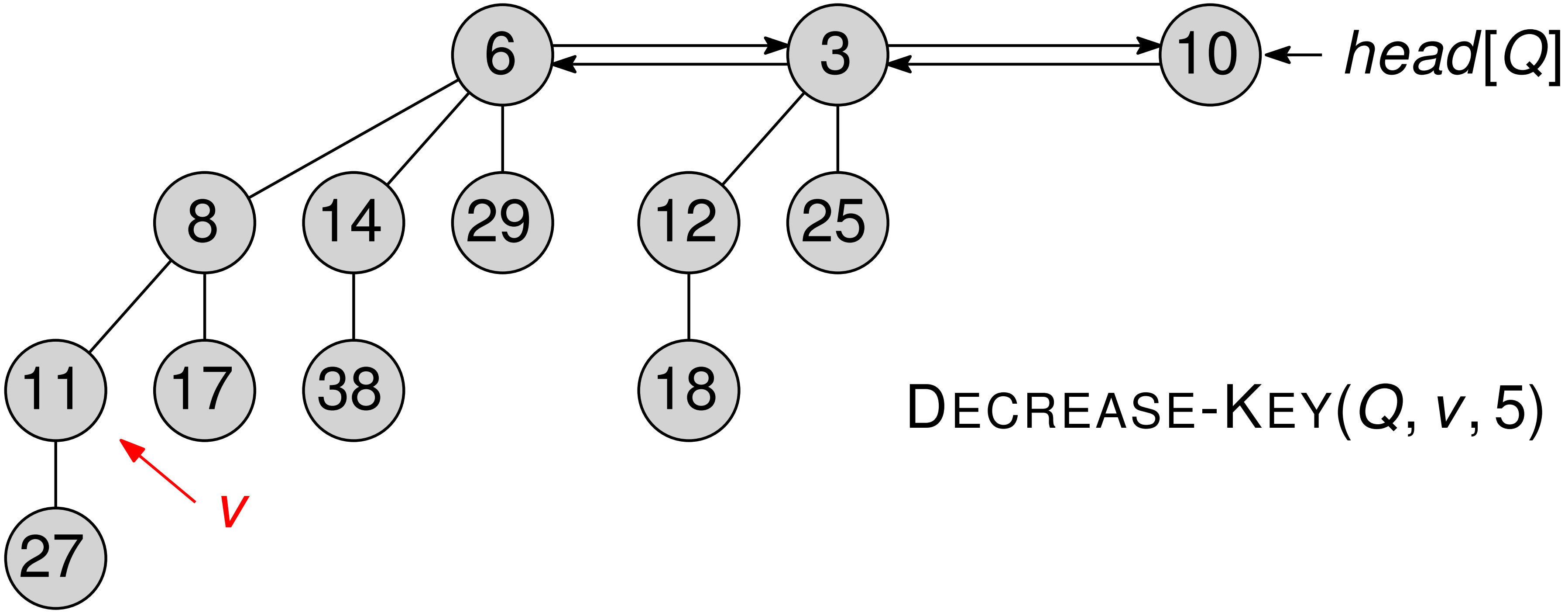
# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time



# Fibonacci Heaps

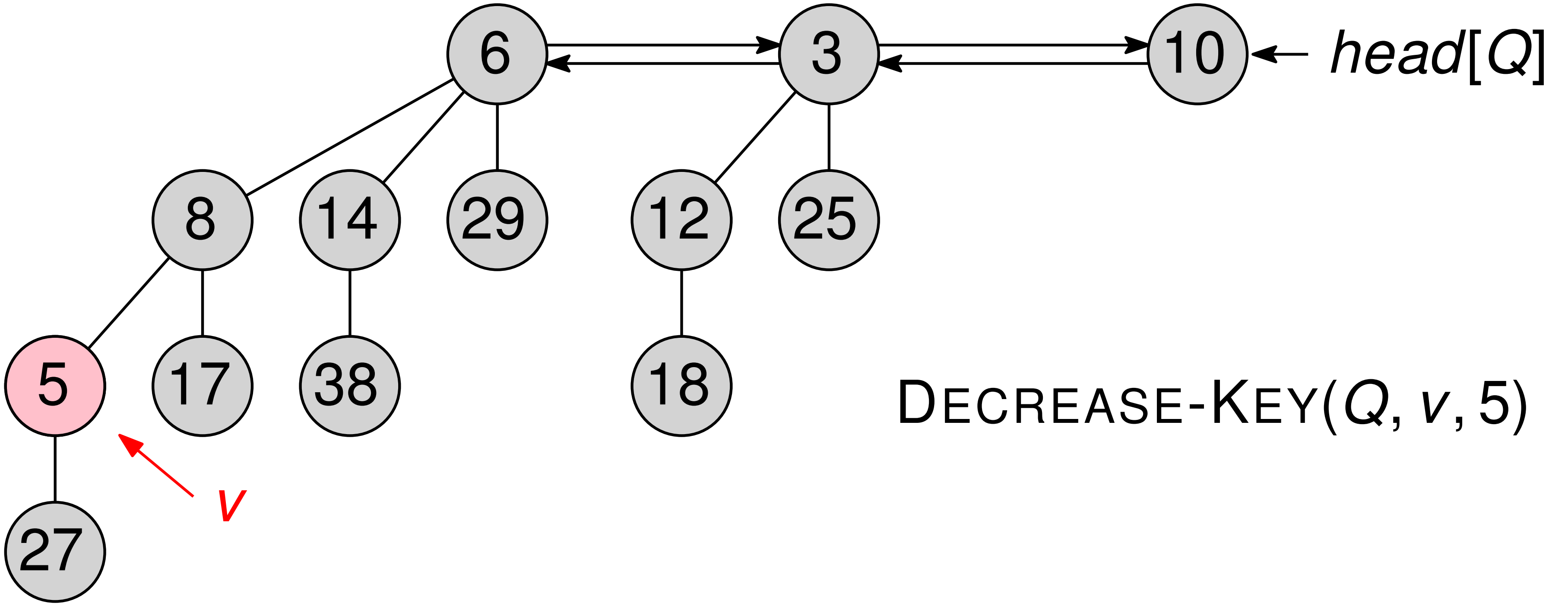
**Goal:** DECREASE-KEY($Q, v, k$) in $O(1)$ (amortized) time




DECREASE-KEY($Q, v, 5$)

# Fibonacci Heaps

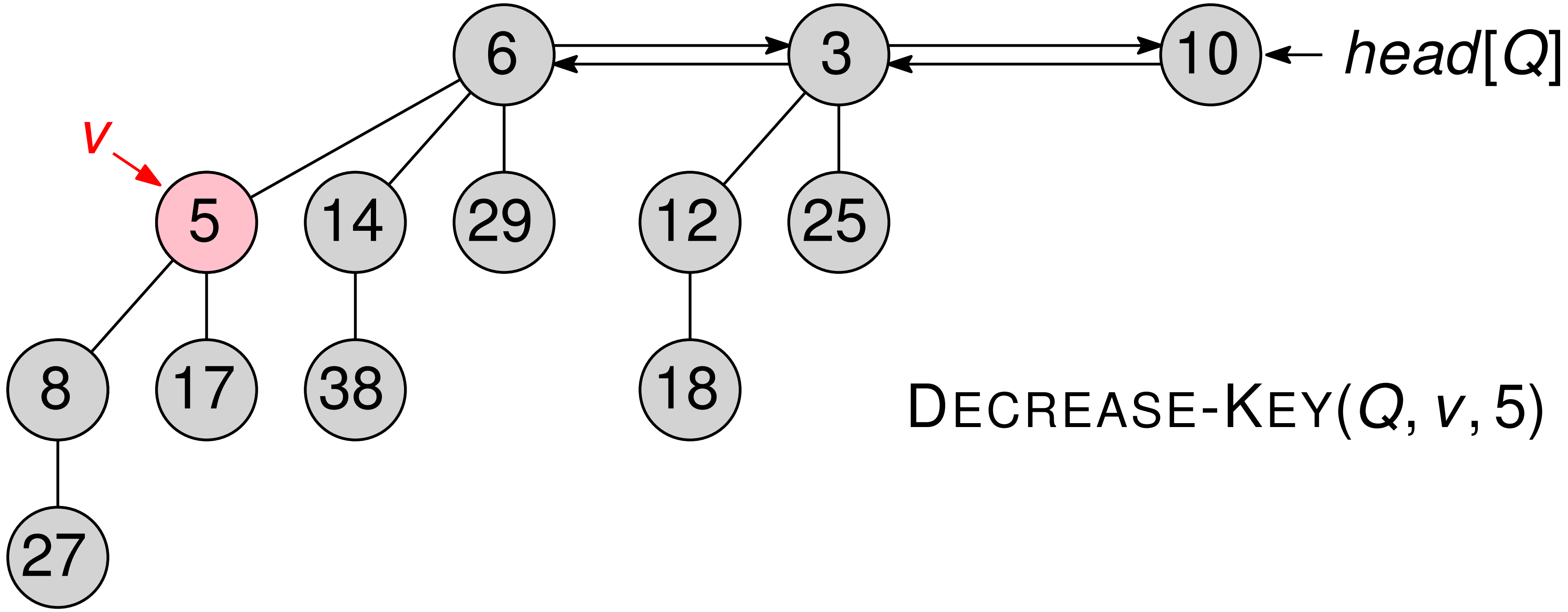
**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time




DECREASE-KEY($Q$, $v$, 5)

# Fibonacci Heaps

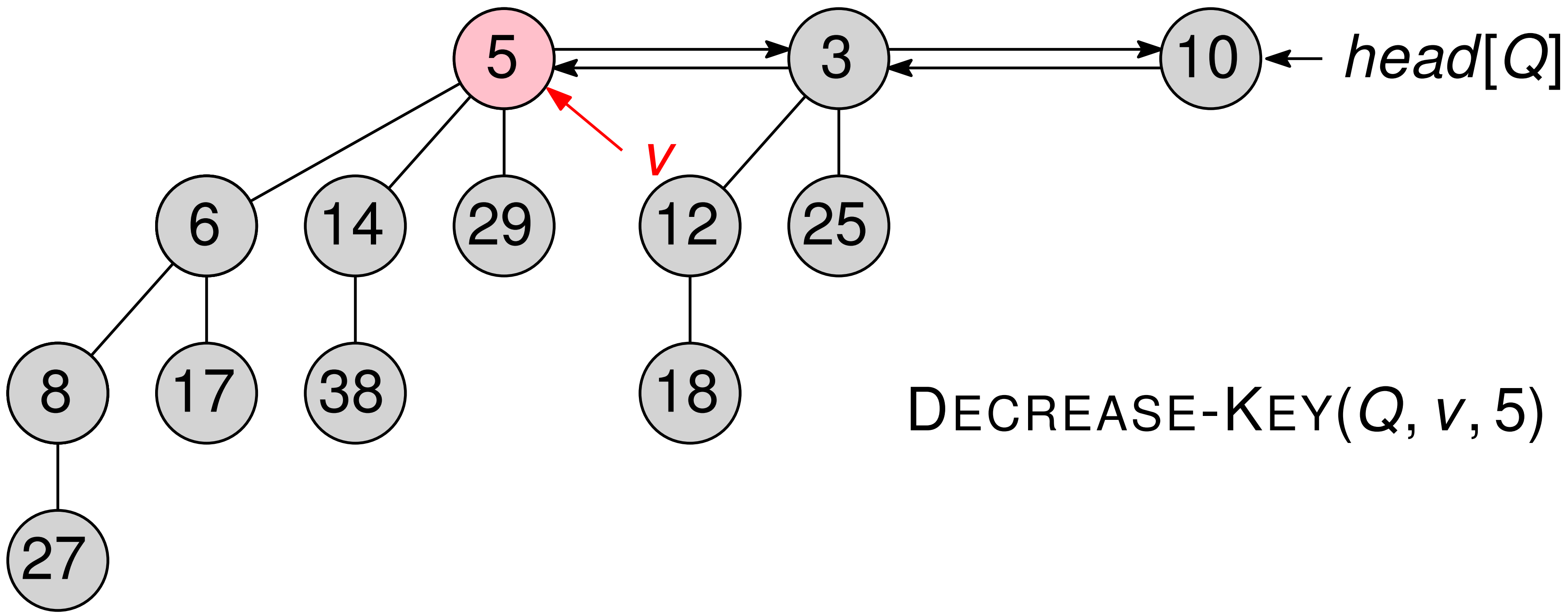
**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time




DECREASE-KEY($Q$, $v$, 5)

# Fibonacci Heaps

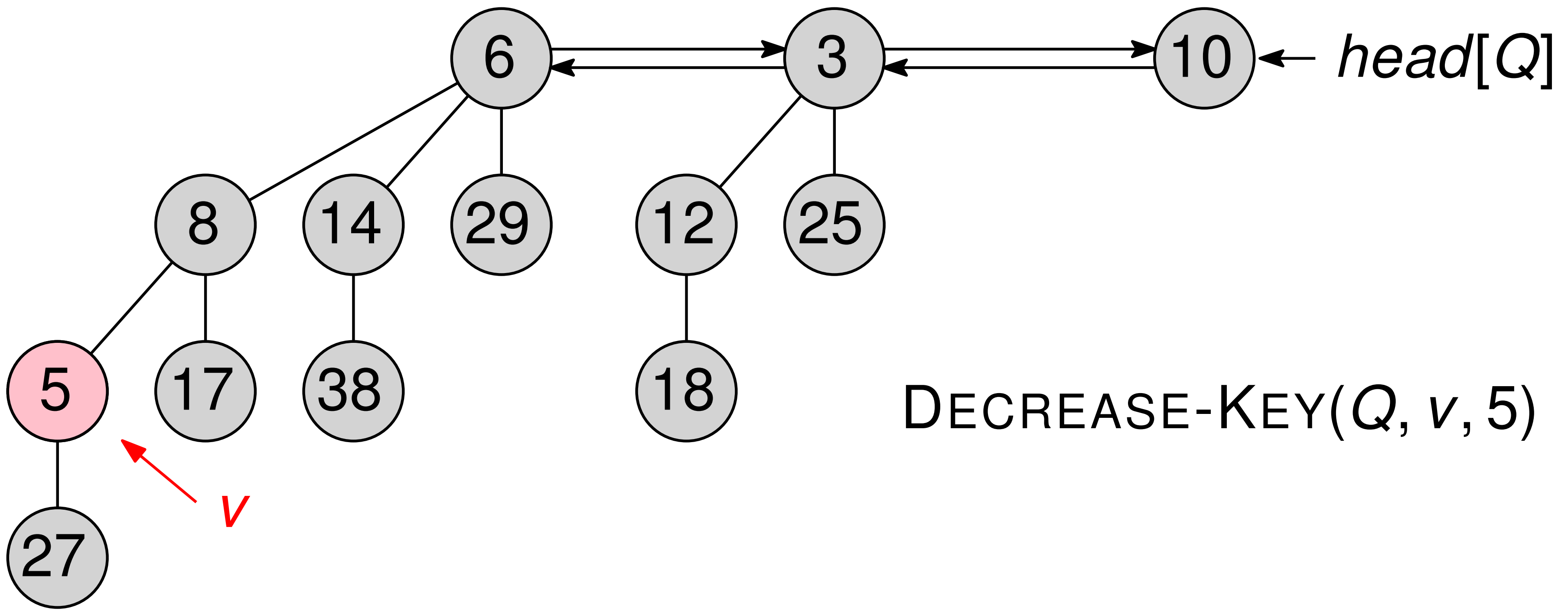
**Goal:** DECREASE-KEY($Q, v, k$) in $O(1)$ (amortized) time




DECREASE-KEY($Q, v, 5$)

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time

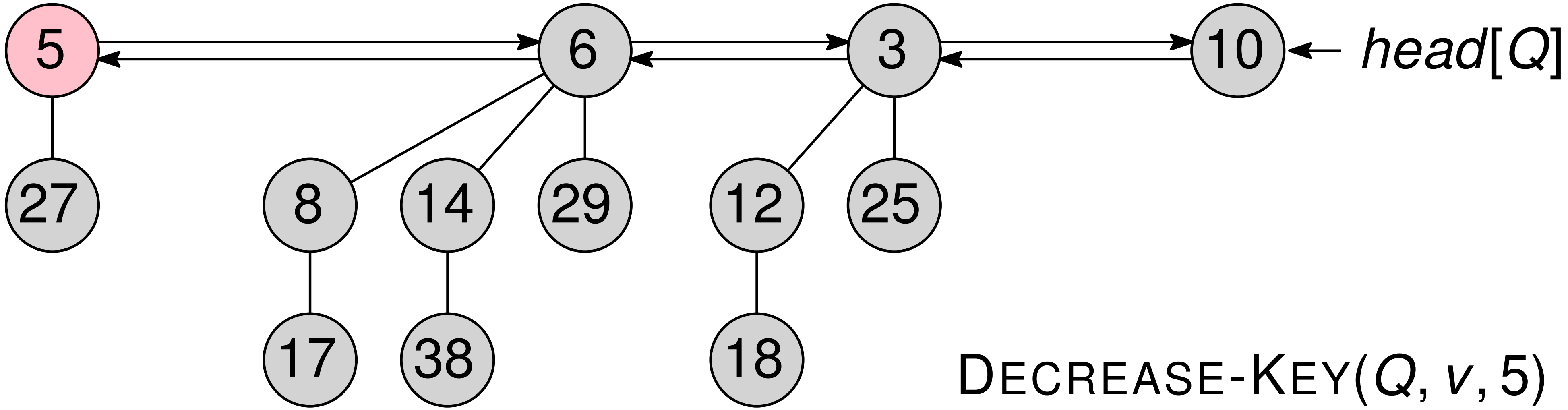



DECREASE-KEY($Q$, $v$, 5)

**Idea:** If the new key is smaller than parent's, DECREASE-KEY($Q$, $v$, $k$) will splice-out $v$ and add it to the root list

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time

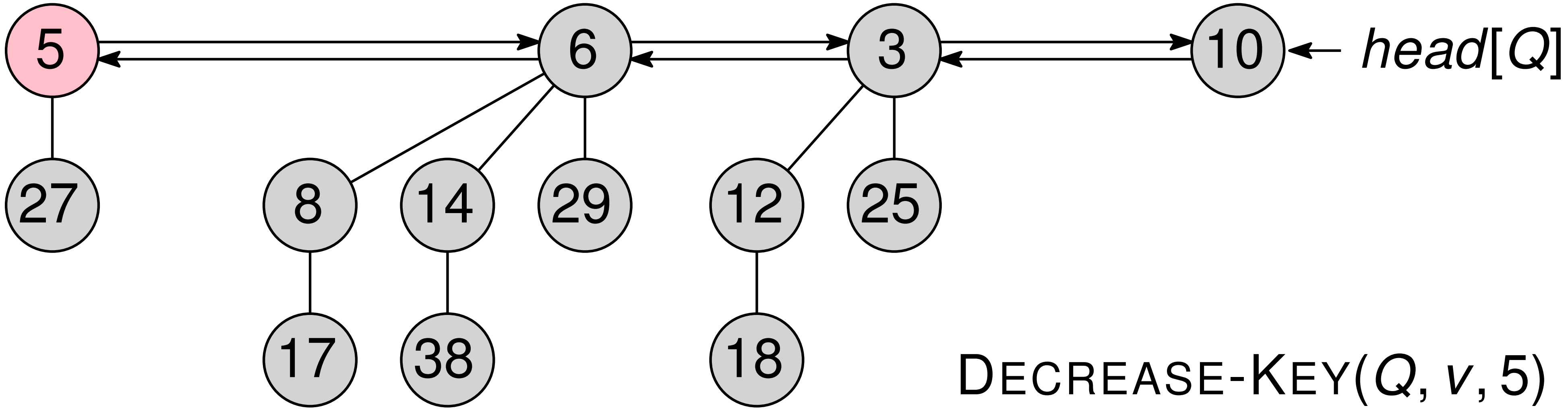



DECREASE-KEY($Q$, $v$, 5)

**Idea:** If the new key is smaller than parent's, DECREASE-KEY($Q$, $v$, $k$) will splice-out $v$ and add it to the root list

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time




DECREASE-KEY($Q$, $v$, 5)

**Idea:** If the new key is smaller than parent's, DECREASE-KEY($Q$, $v$, $k$) will splice-out $v$ and add it to the root list

Almost!

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q$, $v$, $k$) in $O(1)$ (amortized) time

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q, v, k$) in $O(1)$ (amortized) time

Collection of heap-ordered (binomial) trees:

- Each tree is heap-ordered
- Arbitrary number of trees in the root list
- Sibling lists are doubly-linked circular lists
- Each node $v$ stores
  - A pointer to parent ($v.parent$)
  - A pointer to one (arbitrary) child ($v.child$)

# Fibonacci Heaps

**Goal:** DECREASE-KEY($Q, v, k$) in $O(1)$ (amortized) time

Collection of heap-ordered (binomial) trees:

- Each tree is heap-ordered
- Arbitrary number of trees in the root list
- Sibling lists are doubly-linked circular lists
- Each node $v$ stores
  - A pointer to parent ($v.parent$)
  - A pointer to one (arbitrary) child ($v.child$)

Each node $v$:

- stores number of children ($v.degree$)
- can be “marked” (`boolean` $v.marked$)
  - If $v$ lost a child since becoming a child of another node
  - If so, next time it loses another child, it will be spliced out and added to root list (and will become unmarked)

# FIB-DECREASE-KEY($Q, x, k$)

**function** FIB-DECREASE-KEY($Q, v, k$) $\triangleright$ Assert $k < v.key$

$v.key = k$

**if** $v.parent \neq$ NIL **and** $v.key < v.parent.key$ **then**

RECURSIVE-CUT($Q, v, v.parent$)

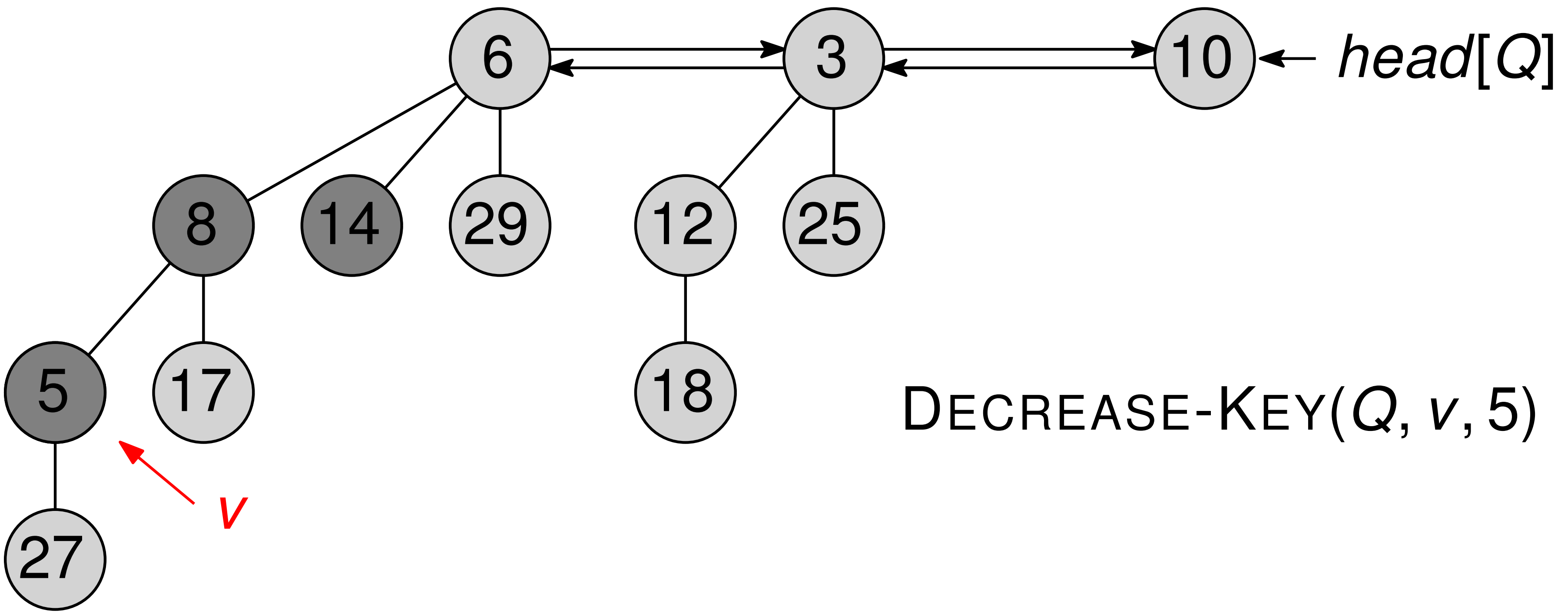
# FIB-DECREASE-KEY($Q, x, k$)

```
function FIB-DECREASE-KEY(Q, v, k)                    ▷ Assert k < v.key
    v.key = k
    if v.parent ≠ NIL and v.key < v.parent.key then
        RECURSIVE-CUT(Q, v, v.parent)

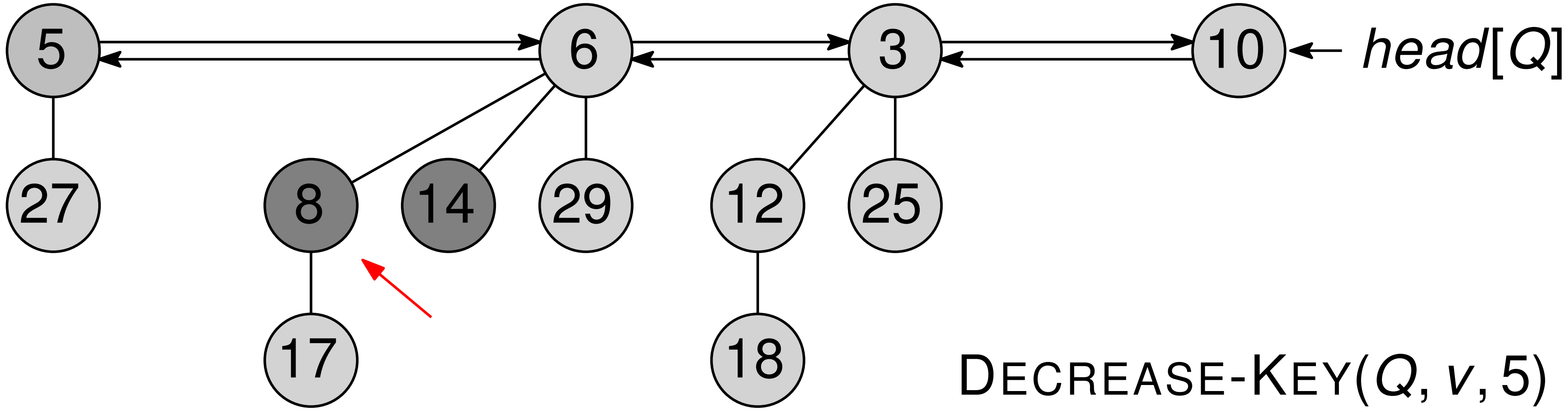
function RECURSIVE-CUT(Q, v, p)                       ▷ p is the parent of v
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL                    ▷ Unmark v

    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE                 ▷ p just lost a child, so mark it
        else                              ▷ p just lost the second child
            RECURSIVE-CUT(Q, p, p.parent) ▷ so add it to the root list too
```

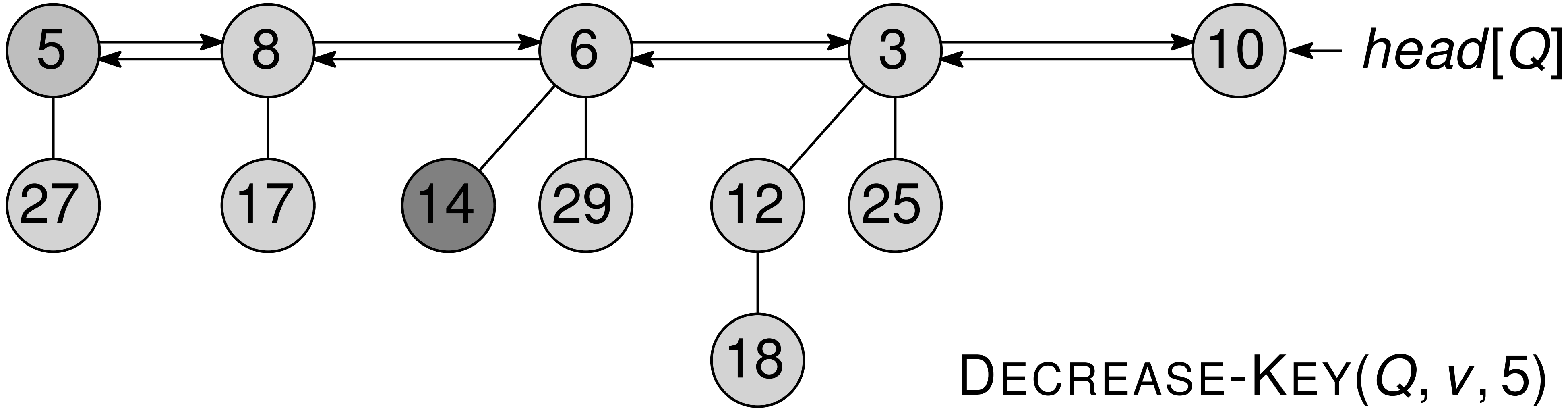
# Example: FIB-DECREASE-KEY($Q$, $v$, $k$)



# Example: FIB-DECREASE-KEY($Q$, $v$, $k$)



# Example: FIB-DECREASE-KEY($Q$, $v$, $k$)




DECREASE-KEY($Q$, $v$, 5)

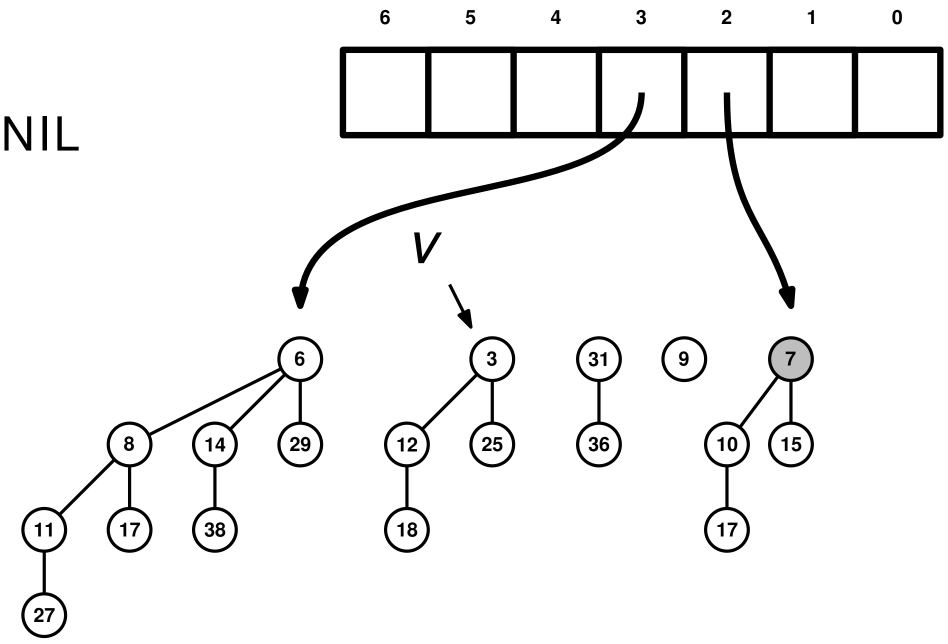
# Unmarking Vertices

Each node *v*:

- stores number of children (*v*.*degree*)
- can be "marked" (`boolean` *v*.*marked*)
  - If *v* lost a child since becoming a child of another node

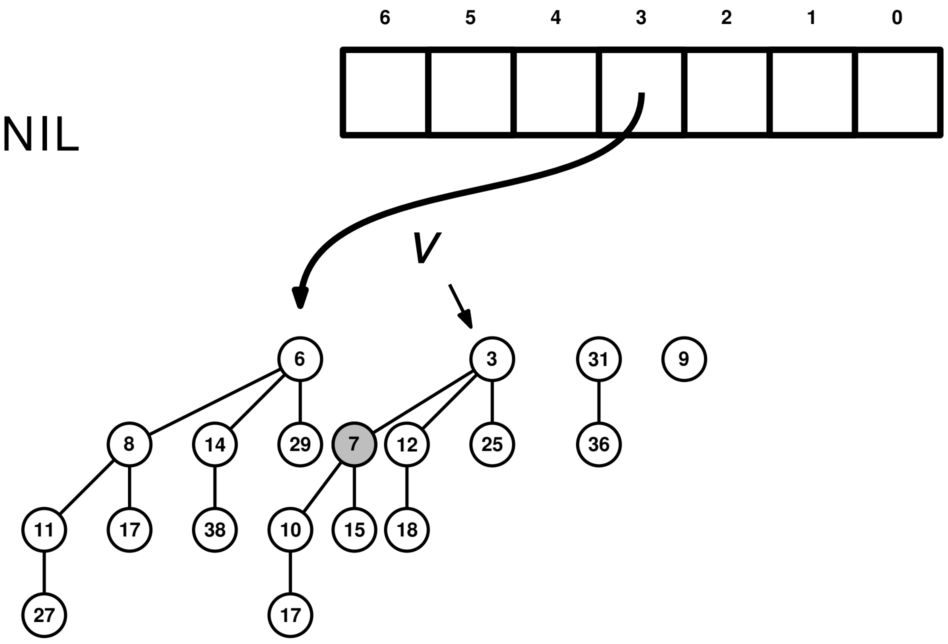
# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
  Initialize log $n$-sized array $A$ to NIL
  **for each** $v$ in root list **do**
    $d = v.degree$
    **while** $A[d] \neq$ NIL **do**
      $v =$ LINK($v, A[d]$)
      $A[d] =$ NIL
      $d = d + 1$
    $A[d] = v$; $v.parent =$ NIL



# CONSOLIDATE(*Q*)

**function** CONSOLIDATE(Q)
  Initialize log *n*-sized array *A* to NIL
  **for each** *v* in root list **do**
    $d = v.degree$
    **while** $A[d] \neq$ NIL **do**
      $v$ = LINK($v, A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL



# CONSOLIDATE($Q$)

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  **for each** $v$ in root list **do**
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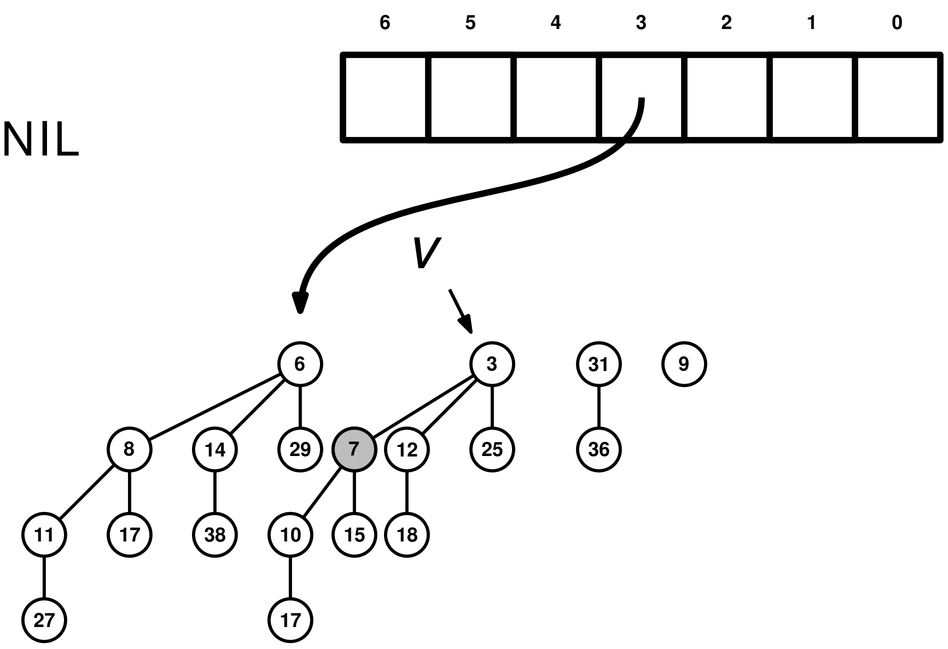
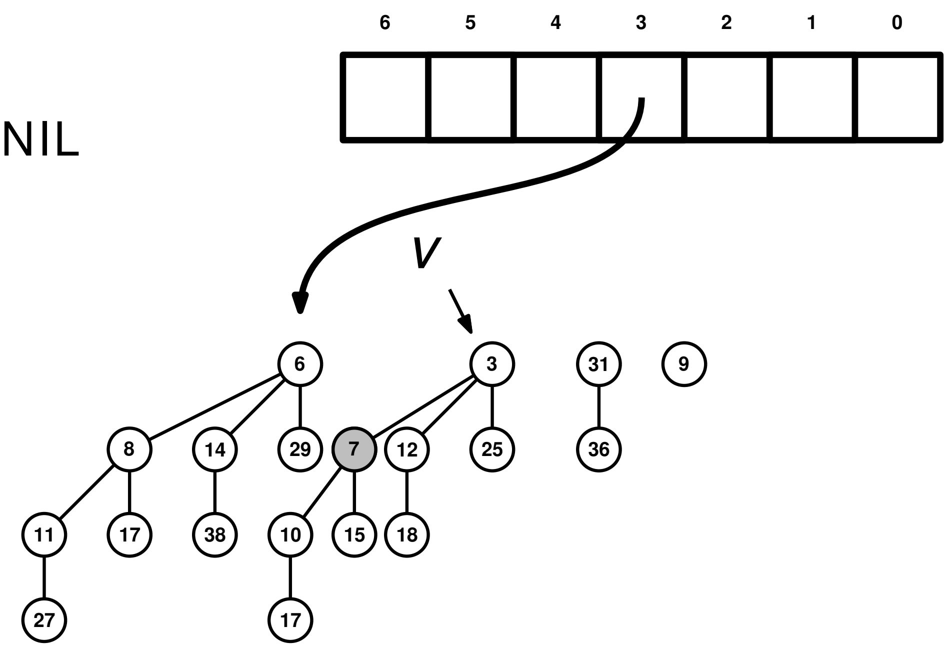
    **while** $A[d] \neq$ NIL **do**
      $v$ = LINK($v$, $A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL




**function** LINK($v$, $w$)
  **if** $w.key < v.key$ **then**
    SWAP($v$, $w$)      ▷ make sure $v$ is smaller
  Add $w$ to the child list of $v$
  $v.degree = v.degree + 1$
  $w.marked$ = FALSE

# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
  Initialize log $n$-sized array $A$ to NIL
  **for each** $v$ in root list **do**
    $d = v.degree$
    **while** $A[d] \neq$ NIL **do**
      $v$ = LINK($v$, $A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL




**function** LINK($v$, $w$)
  **if** $w.key < v.key$ **then**
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  Add $w$ to the child list of $v$
  $v.degree = v.degree + 1$
→ $w.marked$ = FALSE

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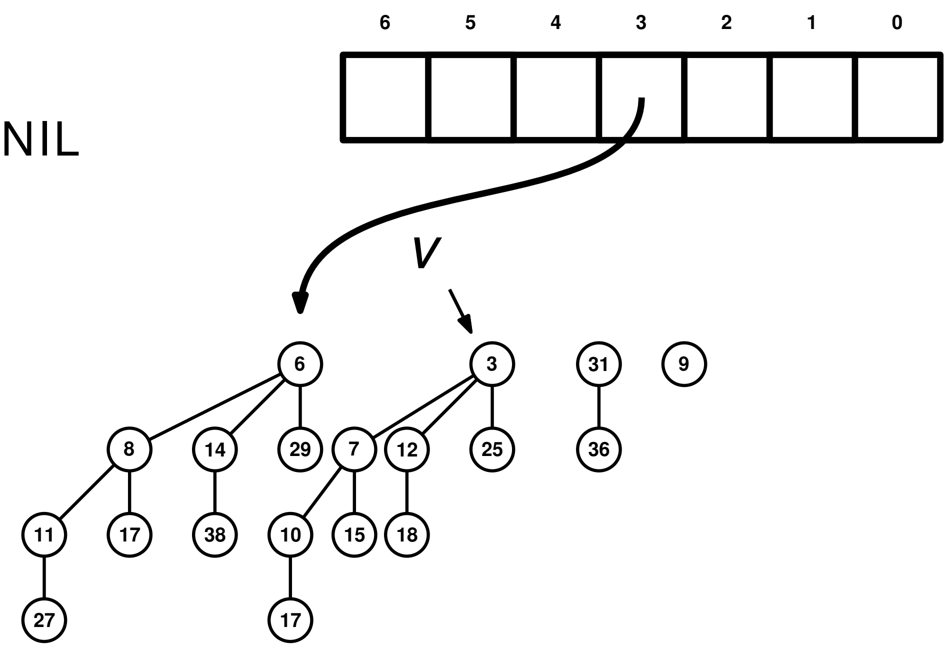
      $v$ = LINK($v$, $A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL

**function** LINK($v$, $w$)
  **if** $w.key < v.key$ **then**
    SWAP($v$, $w$)   ▷ make sure $v$ is smaller
  Add $w$ to the child list of $v$
  $v.degree = v.degree + 1$
→ $w.marked$ = FALSE



# FIB-DECREASE-KEY($Q$, $v$, $k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

# FIB-DECREASE-KEY($Q$, $v$, $k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

Let $t'$ be the number of trees added to the root list.

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

# FIB-DECREASE-KEY($Q$, $v$, $k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

Let $t'$ be the number of trees added to the root list.

- $t_i = t_{i-1} + t'$
- $m_i \leq m_{i-1} - (t' - 1)$

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

# FIB-DECREASE-KEY($Q$, $v$, $k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

Let $t'$ be the number of trees added to the root list.

- $t_i = t_{i-1} + t'$
- $m_i \leq m_{i-1} - (t' - 1)$

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

$$\Delta\Phi = k(t_i - t_{i-1}) + 2k(m_i - m_{i-1}) \leq k(t' + 2 \cdot (1 - t')) = k(2 - t')$$

# FIB-DECREASE-KEY($Q, v, k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

Let $t'$ be the number of trees added to the root list.

- $t_i = t_{i-1} + t'$
- $m_i \leq m_{i-1} - (t' - 1)$

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

$$\Delta\Phi = k(t_i - t_{i-1}) + 2k(m_i - m_{i-1}) \leq k(t' + 2 \cdot (1 - t')) = k(2 - t')$$

Actual cost: $c_i = O(t')$, i.e., $c_i \leq \bar{k} \cdot t'$ for some constant $\bar{k}$

# FIB-DECREASE-KEY($Q$, $v$, $k$) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- $t_i$ = # of root list trees
- $m_i$ = # of marked nodes
- $k$ = constant TBD later

Let $t'$ be the number of trees added to the root list.

- $t_i = t_{i-1} + t'$
- $m_i \leq m_{i-1} - (t' - 1)$

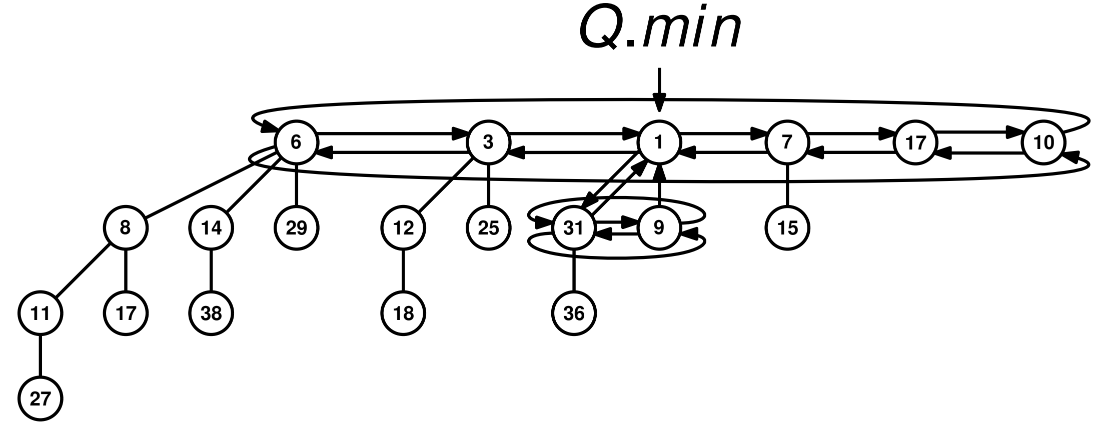
```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL
    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE
        else
            RECURSIVE-CUT(Q, p, p.parent)
```

$$\Delta\Phi = k(t_i - t_{i-1}) + 2k(m_i - m_{i-1}) \leq k(t' + 2 \cdot (1 - t')) = k(2 - t')$$

Actual cost: $c_i = O(t')$, i.e., $c_i \leq \bar{k} \cdot t'$ for some constant $\bar{k}$

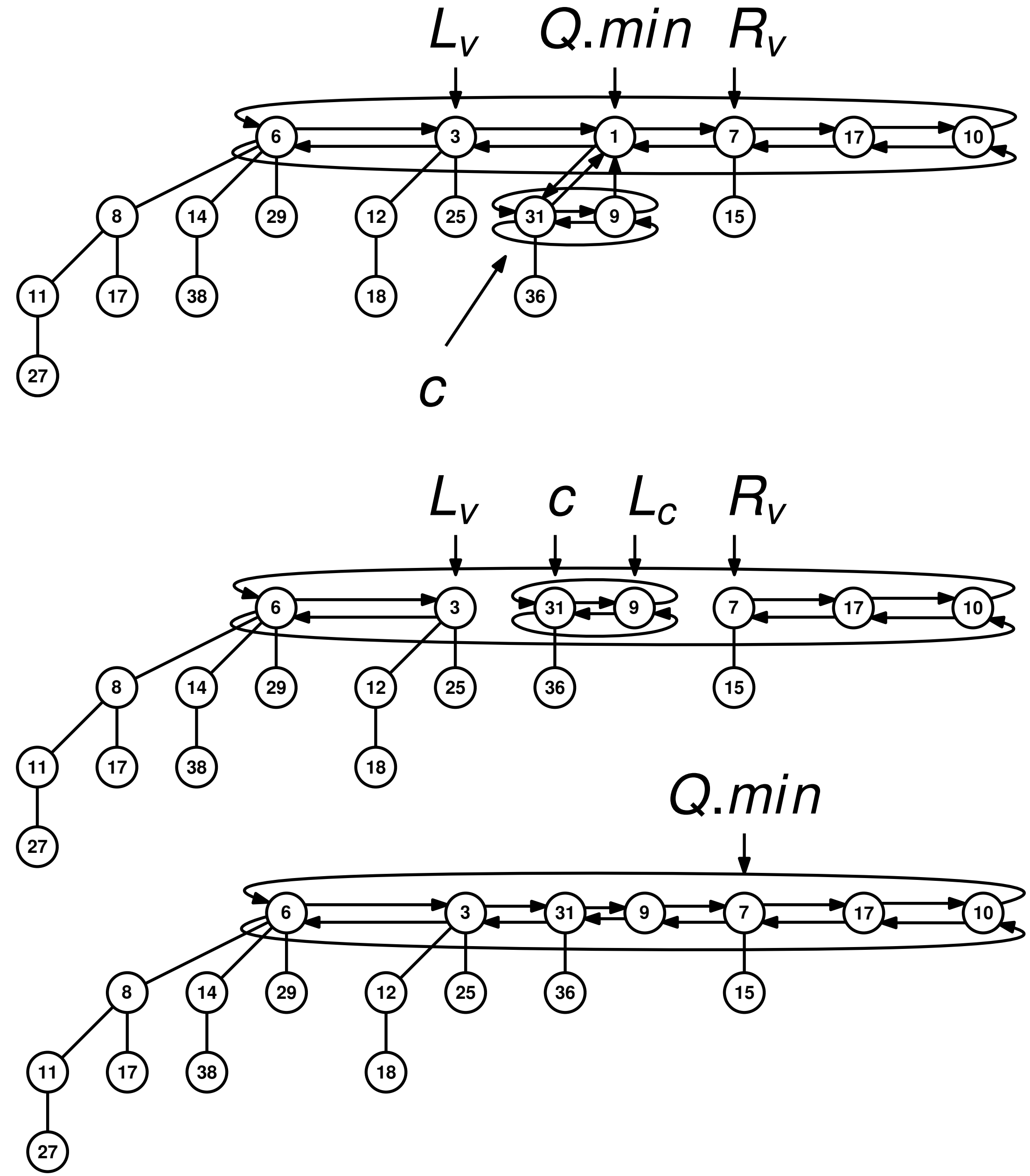
By setting $k = \bar{k}$, we get:
$\hat{c}_i = c_i + \Delta\Phi \leq \bar{k} \cdot t' + \bar{k}(2 - t') = 2\bar{k} = O(1)$

# EXTRACT-MIN(*Q*)



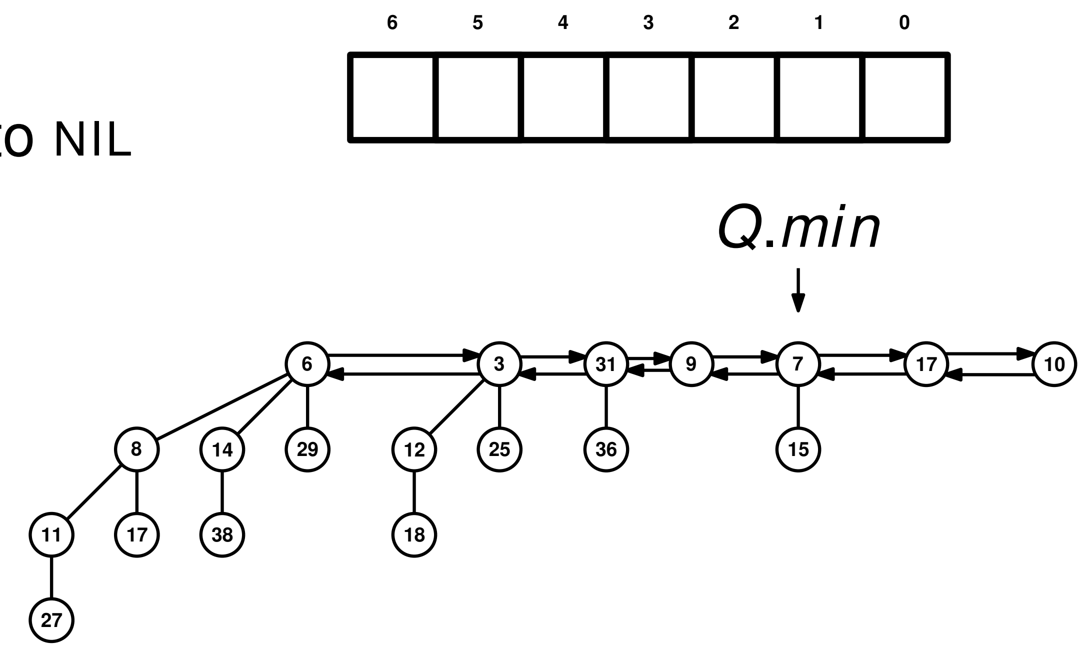
# EXTRACT-MIN($Q$)

**function** EXTRACT-MIN($Q$)
  $v$ = MINIMUM($Q$)
  Extract $v$ from root list
  Add $v$'s children into root list
  CONSOLIDATE($Q$)
  **return** $v$



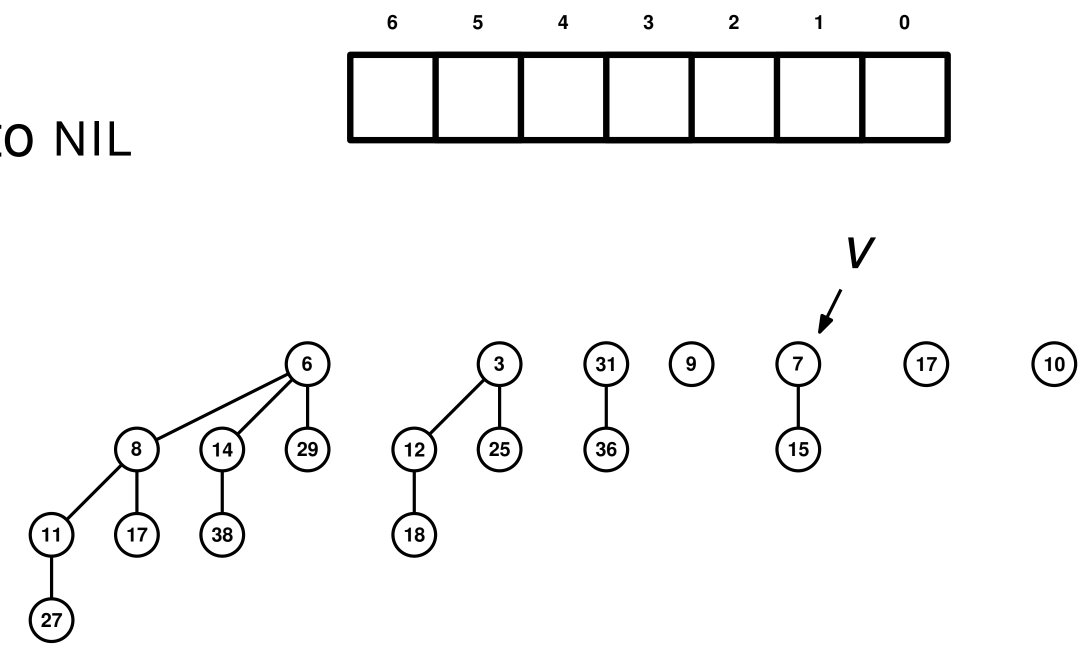
# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
    Initialize $O(\log n)$-sized array $A$ to NIL
    **for each** $v$ in root list **do**
        $d = v.degree$
        **while** $A[d] \neq$ NIL **do**
            $v$ = LINK($v$, $A[d]$)
            $A[d]$ = NIL
            $d = d + 1$
        $A[d] = v$; $v.parent$ = NIL



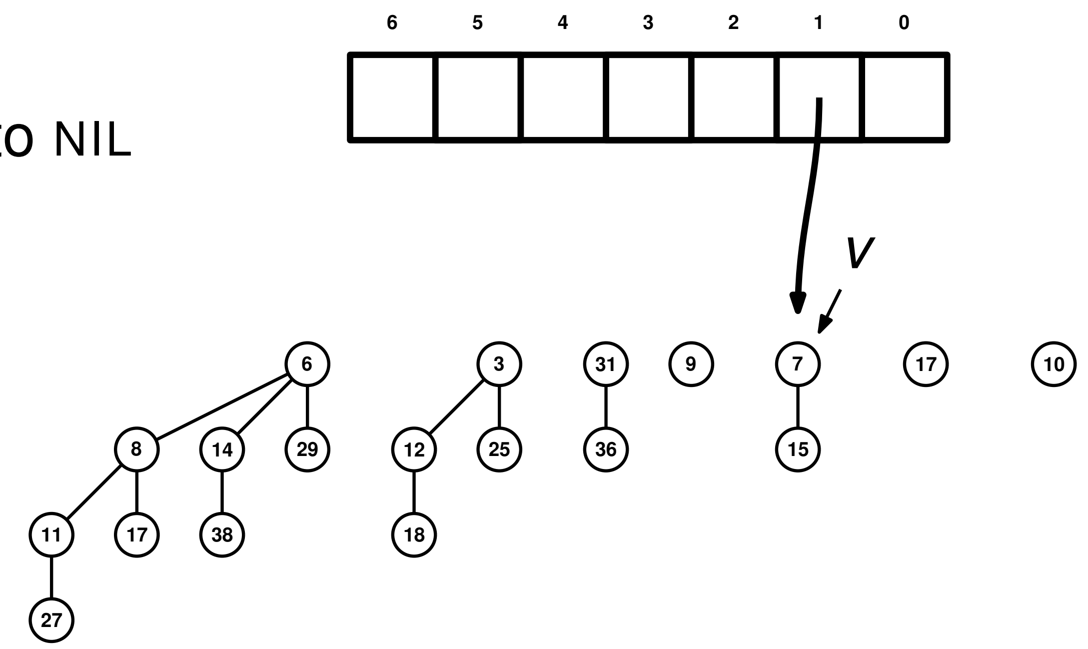
# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
  Initialize $O(\log n)$-sized array $A$ to NIL
  → **for each** $v$ in root list **do**
    $d = v.degree$
    **while** $A[d] \neq$ NIL **do**
      $v$ = LINK($v$, $A[d]$)
      $A[d]$ = NIL
      $d = d + 1$
    $A[d] = v$; $v.parent$ = NIL



# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
    Initialize $O(\log n)$-sized array $A$ to NIL
    **for each** $v$ in root list **do**
        $d = v.degree$
        **while** $A[d] \neq$ NIL **do**
            $v$ = LINK($v$, $A[d]$)
            $A[d]$ = NIL
            $d = d + 1$
→      $A[d] = v$; $v.parent$ = NIL



# CONSOLIDATE($Q$)

**function** CONSOLIDATE(Q)
  Initialize $O(\log n)$-sized array $A$ to NIL
  **for each** $v$ in root list **do**
    $d = v.degree$
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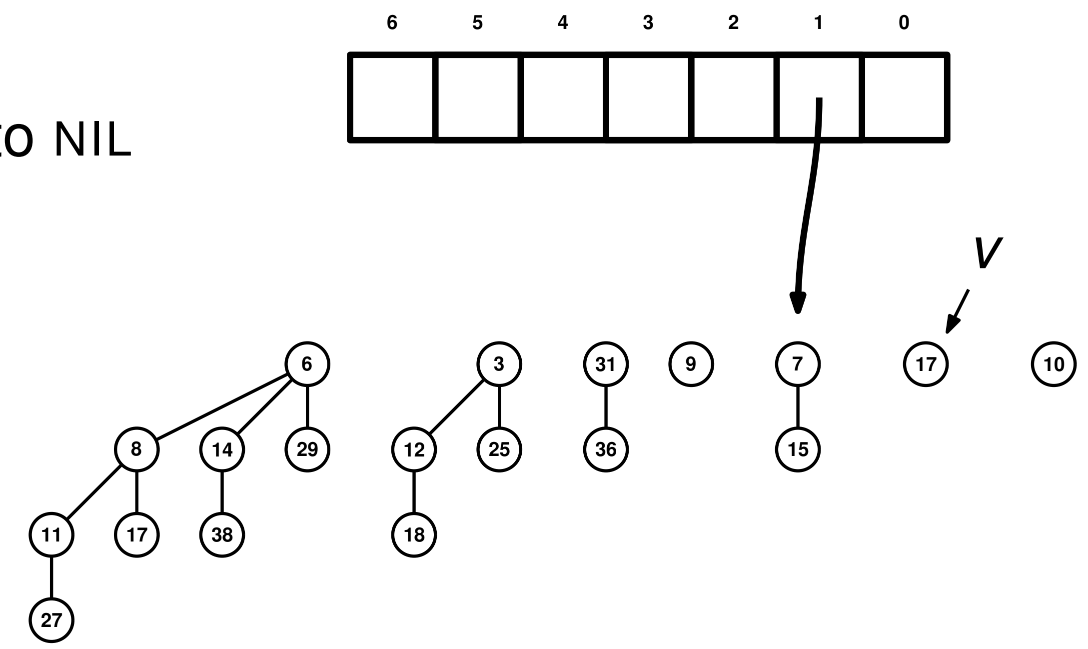

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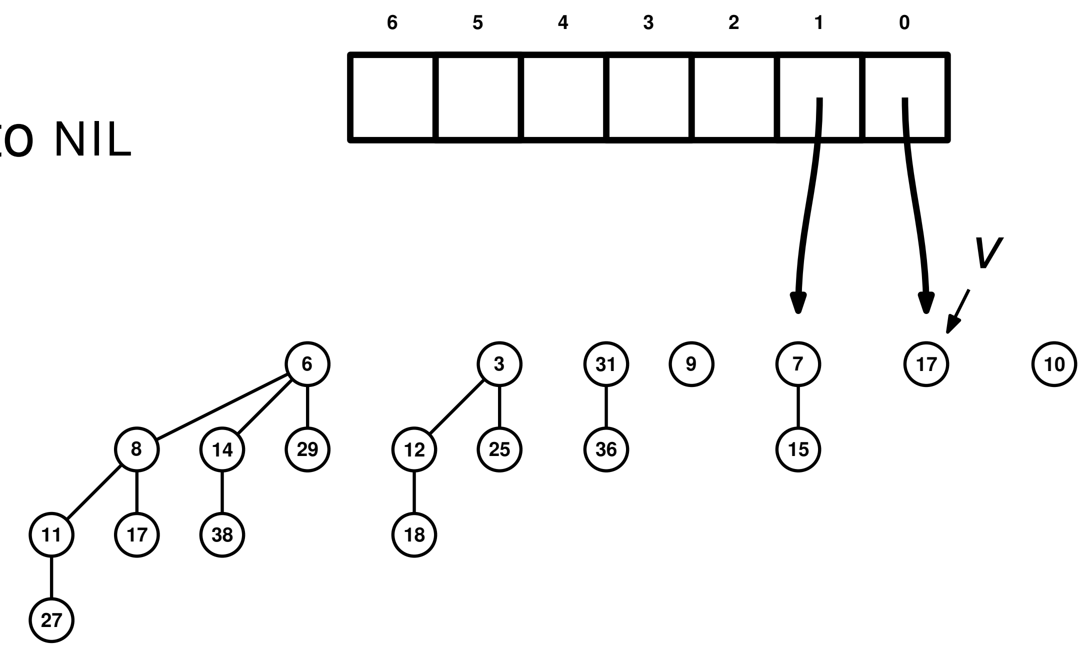

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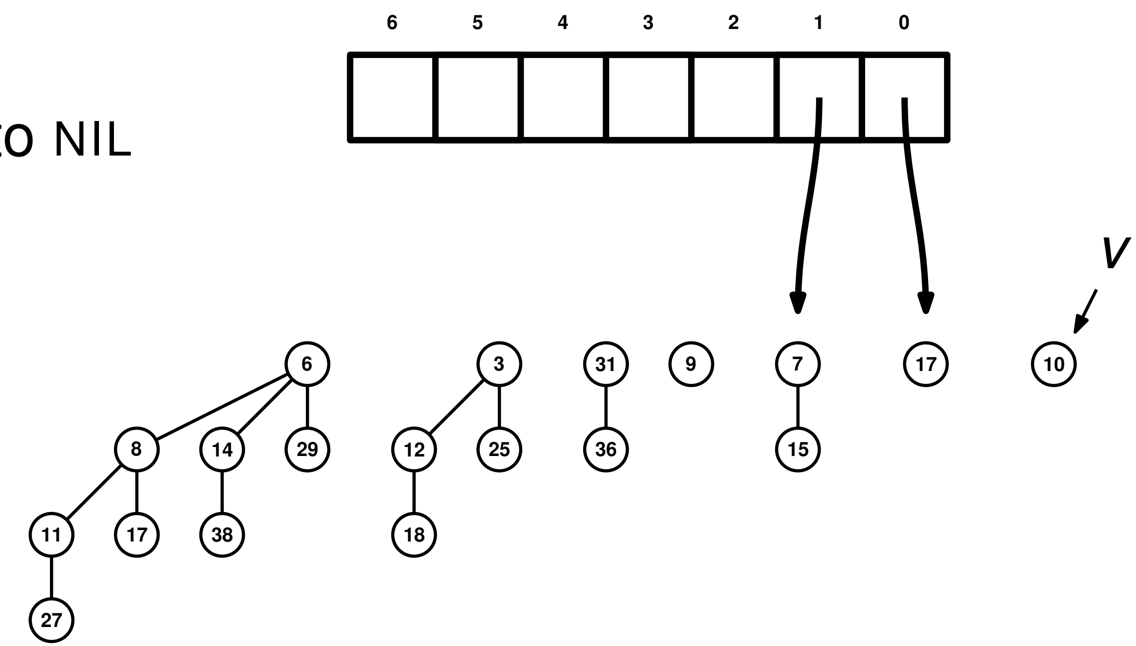

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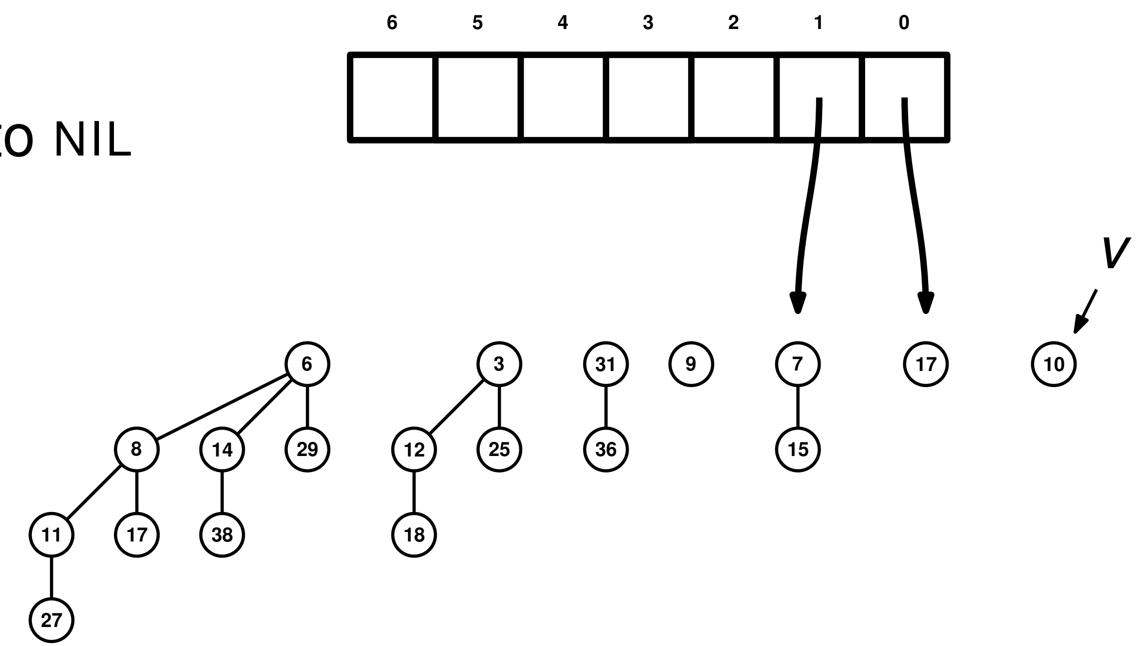

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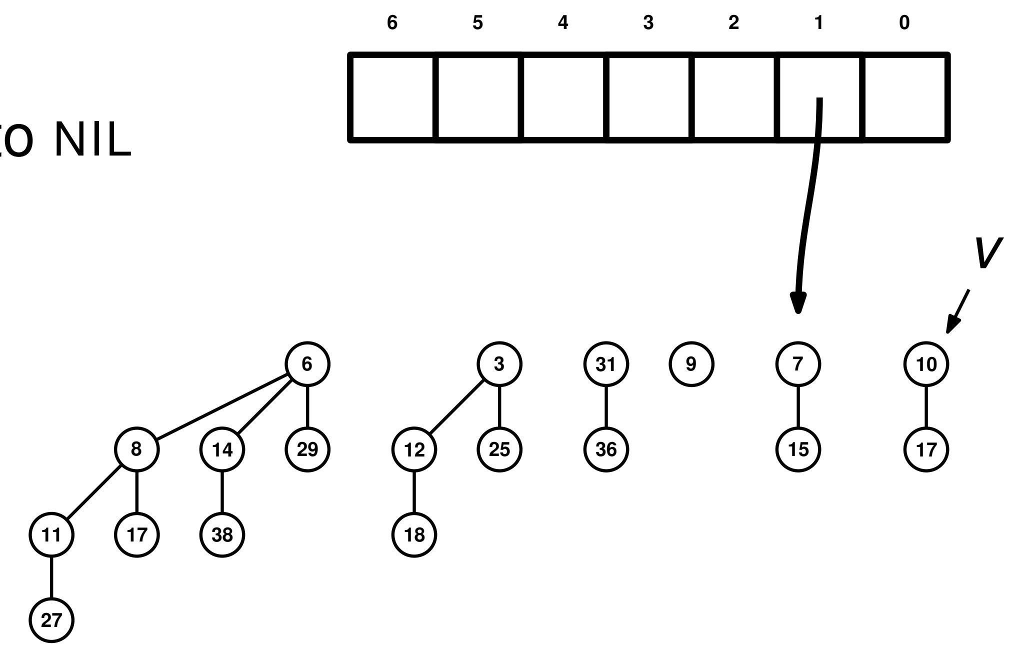

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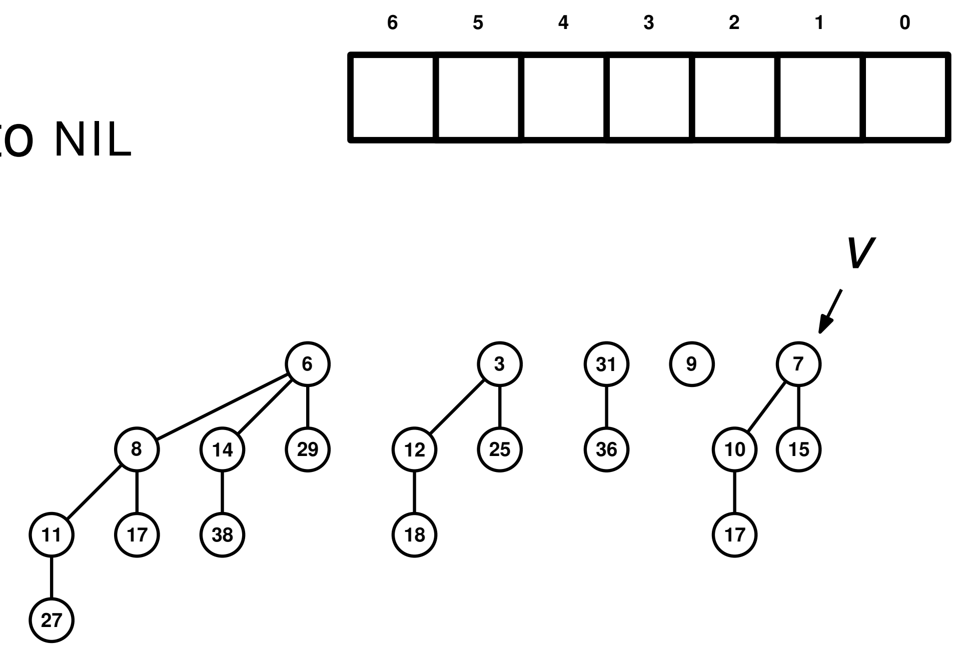

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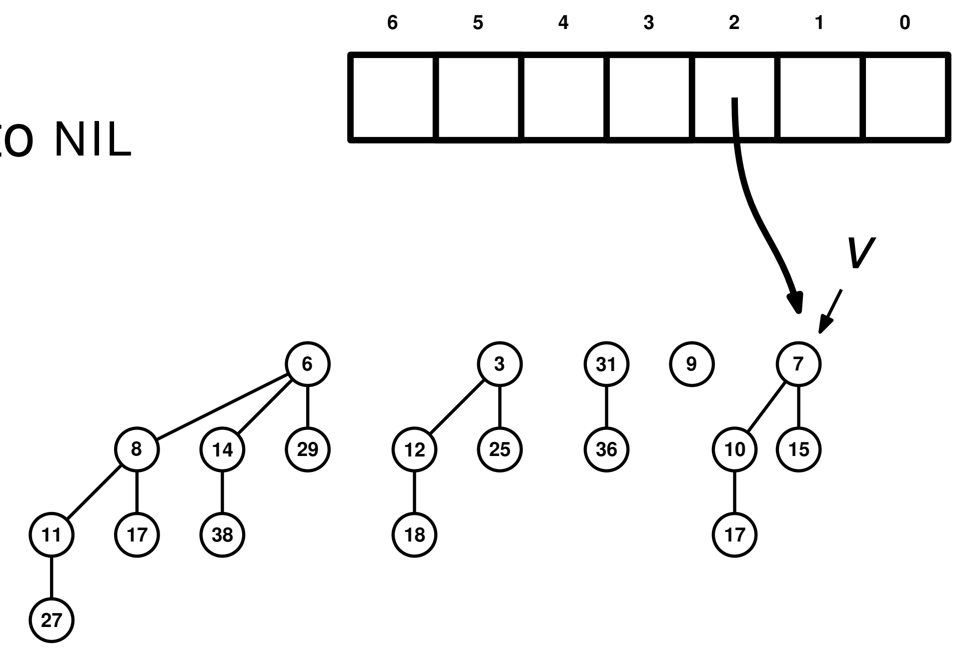

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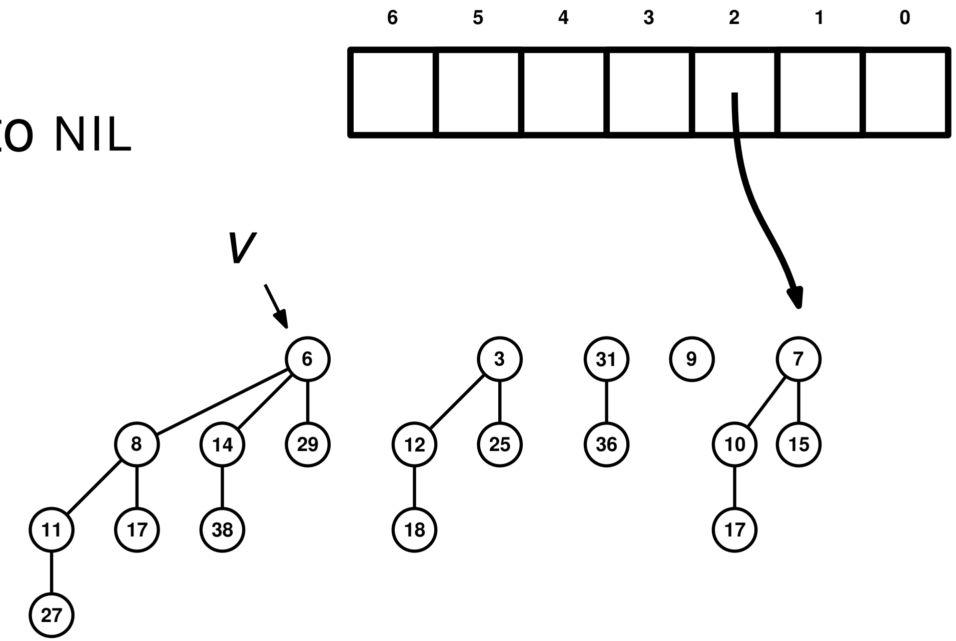

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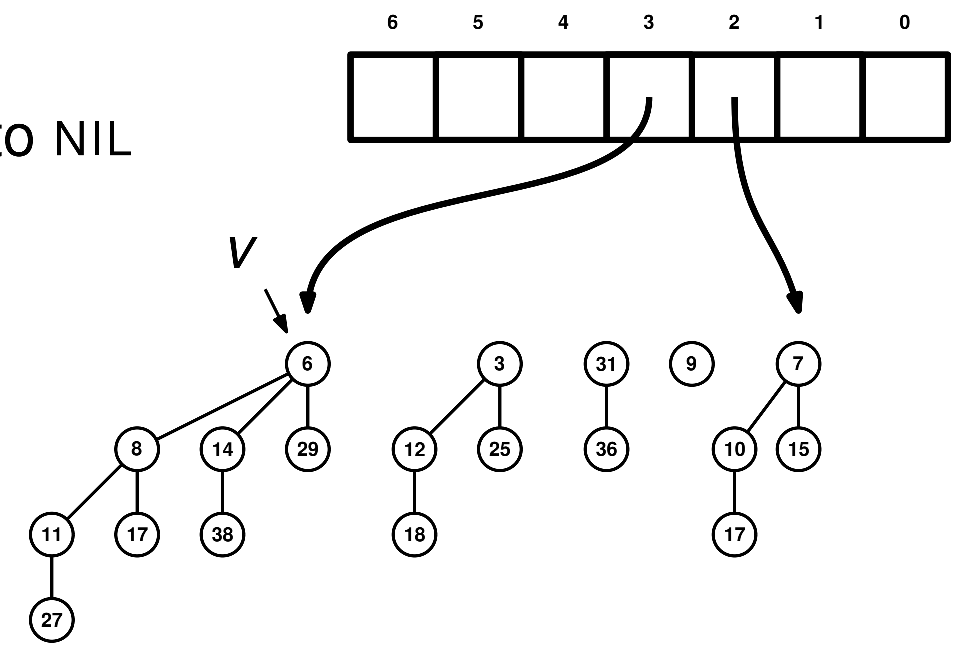

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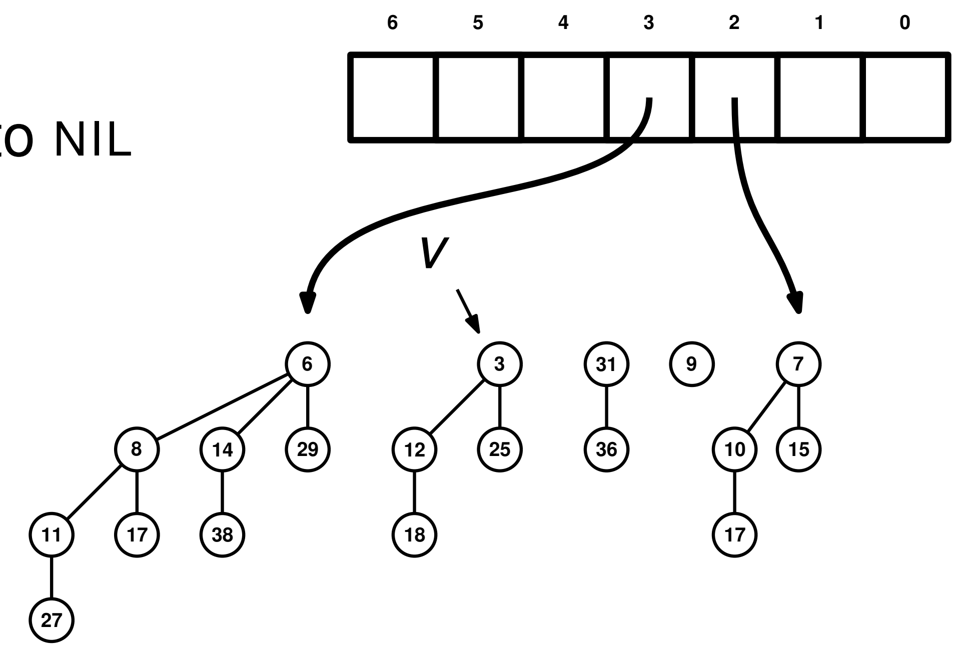

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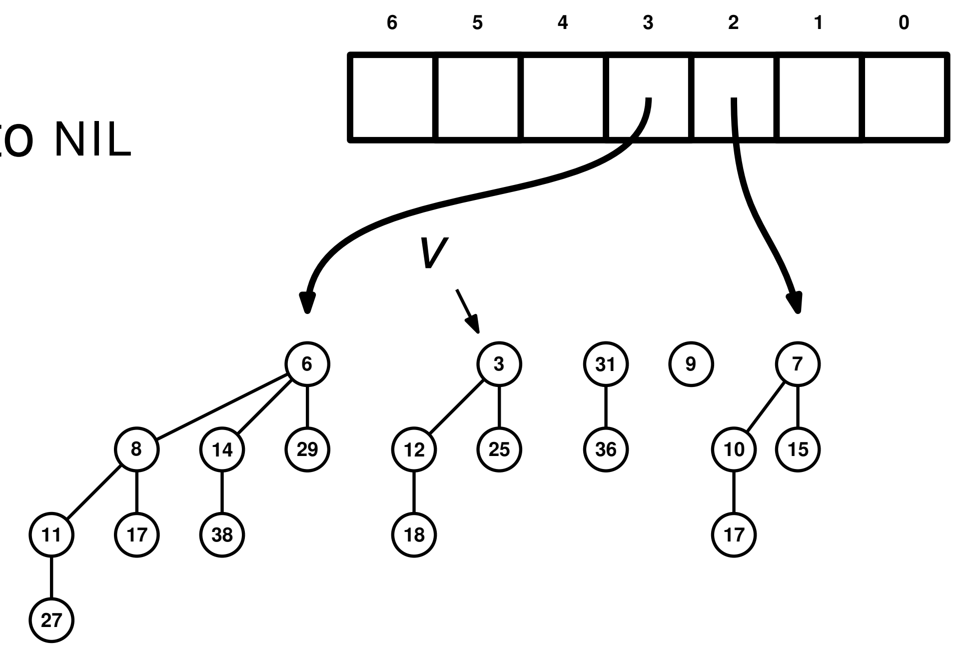

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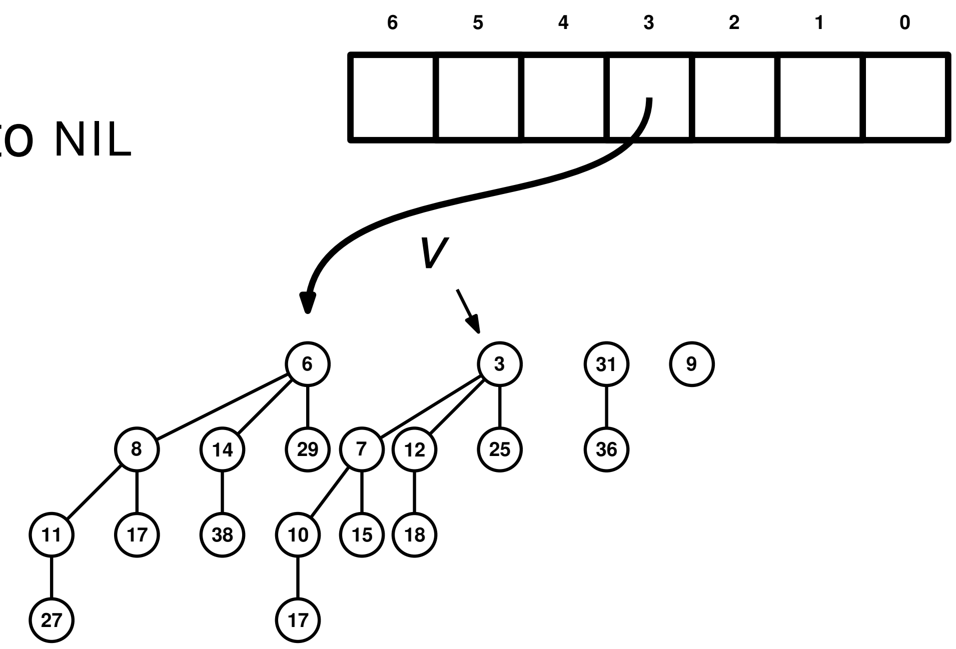

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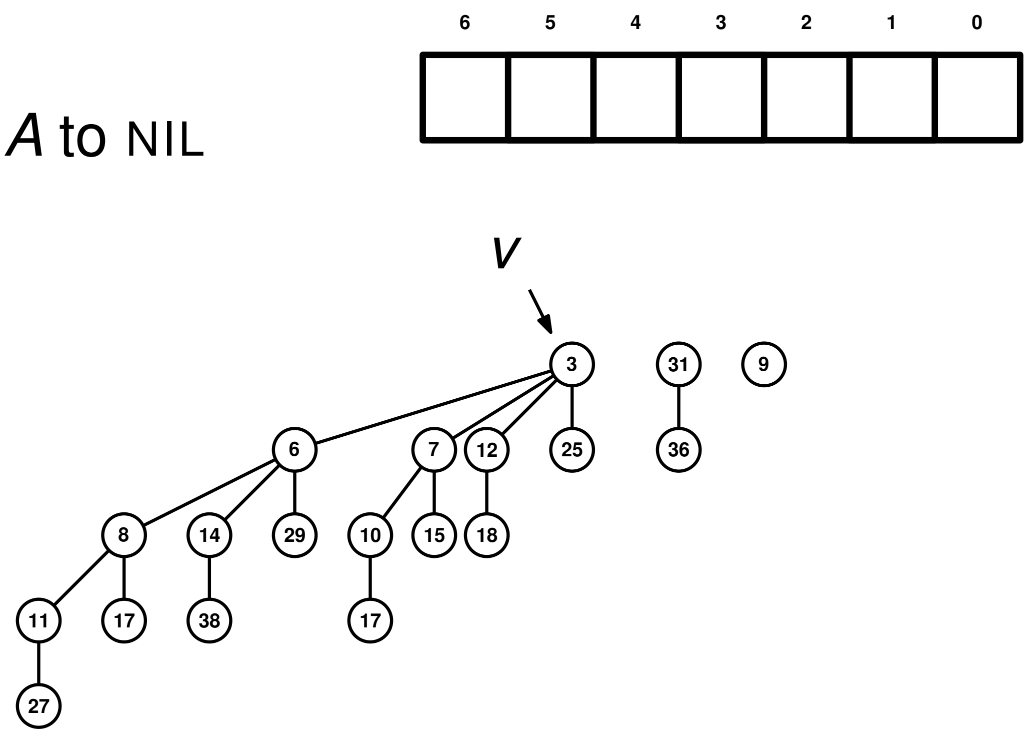

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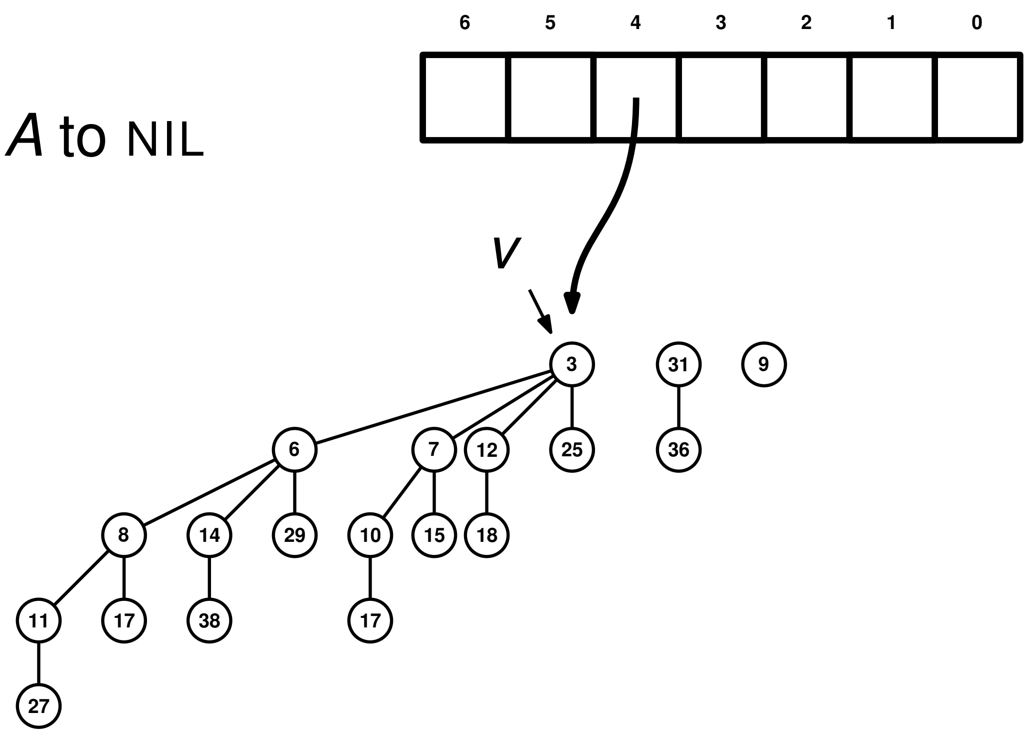

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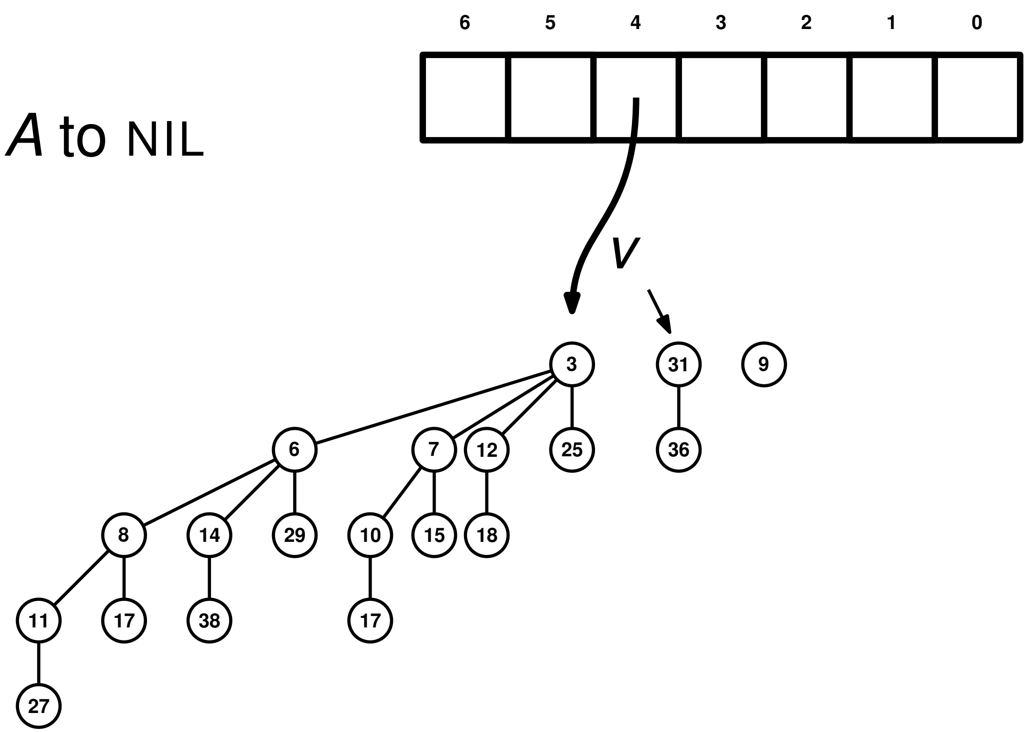

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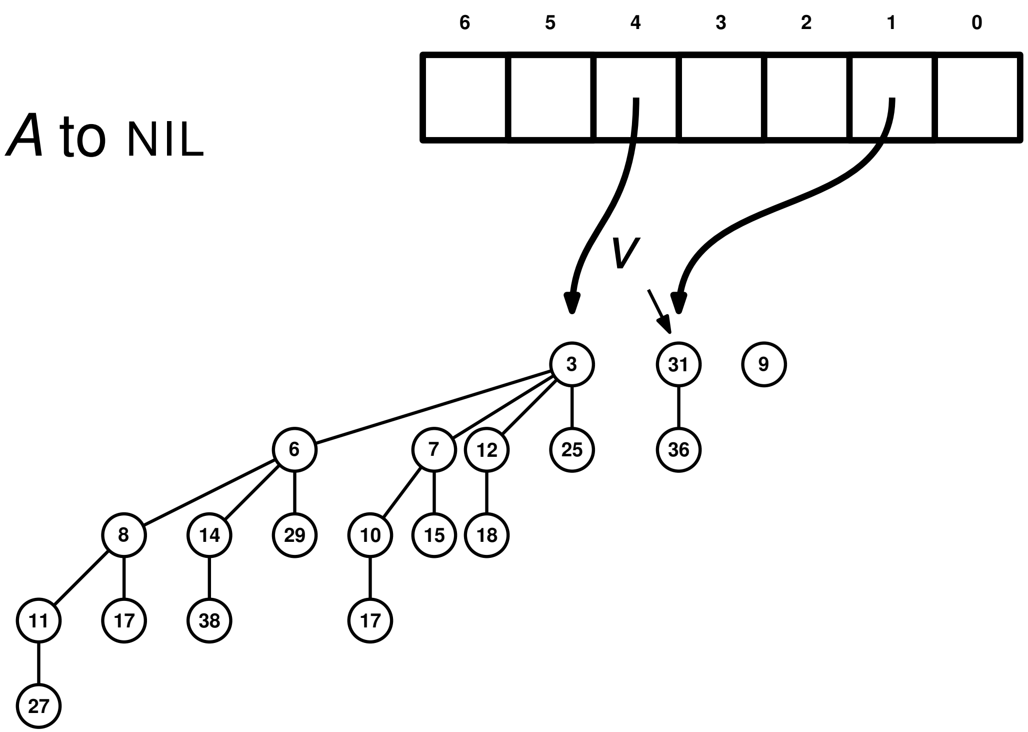

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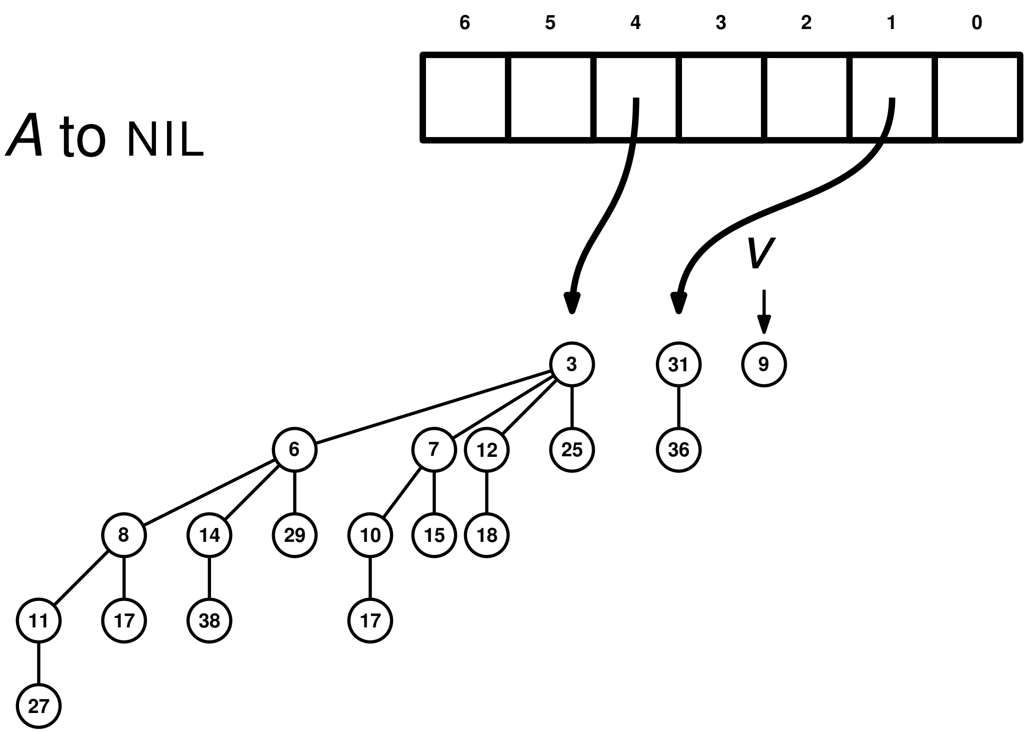

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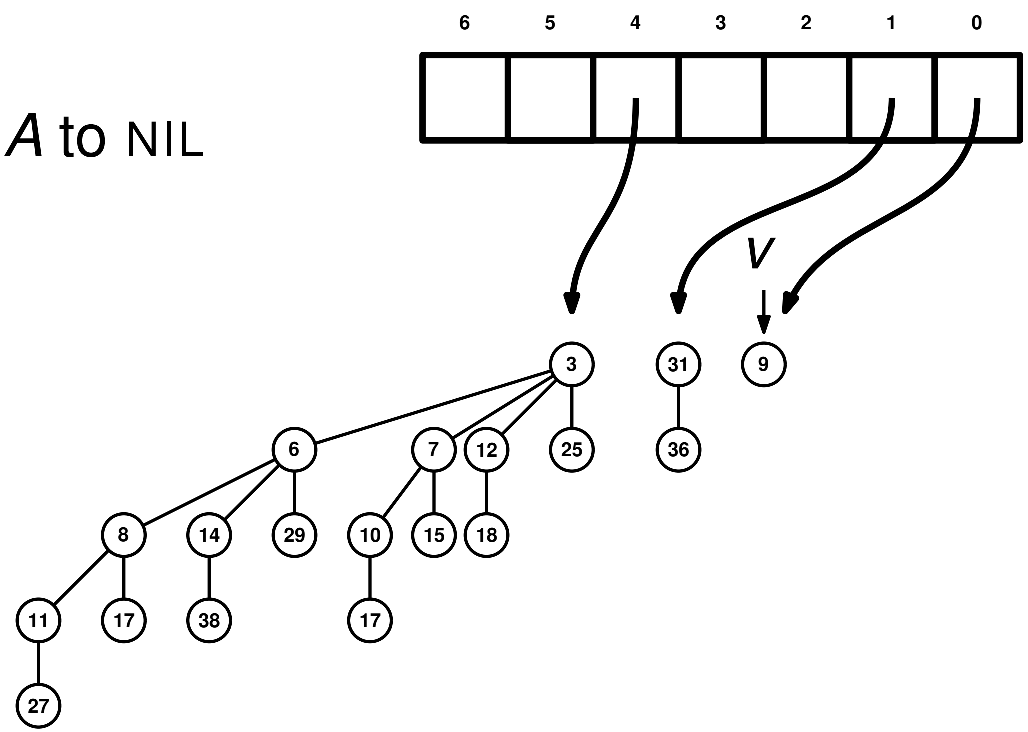

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  $min = +\infty$
  **for** $i = 0$ **to** $|A| - 1$ **do**
    **if** $A[i] \neq$ NIL **then**
      Add $A[i]$ to the root list
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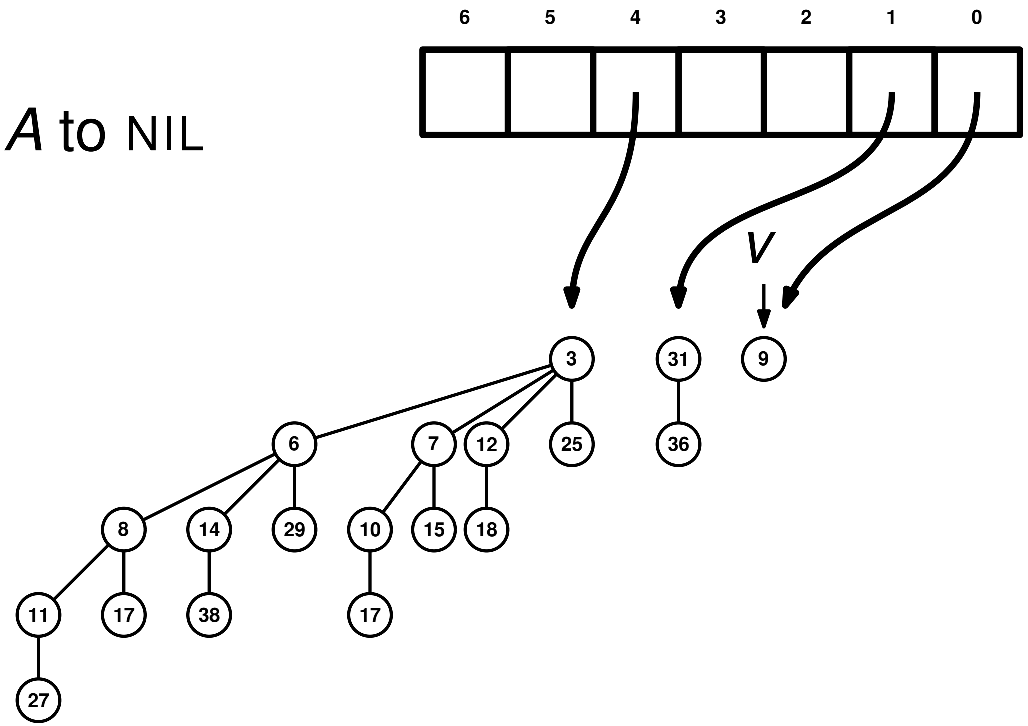

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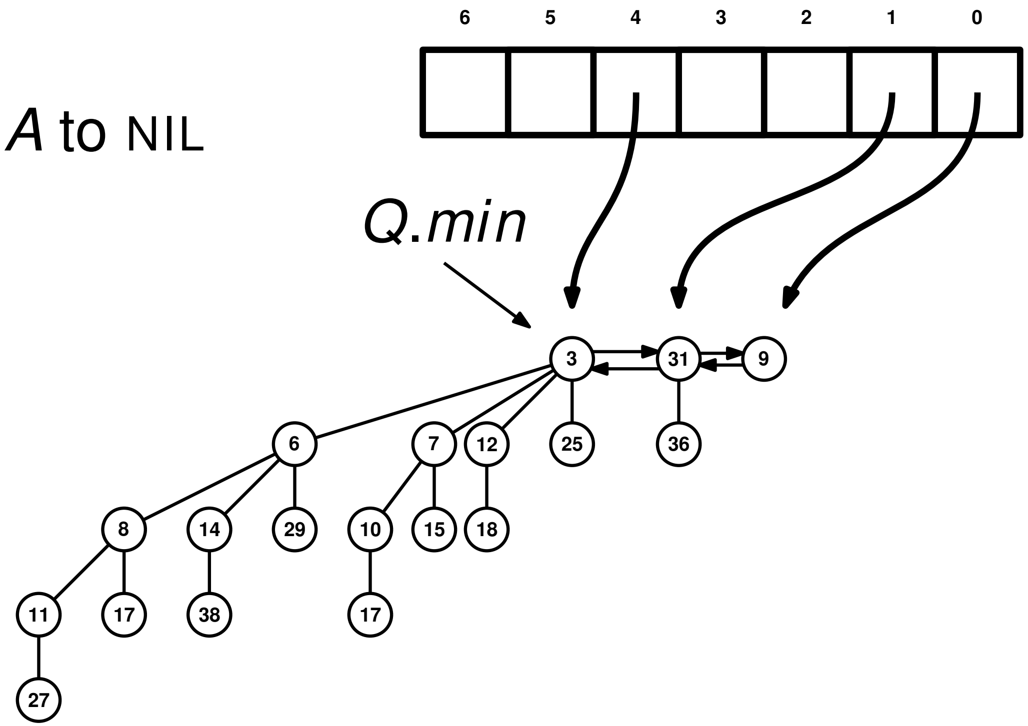

# Bounding maximum degree

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**Lemma 1.** *Let $k = v.degree$. Then for every node $v$ in Fibonacci heap,* $\textsc{size}(v) \geq F_{k+2} \geq \phi^k$, *where* $\phi = \frac{1+\sqrt{5}}{2}$

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**Corollary 1.** $k \leq \log_\phi n$

# Analysis of EXTRACT-MIN($Q$)

Potential: $\Phi_i = \bar{k}(t_i + 2m_i)$

- $t_i$ = # of root list trees
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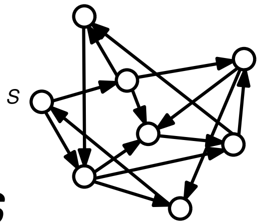
for any $\bar{k} \geq 1$

because there are $t_i \leq \log_\phi n$ trees in the root list after consolidation

# Application: Single Source Shortest Paths

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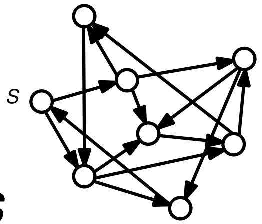


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Dijkstra's Algorithm

Use prioirity queue using

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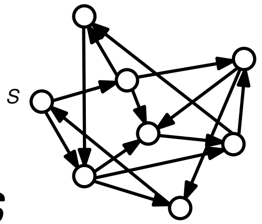


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## Dijkstra's Algorithm

| Use prioiritiy queue using | Binary | Binomial | Fibonacci |
|---|---|---|---|
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| $n$ calls to EXTRACT-MIN | $O(\log n)$ | $O(\log n)$ | $O(\log n)^*$ |
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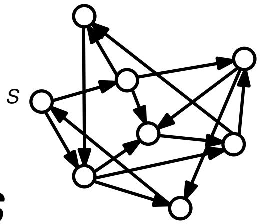


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Binary Binomial Fibonacci

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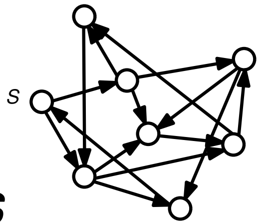


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|---|---|---|
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