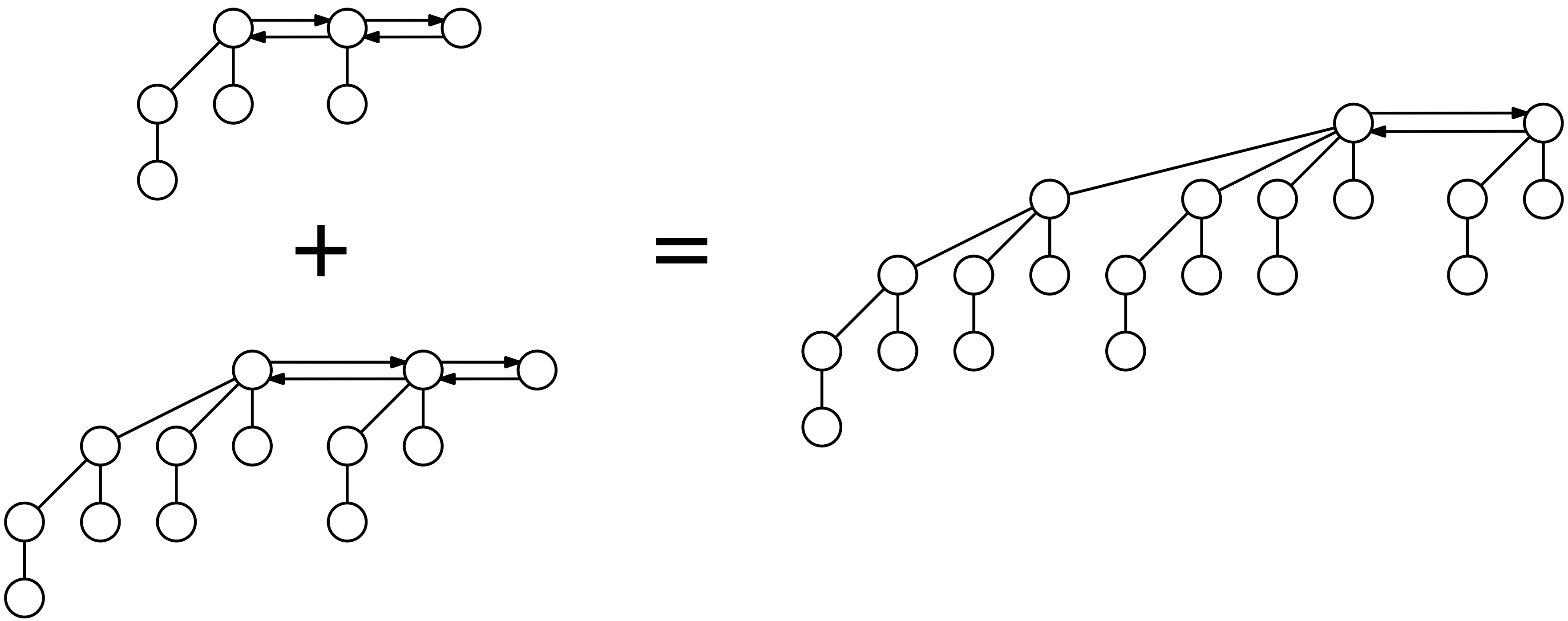
INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

AlgoPARC

ICS 621: Analysis of Algorithms

Prof. Nodari Sitchinava



Mergeable Priority Queues: Binomial Heaps

Priority Queue

PQ Abstract Data Type (ADT):

- MAKE()
- INSERT(Q, x)
- MINIMUM(Q)
- EXTRACT-MIN(Q)
- DECREASE-KEY(Q, x, k)
- DELETE(Q, x)

Priority Queue

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- UNION(Q_1, Q_2)

Binary Heap (from ICS 311)

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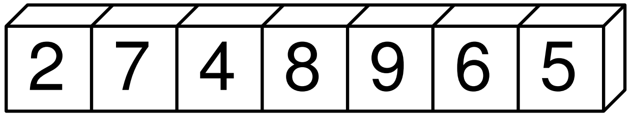

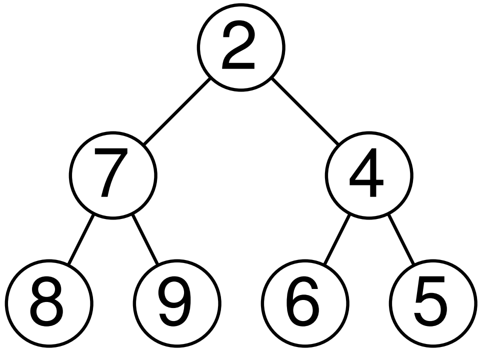

Binary Heap (from ICS 311)

- MAKE() $O(1)$
- INSERT(Q, x) $O(\log n)$
- MINIMUM(Q) $O(1)$
- EXTRACT-MIN(Q) $O(\log n)$
- DECREASE-KEY(Q, x, k) $O(\log n)$
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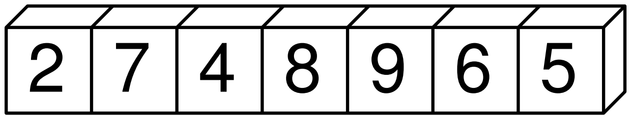

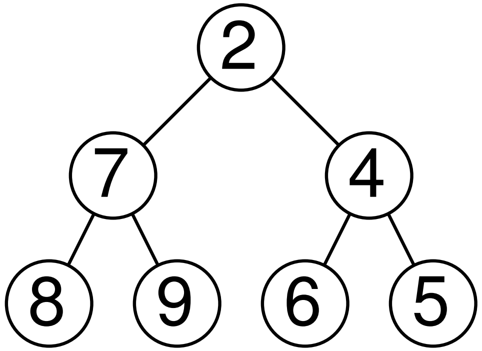

Binary Heap (from ICS 311)

■ MAKE()	$O(1)$
■ INSERT(Q, x)	$O(\log n)$
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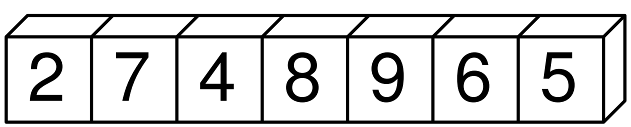
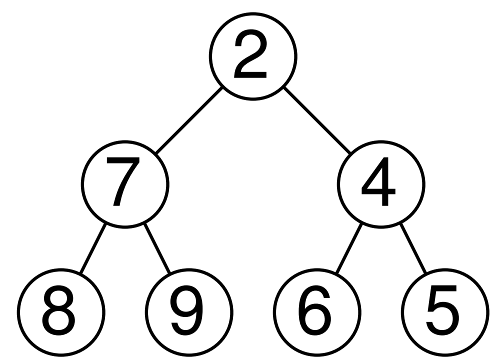
■ DECREASE-KEY(Q, x, k)	$O(\log n)$
■ DELETE(Q, x)	$O(\log n)$
■ UNION(Q_1, Q_2)	$O(n)$





Today: Binomial Heap

- MAKE()
- INSERT(Q, x)
- MINIMUM(Q)
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- DECREASE-KEY(Q, x, k)
- DELETE(Q, x)

- UNION(Q_1, Q_2)

Today: Binomial Heap

	Standard
■ MAKE()	$O(1)$
■ INSERT(Q, x)	~~$O(\log n)$~~ $O(1)^*$
■ MINIMUM(Q)	$O(1)$
■ EXTRACT-MIN(Q)	$O(\log n)$
■ DECREASE-KEY(Q, x, k)	$O(\log n)$
■ DELETE(Q, x)	$O(\log n)$
■ UNION(Q_1, Q_2)	~~$O(n)$~~ $O(\log n)$

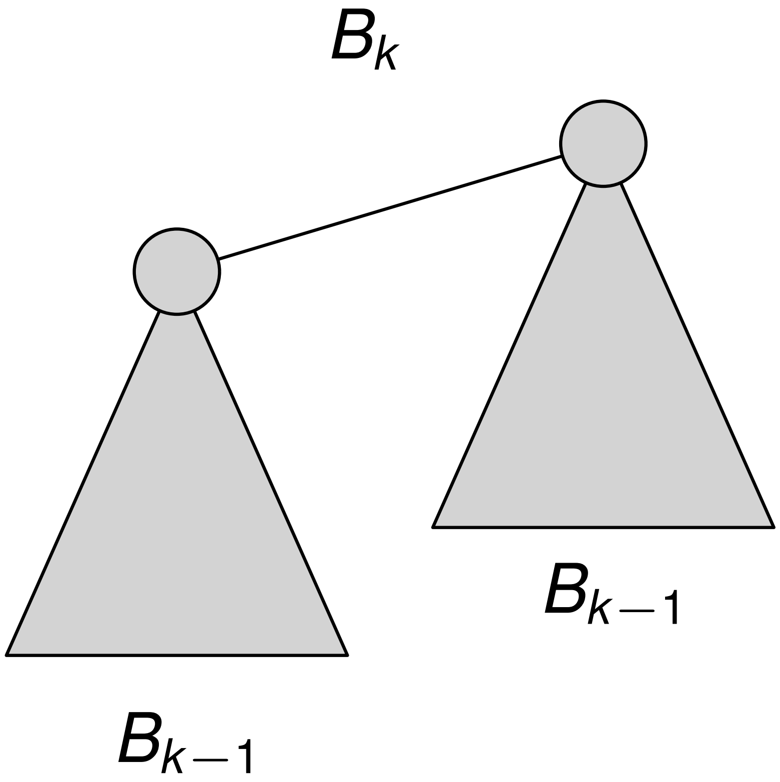
* Amortized cost

Today: Binomial Heap

	Standard	Lazy
MAKE()	$O(1)$	
INSERT(Q, x)	~~$O(\log n)$~~ $O(1)^*$	$O(1)$
MINIMUM(Q)	$O(1)$	$O(1)$
EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)^*$
DECREASE-KEY(Q, x, k)	$O(\log n)$	
DELETE(Q, x)	$O(\log n)$	$O(\log n)^*$
UNION(Q_1, Q_2)	~~$O(n)$~~ $O(\log n)$	$O(1)$

* Amortized cost

Reminder: Binomial Tree



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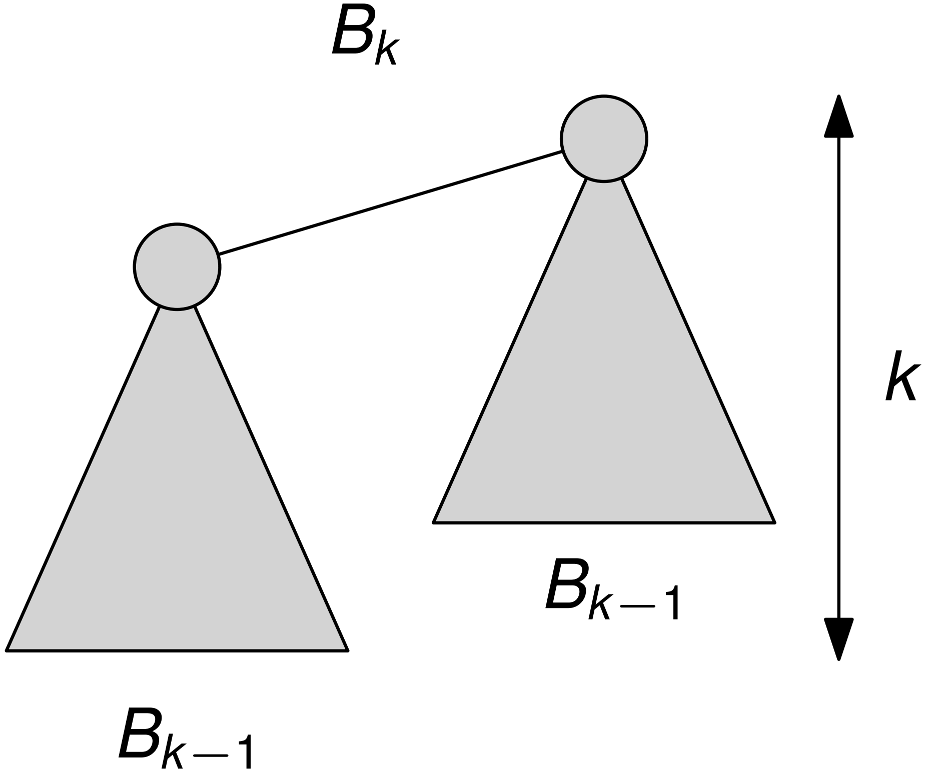
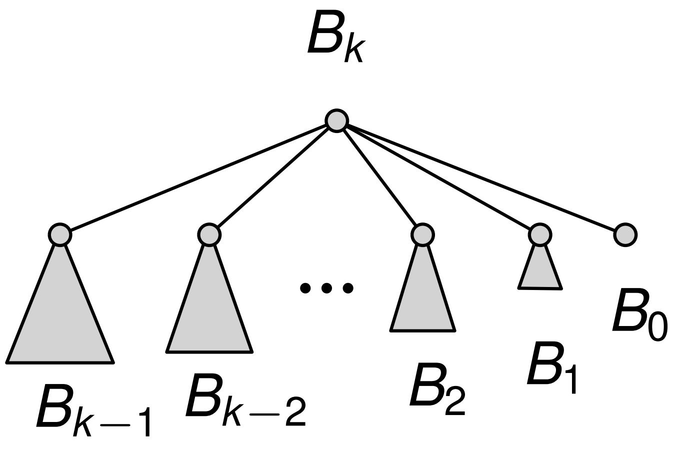
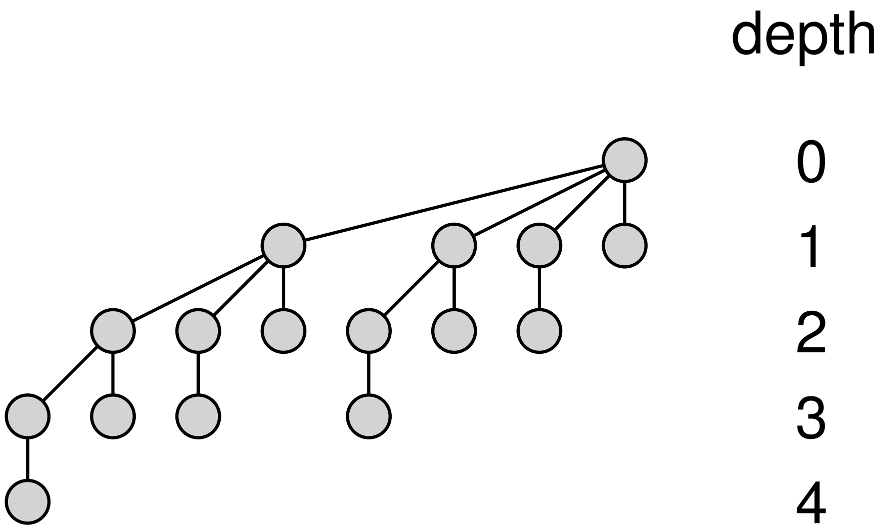


1. $size$ = 2^k nodes
2. $height$ = k
3. $\binom{k}{i}$ nodes at depth i
4. $degree(root)$ = k



Tree representation



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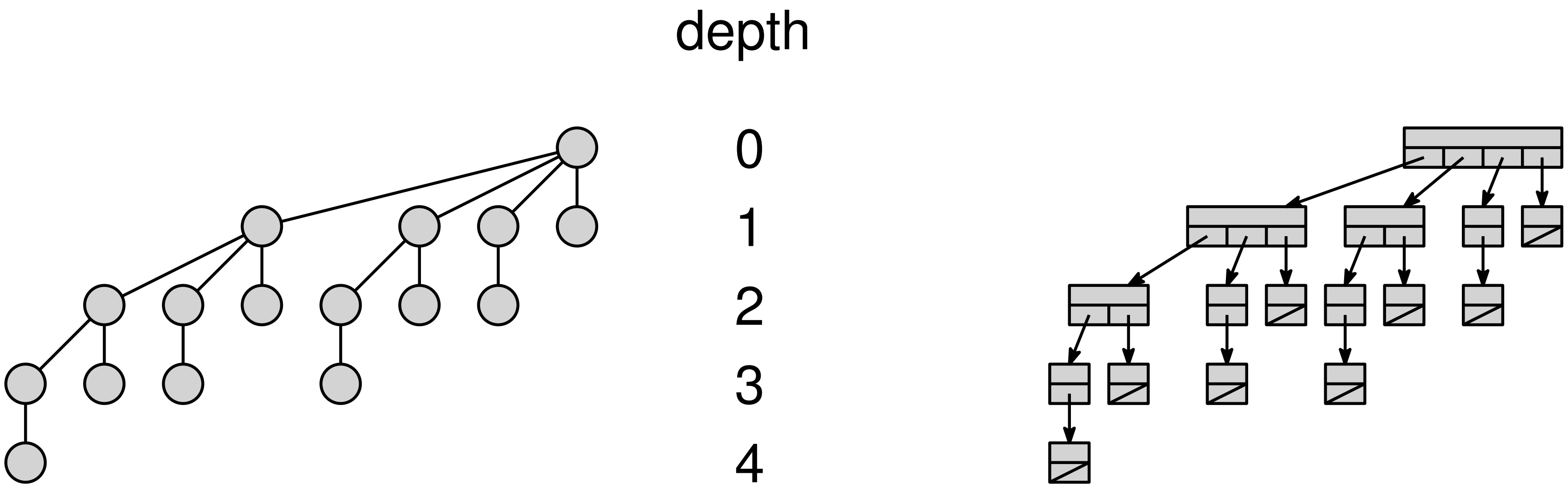

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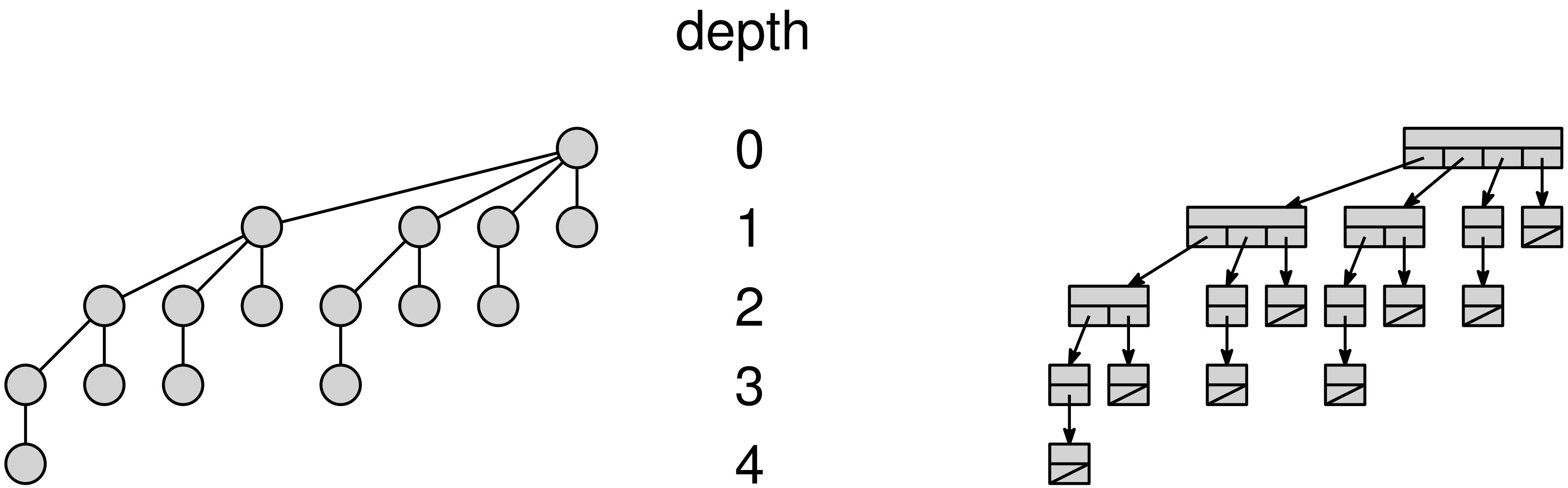
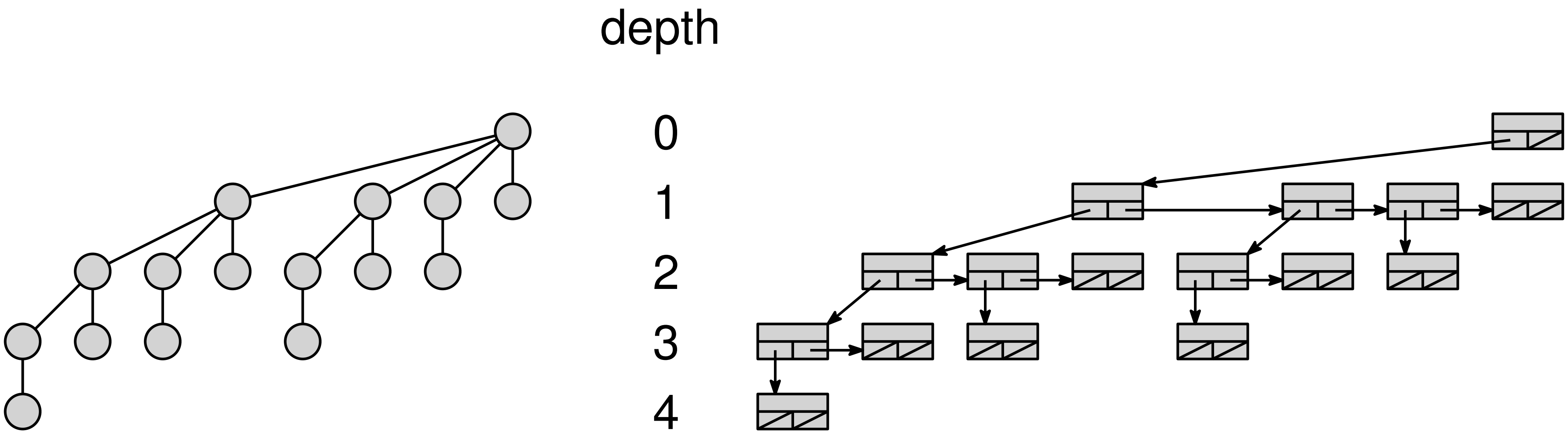


Variable number of children
at each node!

Tree representation

Left child, right sibling representation



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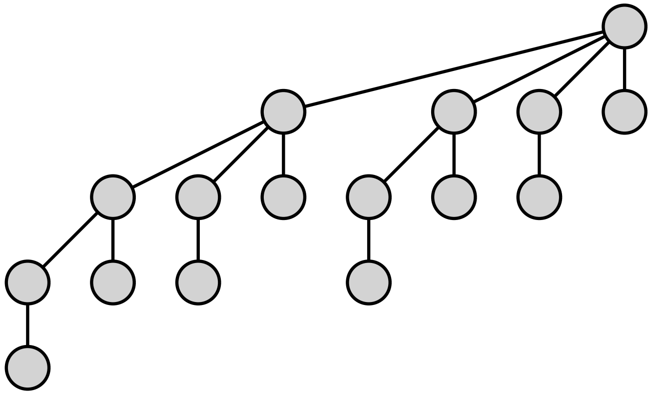
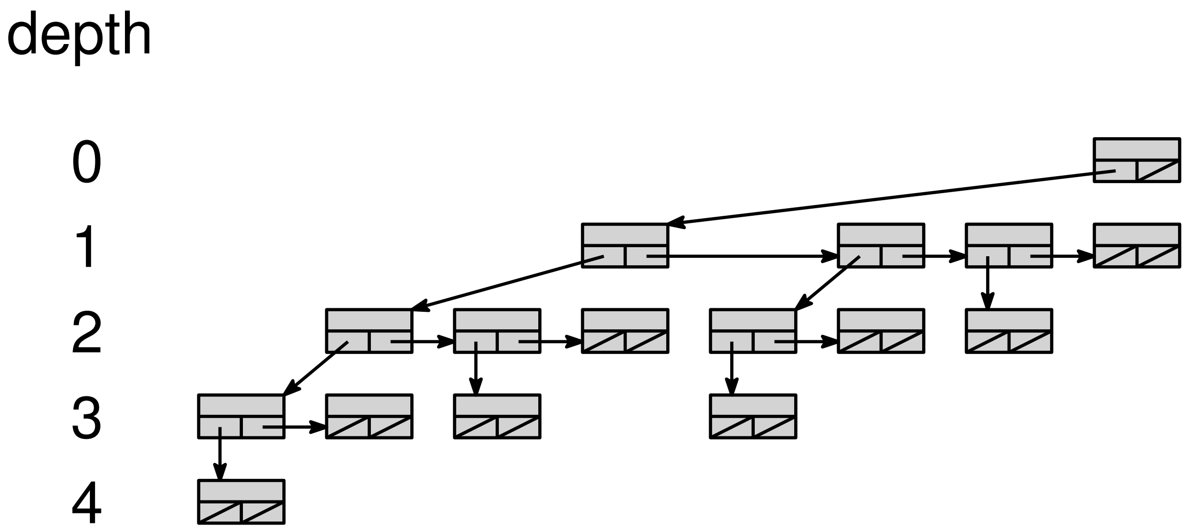
Left child, right sibling representation



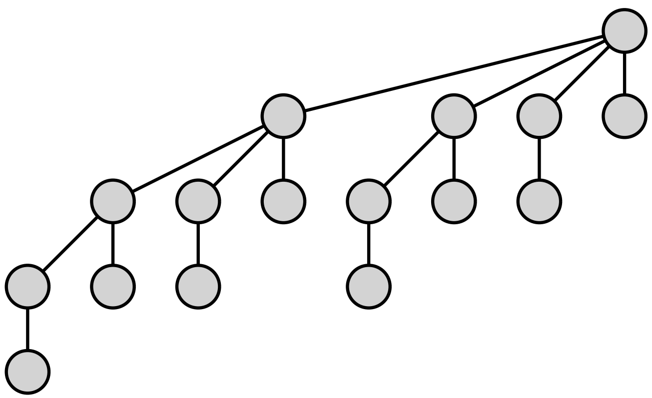


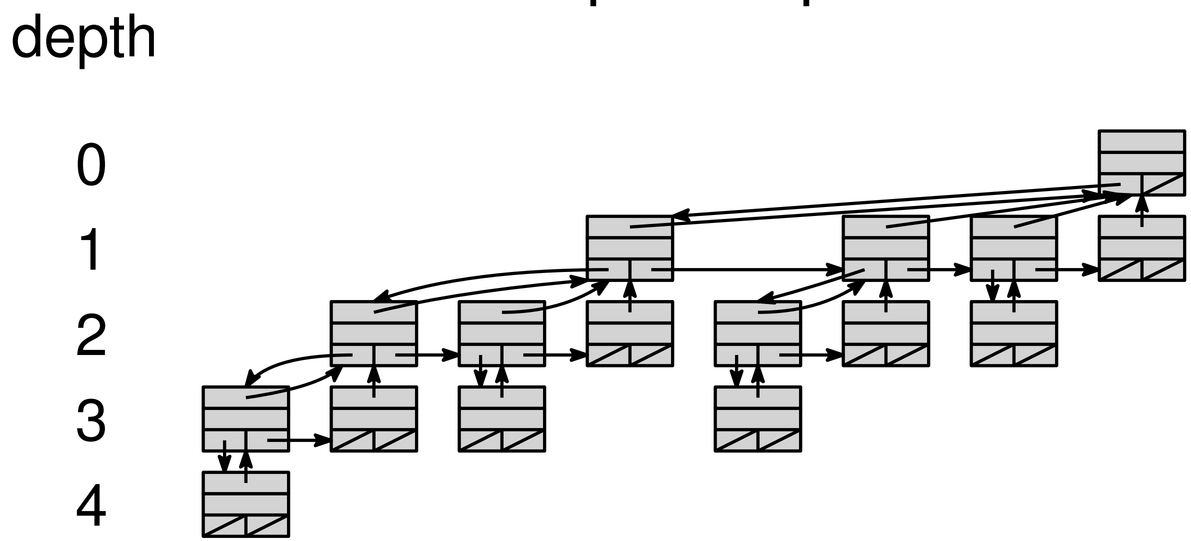
Each node is identical

Tree representation

Left child, right sibling representation

- add parent pointers





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Binomial heap

Collection of heap-ordered binomial trees:

- Each tree is heap-ordered
- At most **one** tree B_k, for $k = 0, 1, 2, \ldots \lfloor \log n \rfloor$

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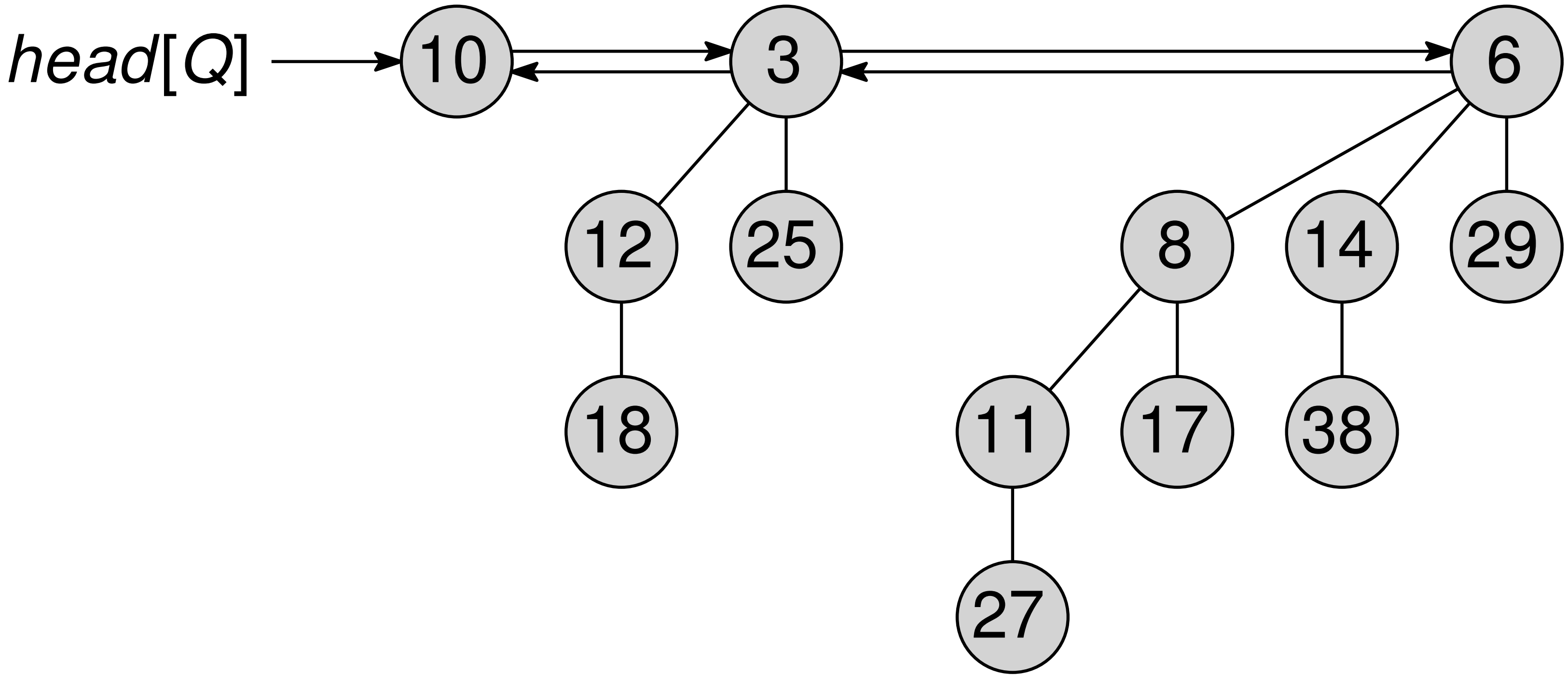

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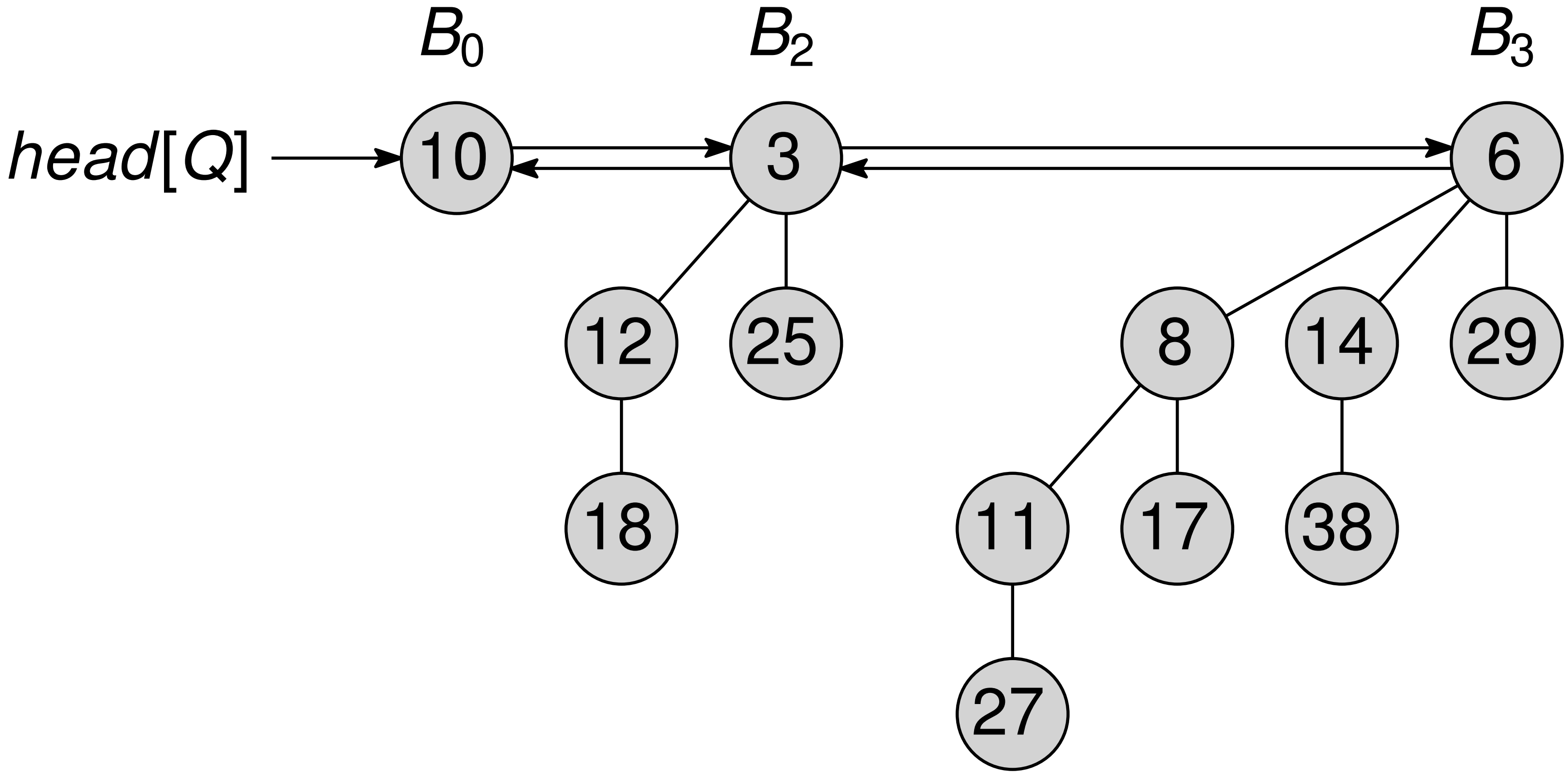
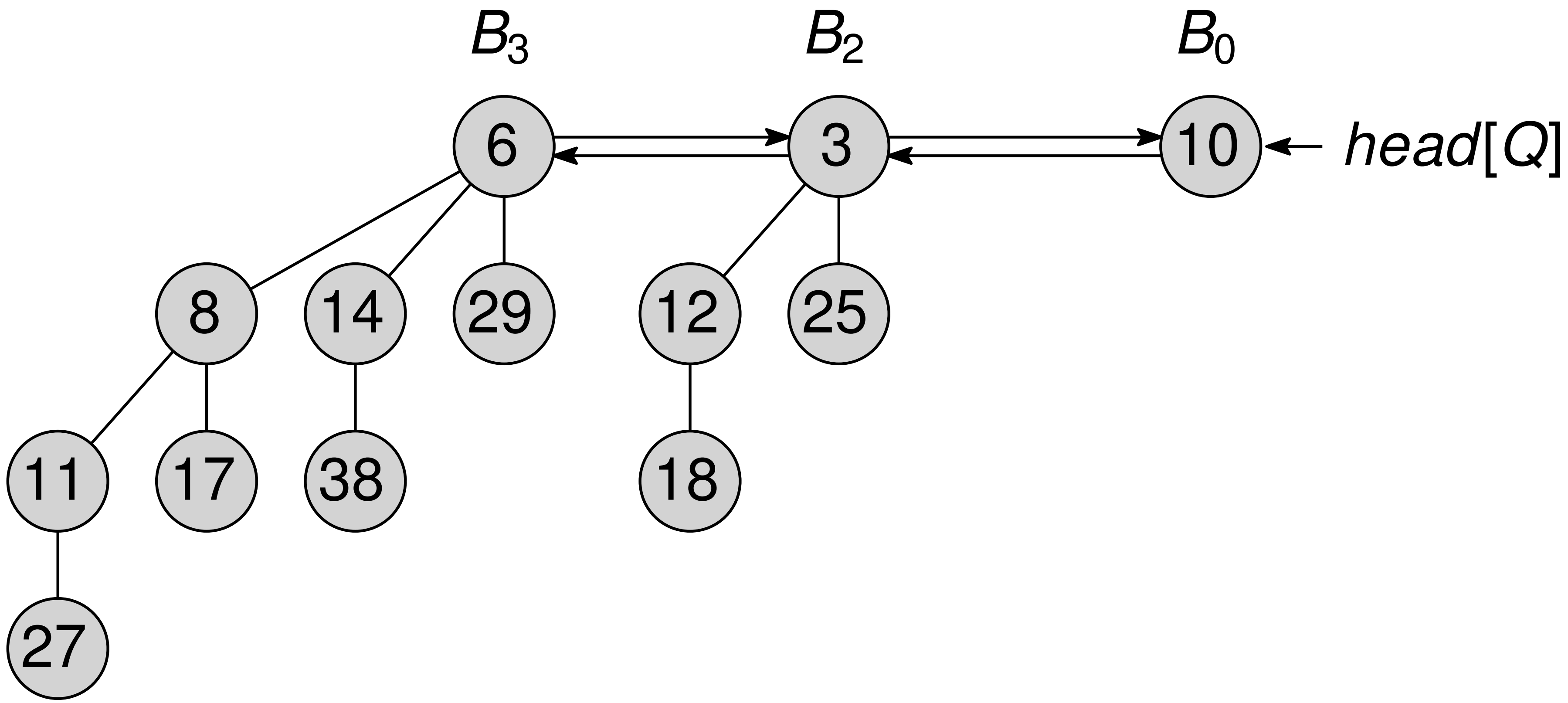


MINIMUM(Q)



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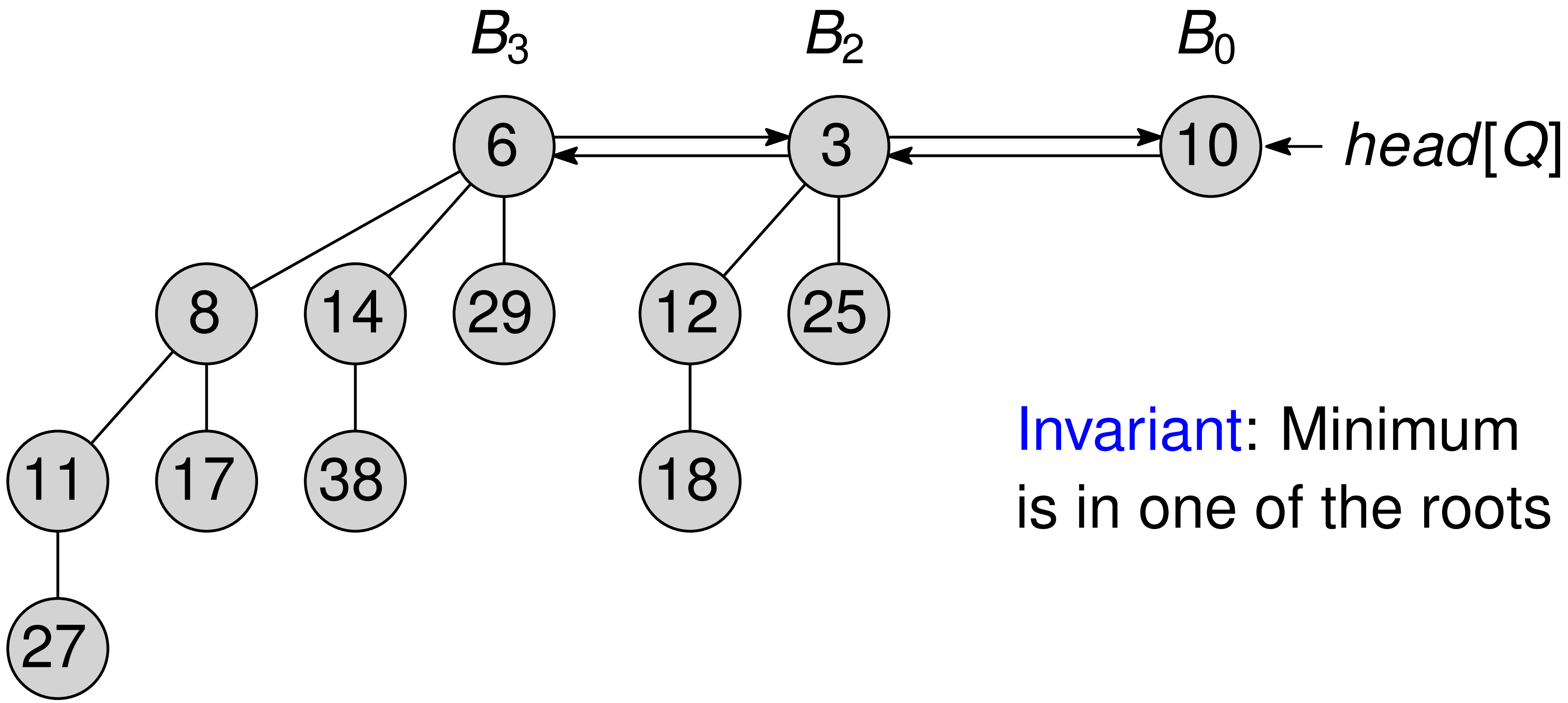

Invariant: Minimum is in one of the roots

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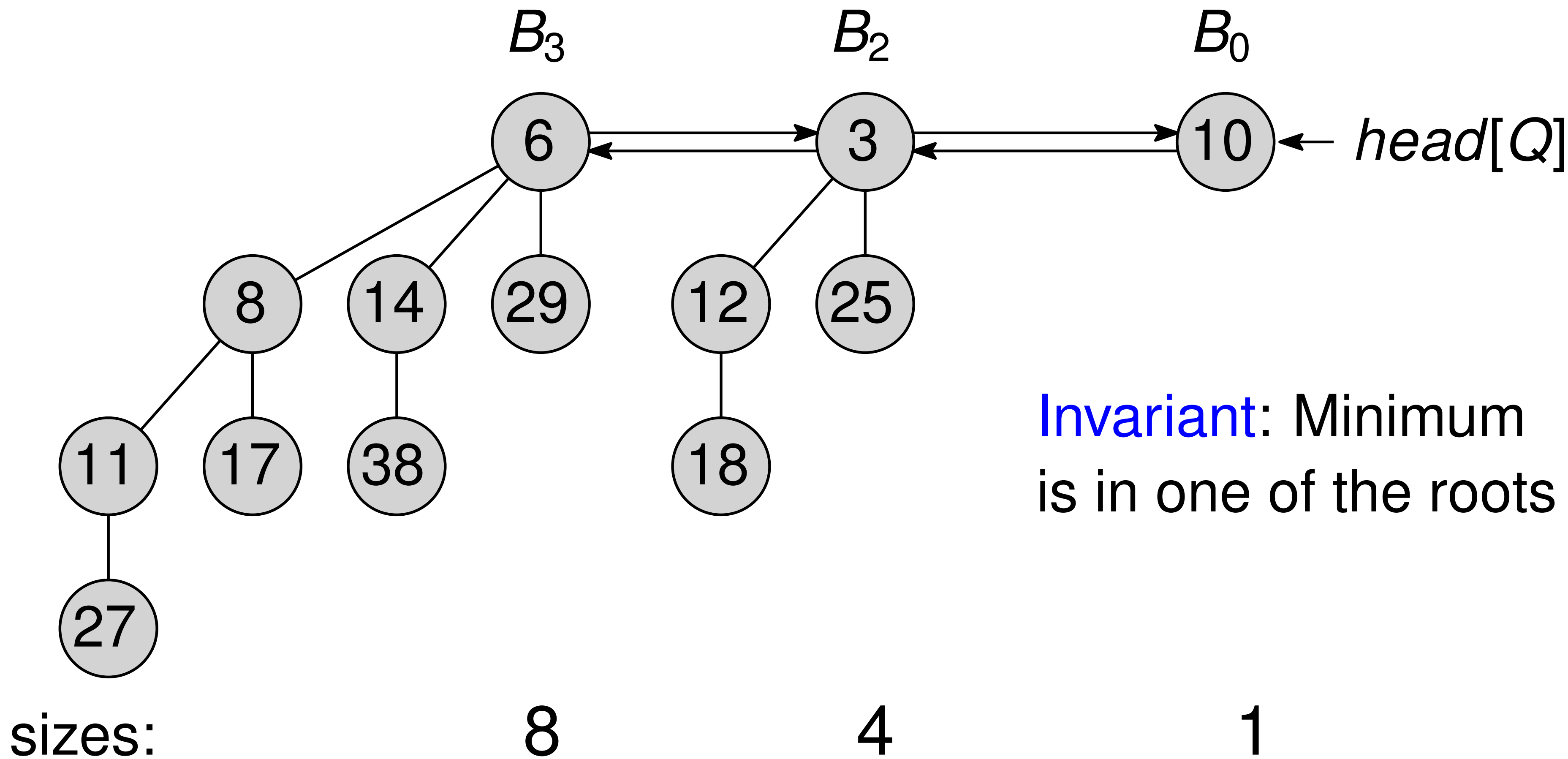

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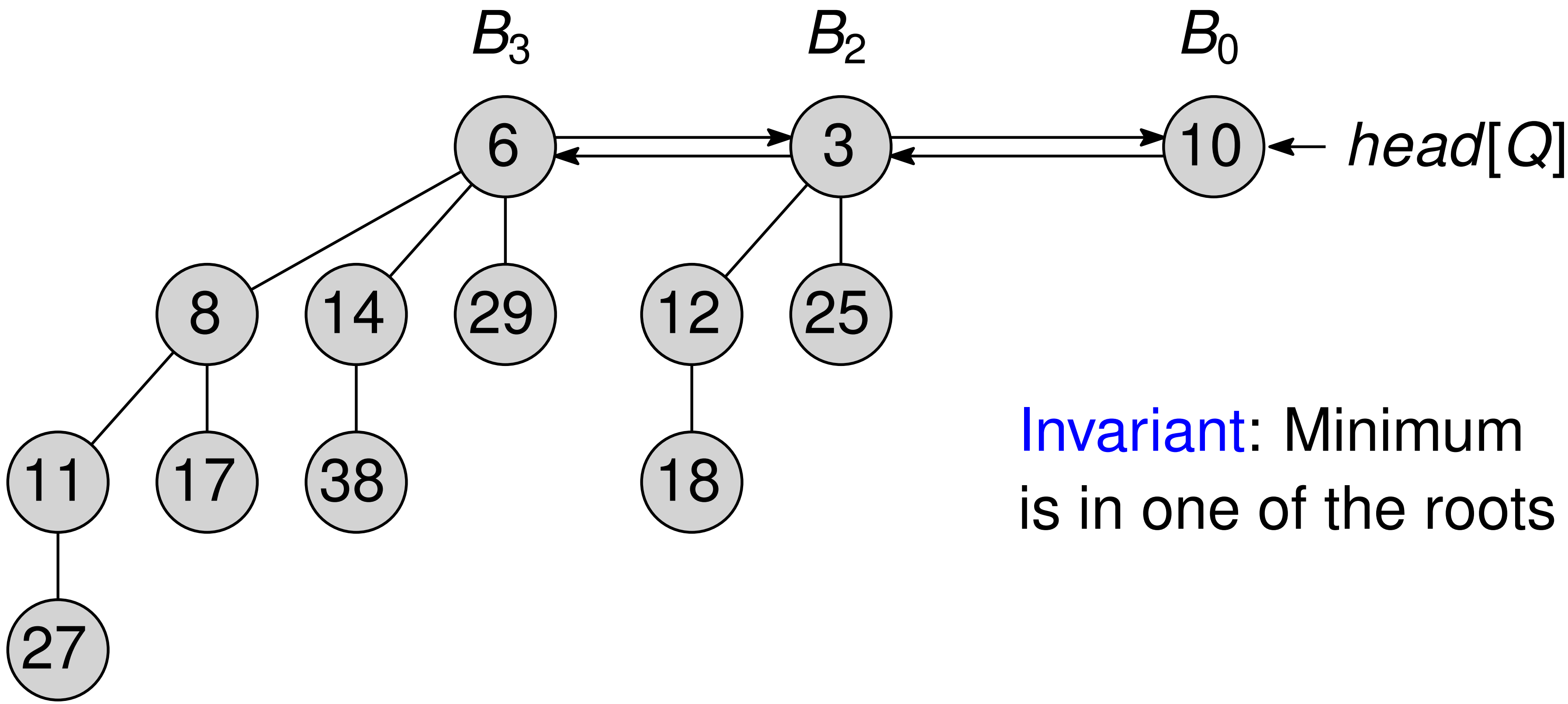
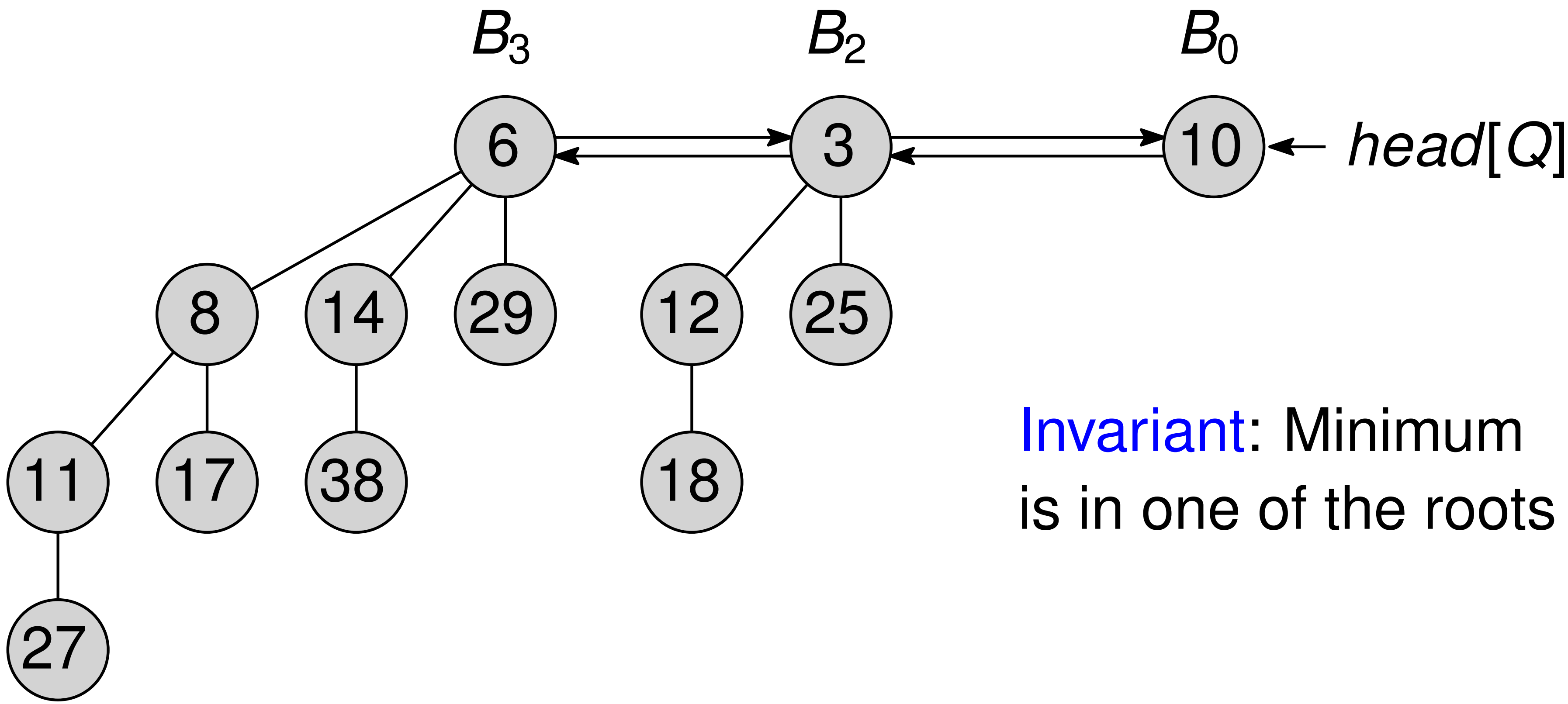


Invariant: Minimum is in one of the roots

sizes: 8 4 1

2^3 2^2 2^0

$\text{MINIMUM}(Q)$

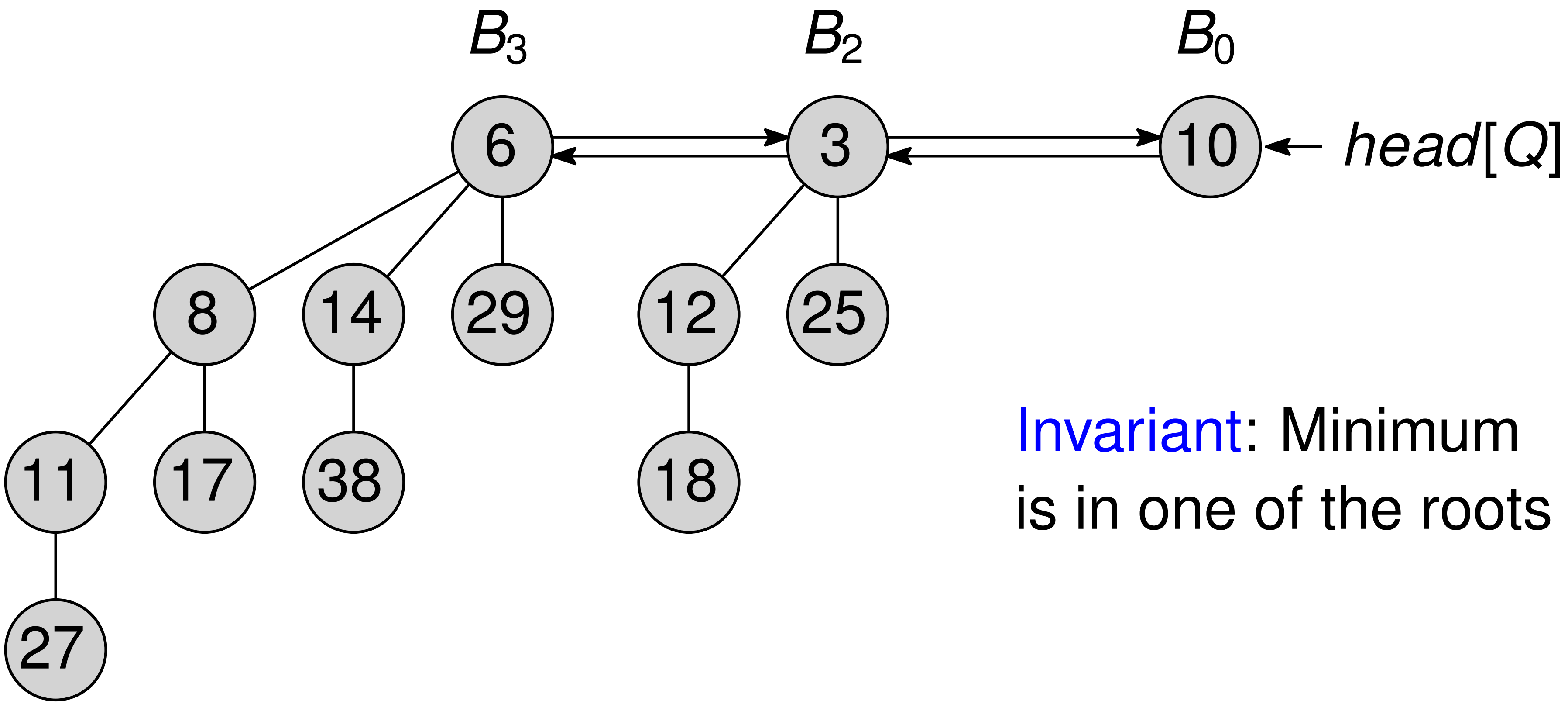


Invariant: Minimum is in one of the roots

sizes: 8 4 1

$N = 13 = 2^3 + 2^2 + 2^0$

MINIMUM(Q)



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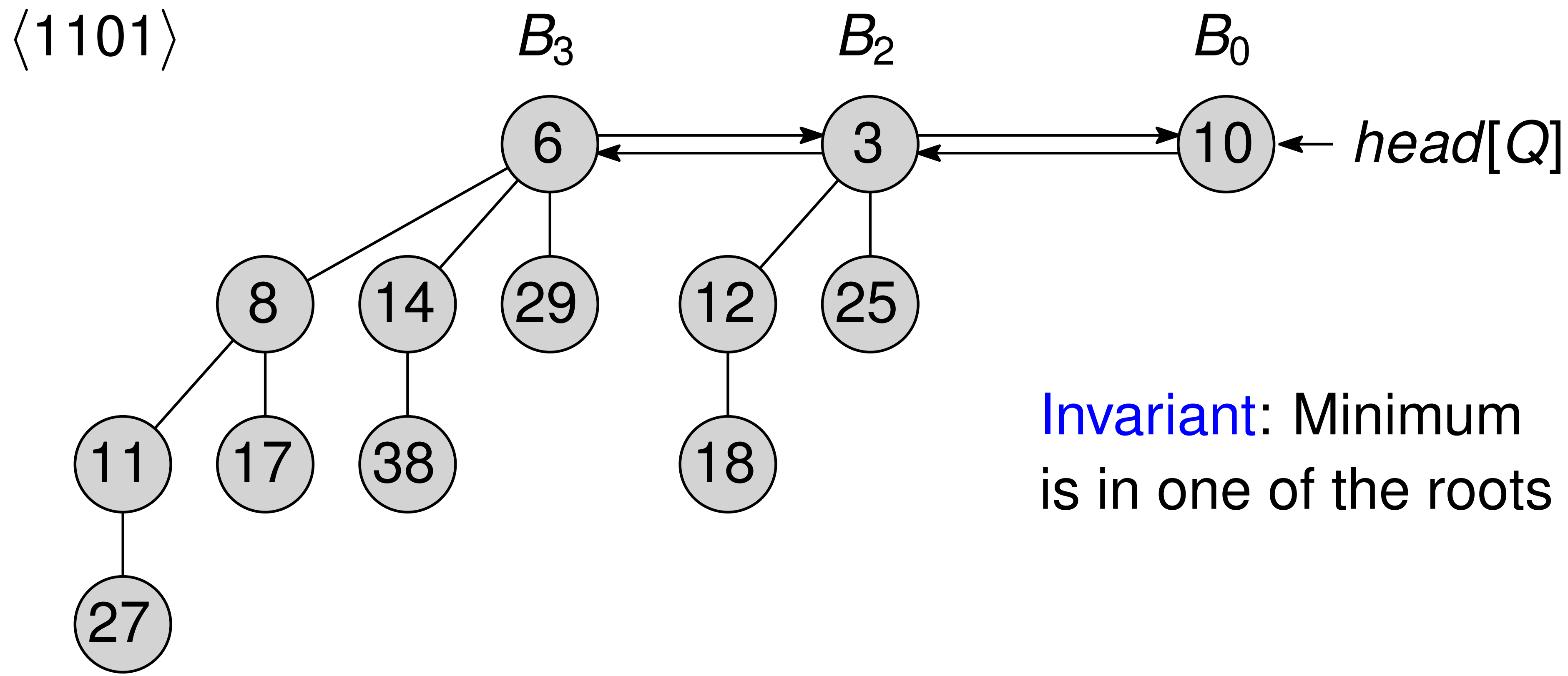

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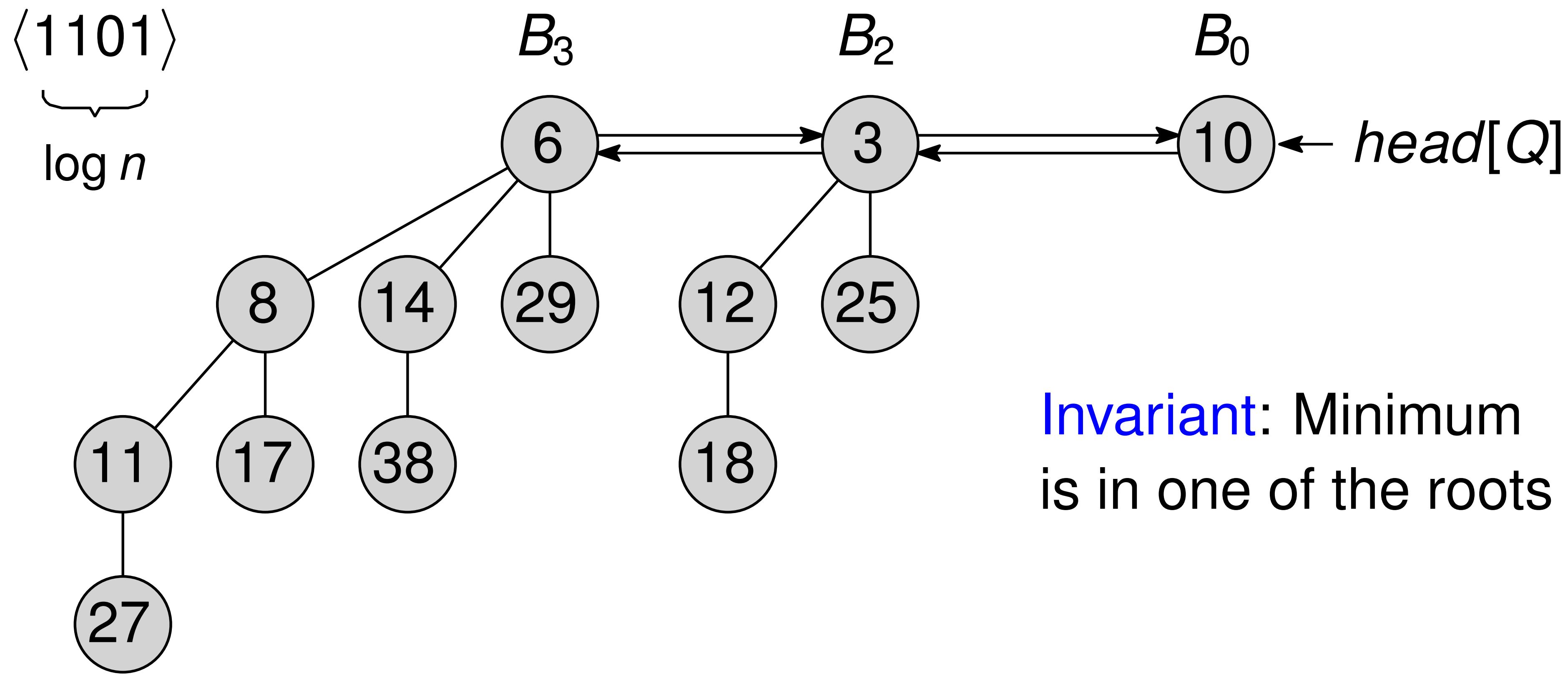

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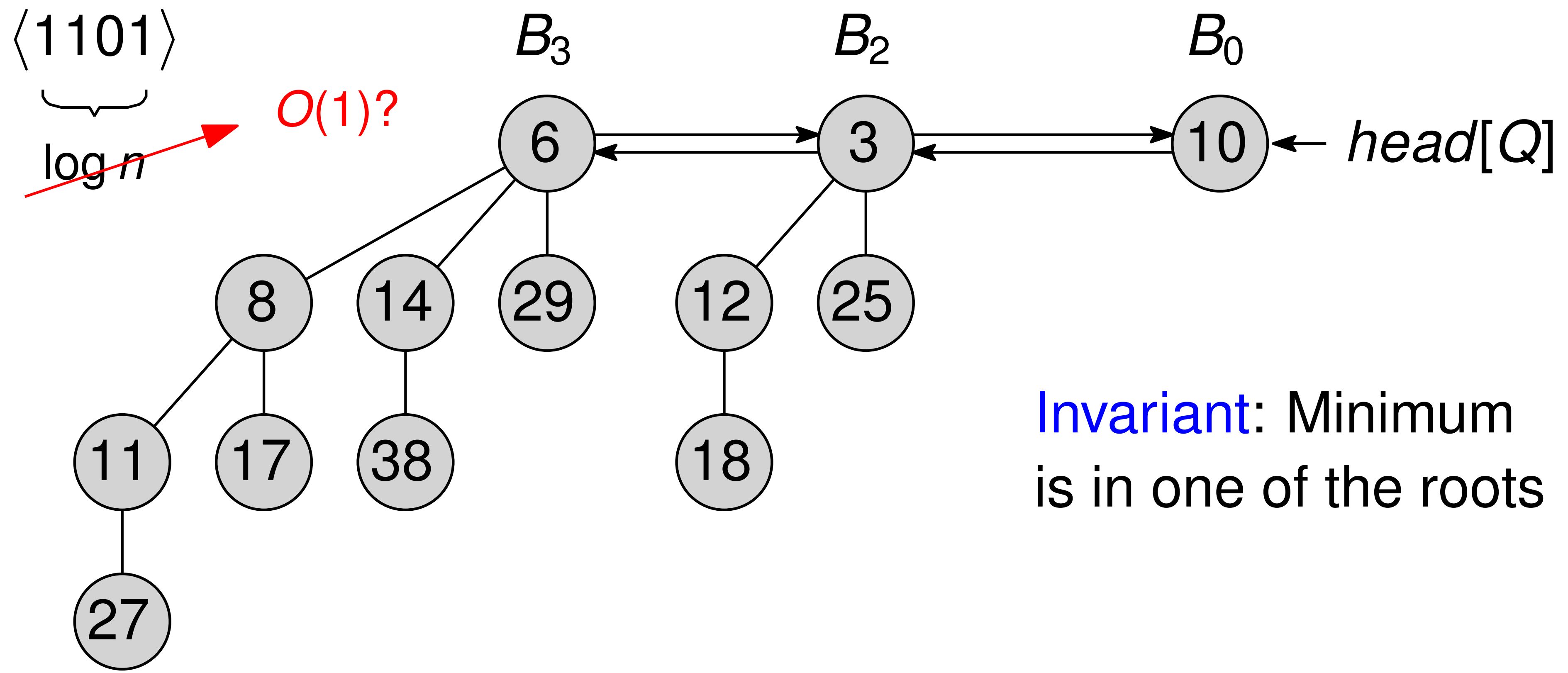

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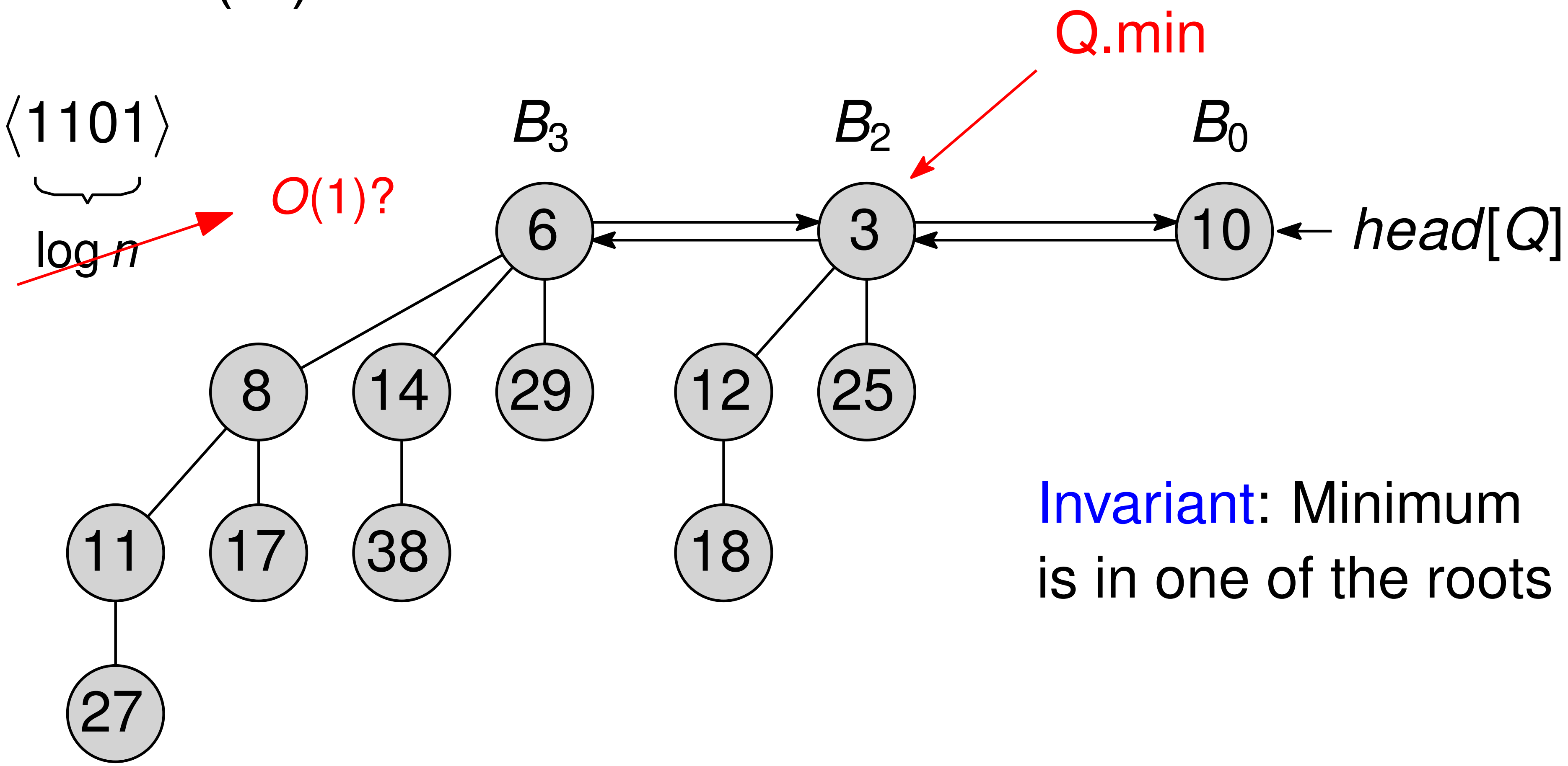

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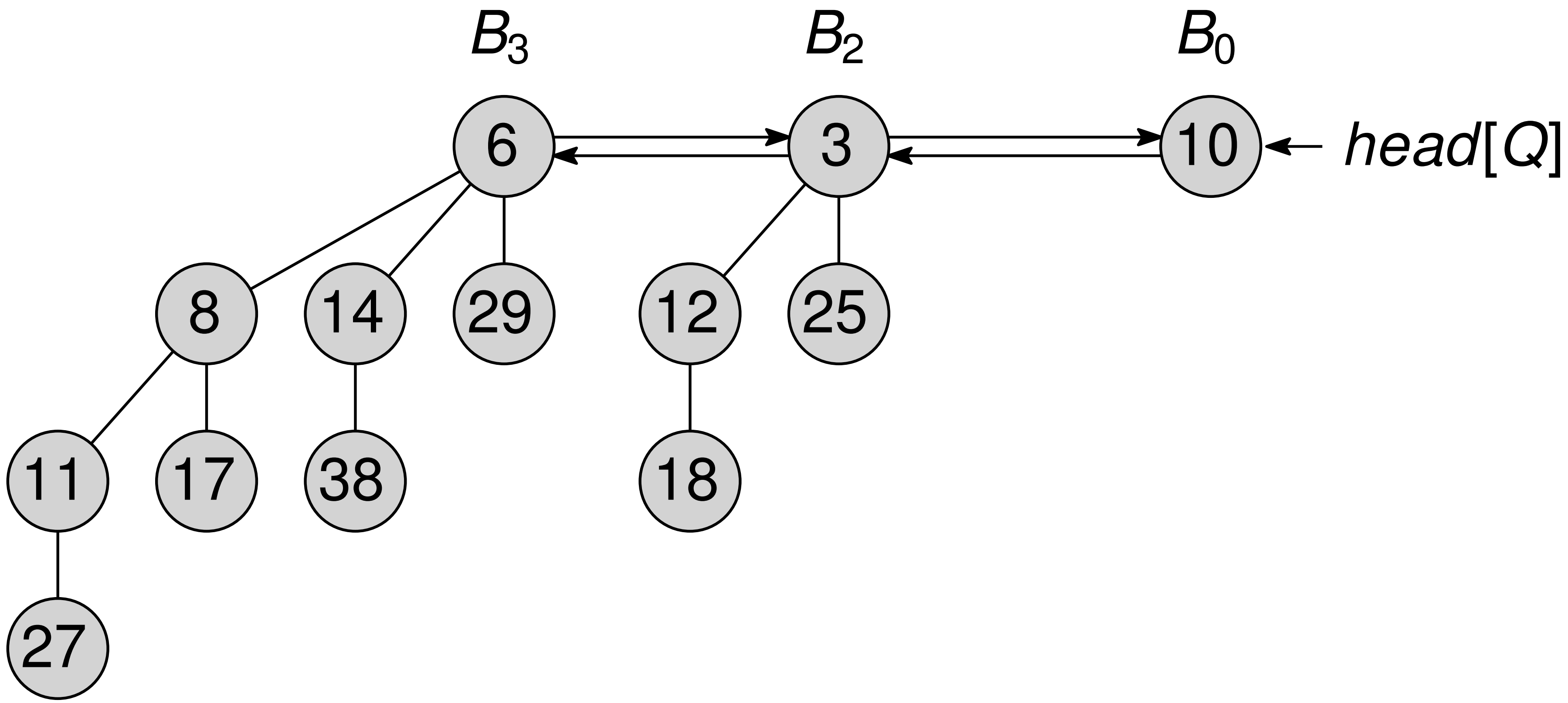
DECREASE-KEY(Q, x, k)



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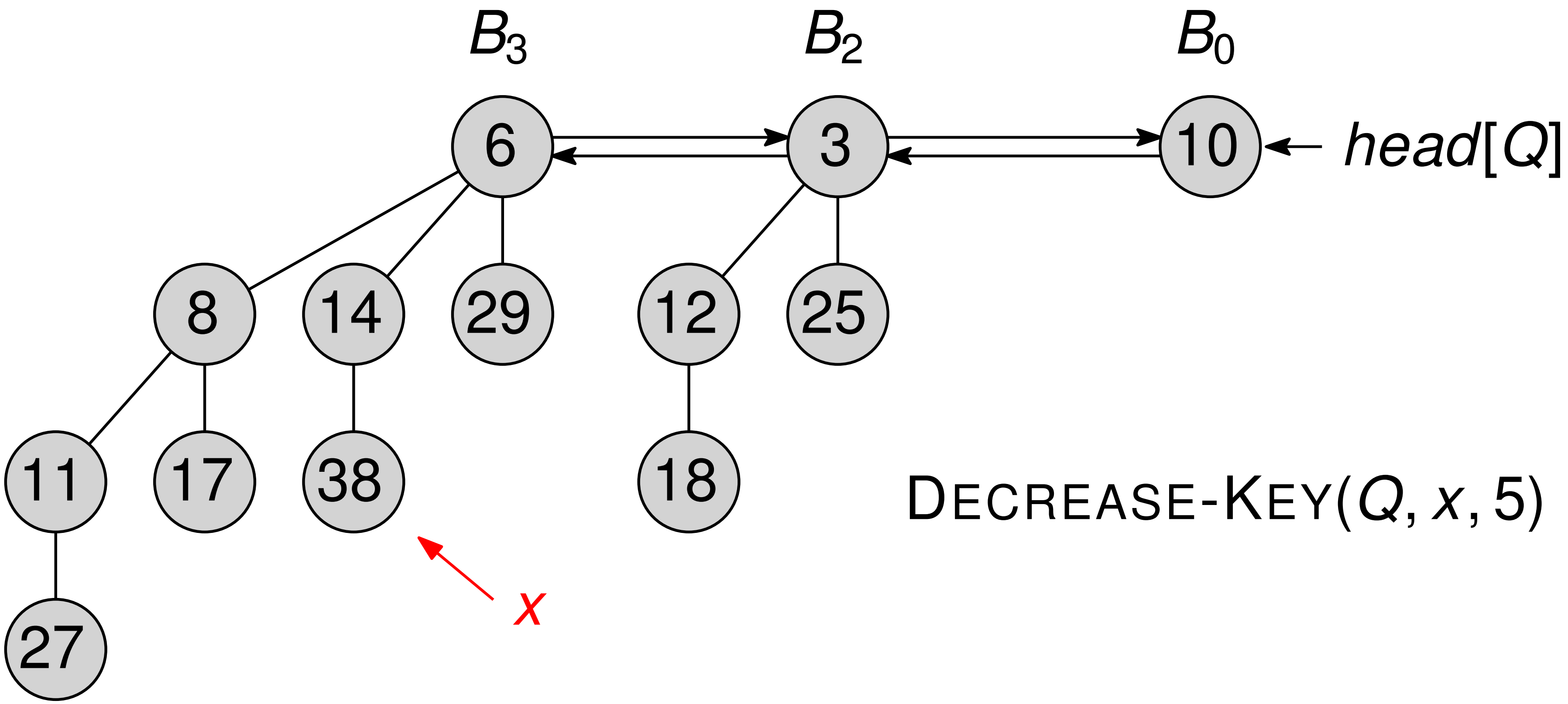

DECREASE-KEY(Q, x, 5)

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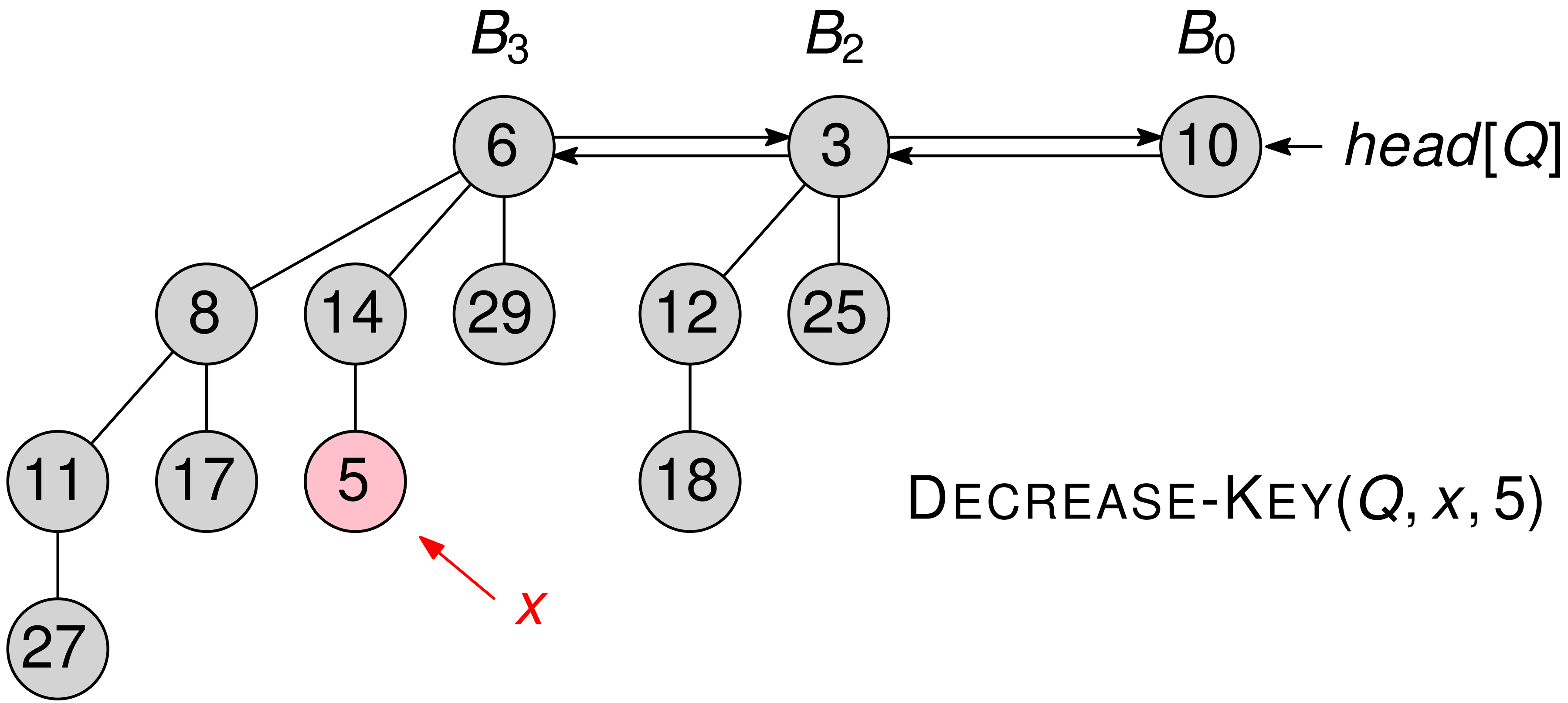

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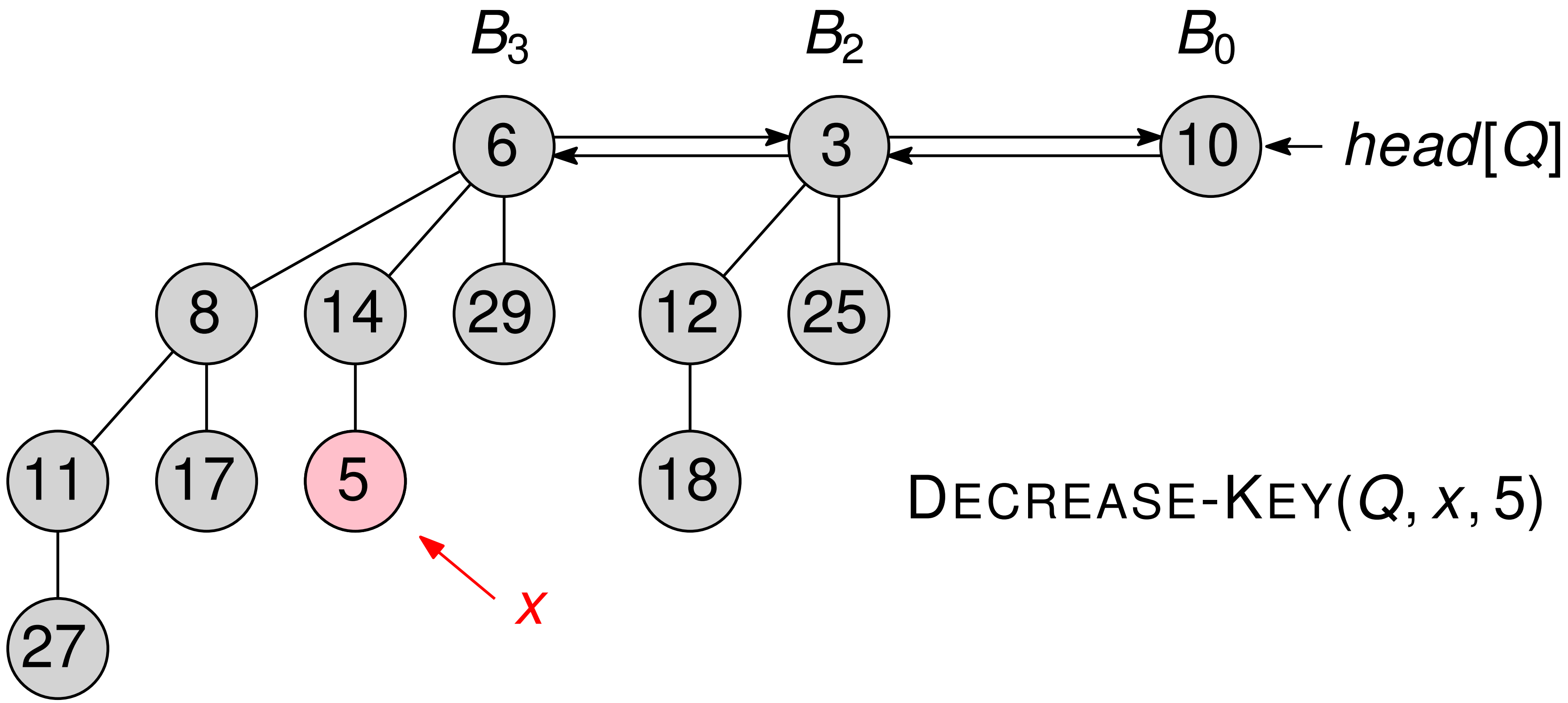
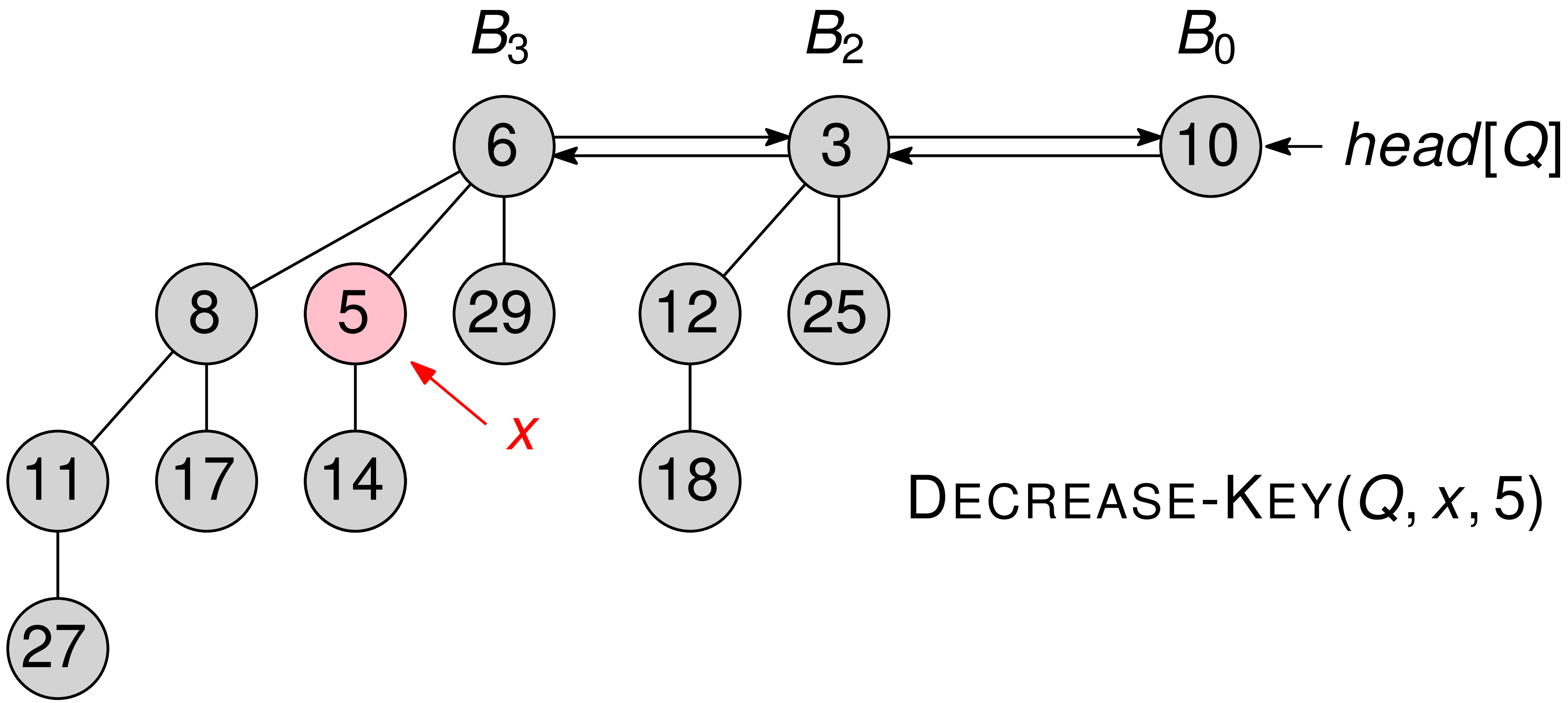


DECREASE-KEY(Q, x, 5)

- Fix heap order

DECREASE-KEY(Q, x, k)



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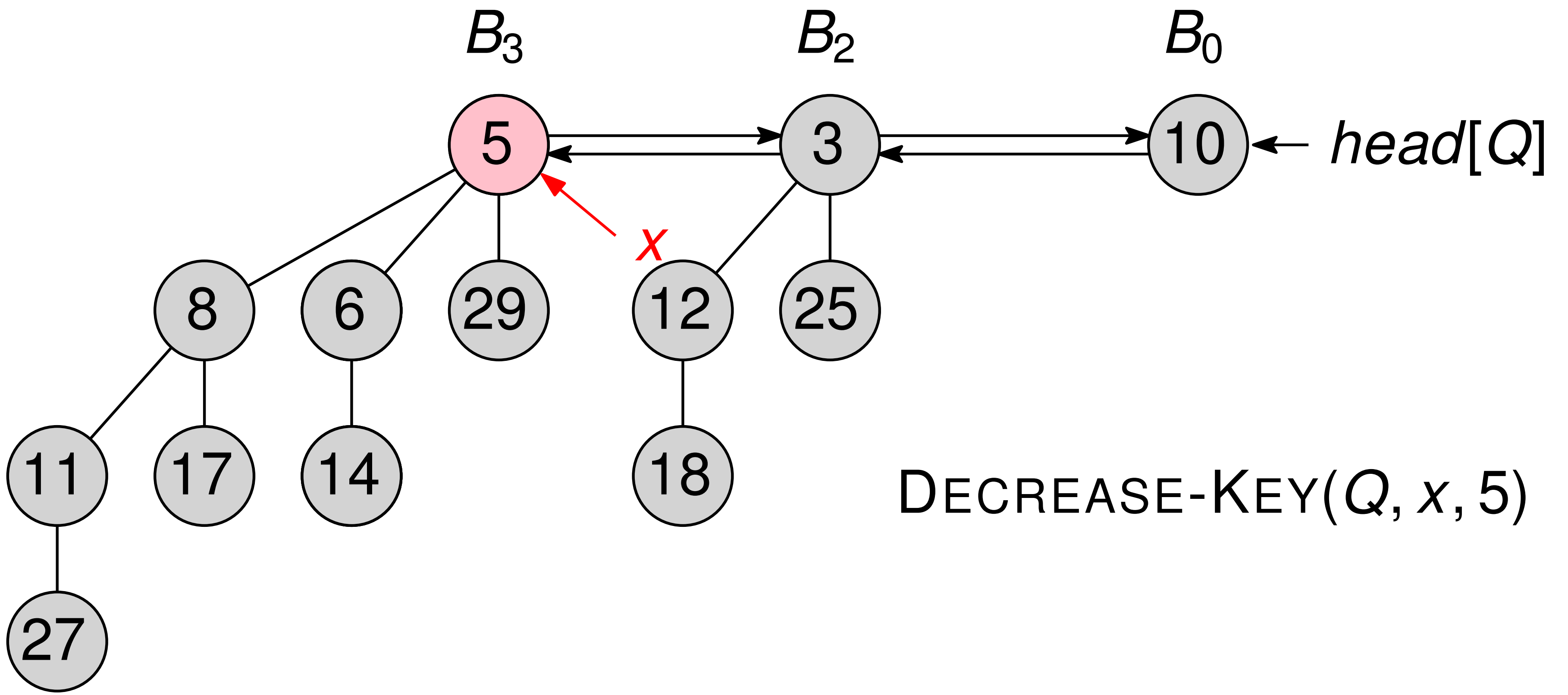

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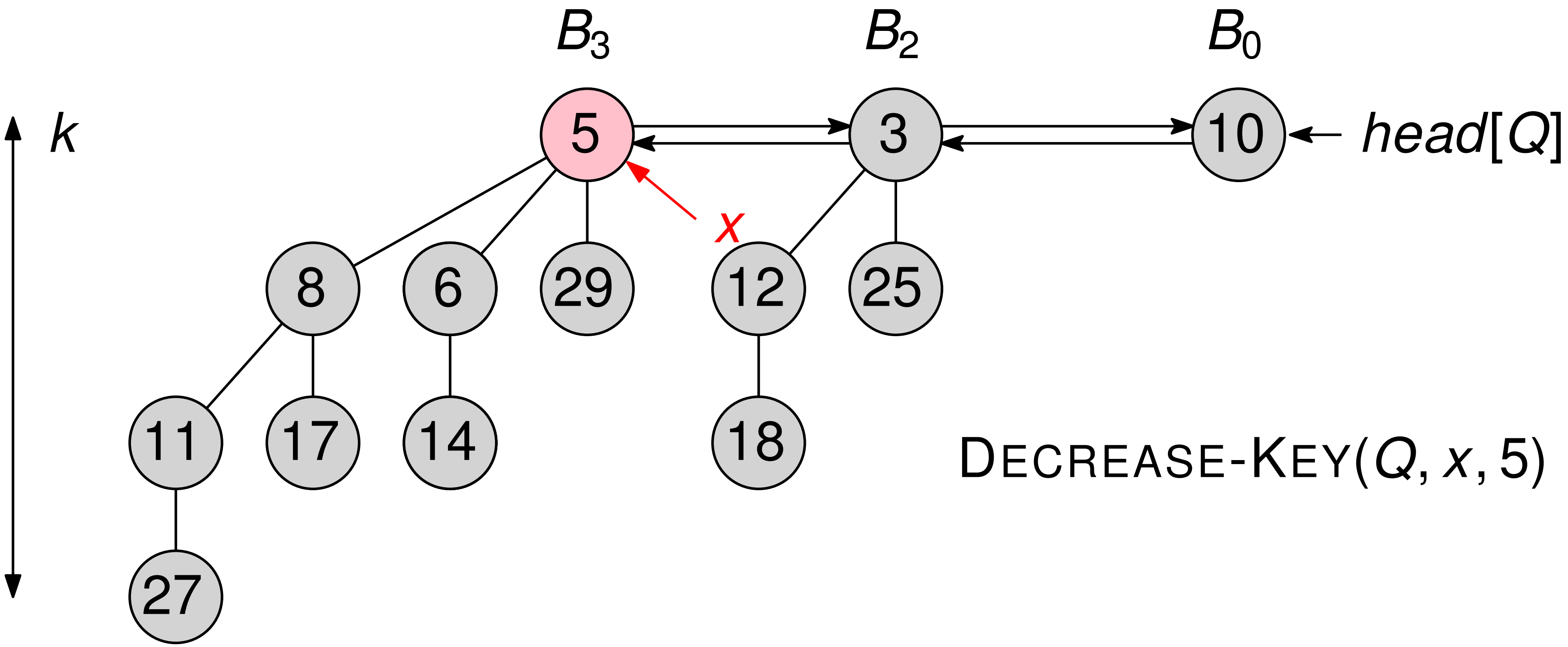
Depth of B_k is k



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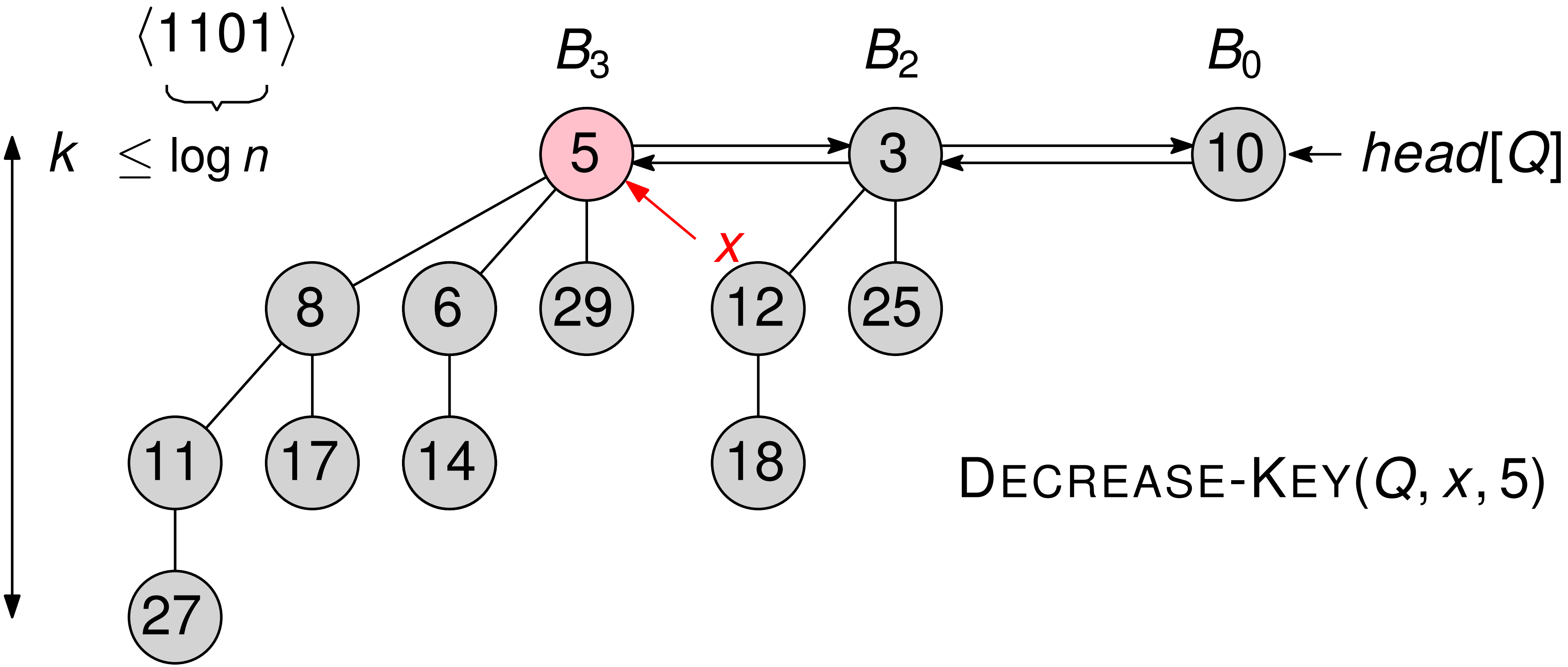

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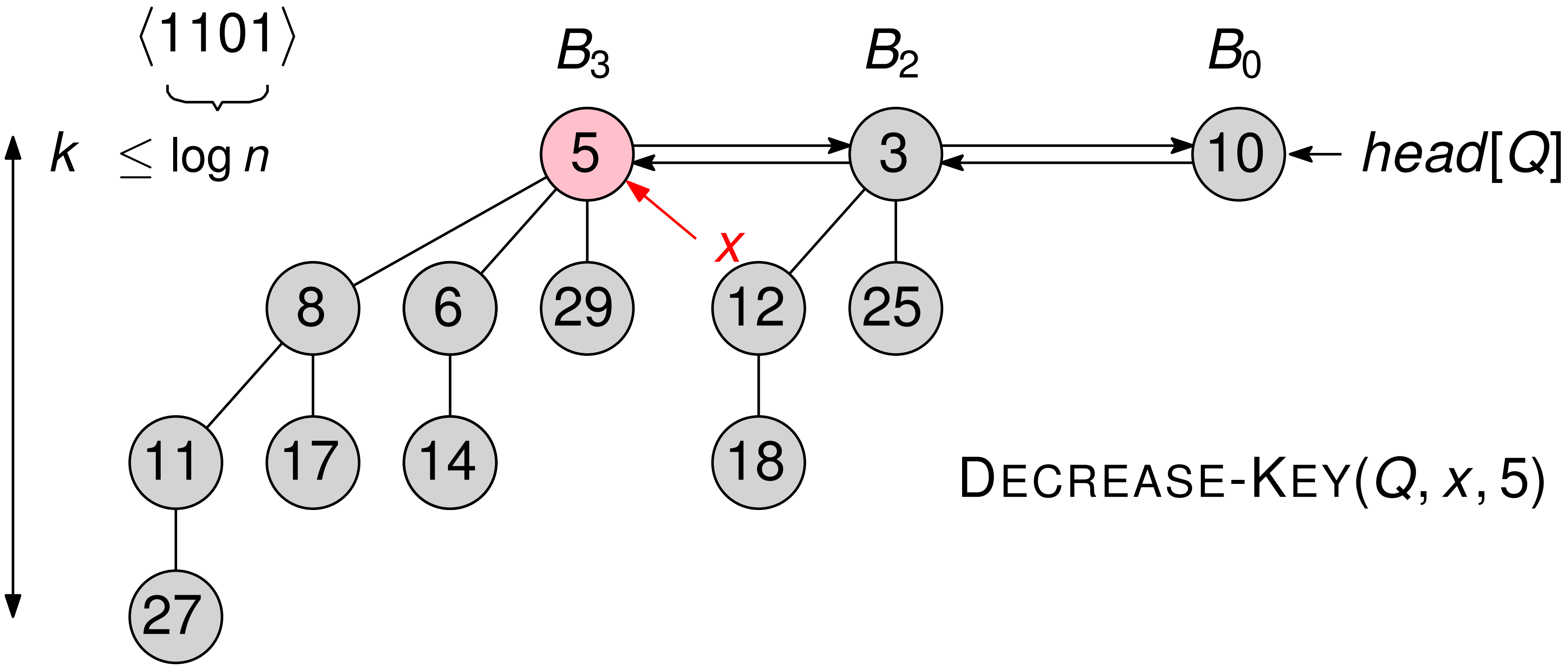
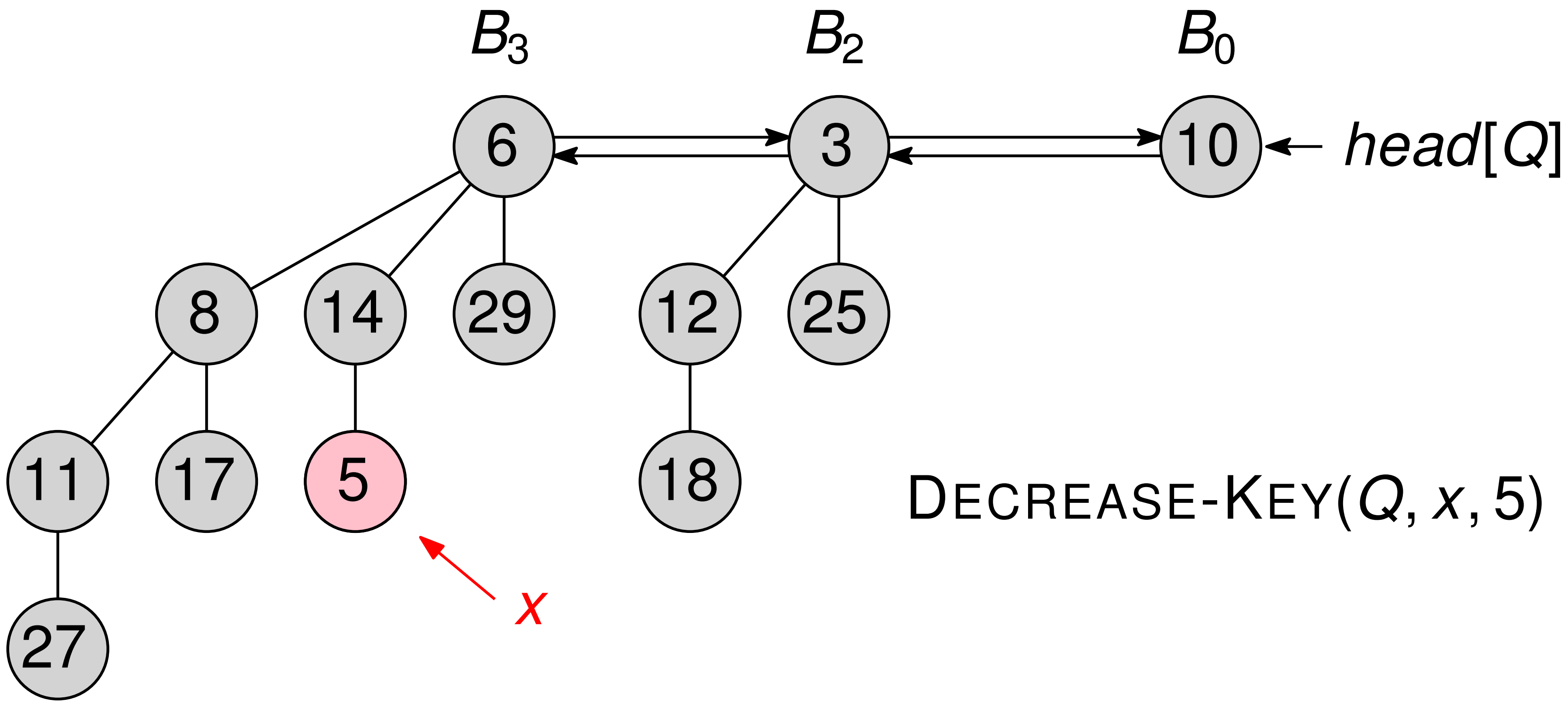


DECREASE-KEY(Q, x, 5)

- Fix heap order

Can you implement DECREASE-KEY(Q, x, k) ?

DECREASEKEY(Q, x, k)



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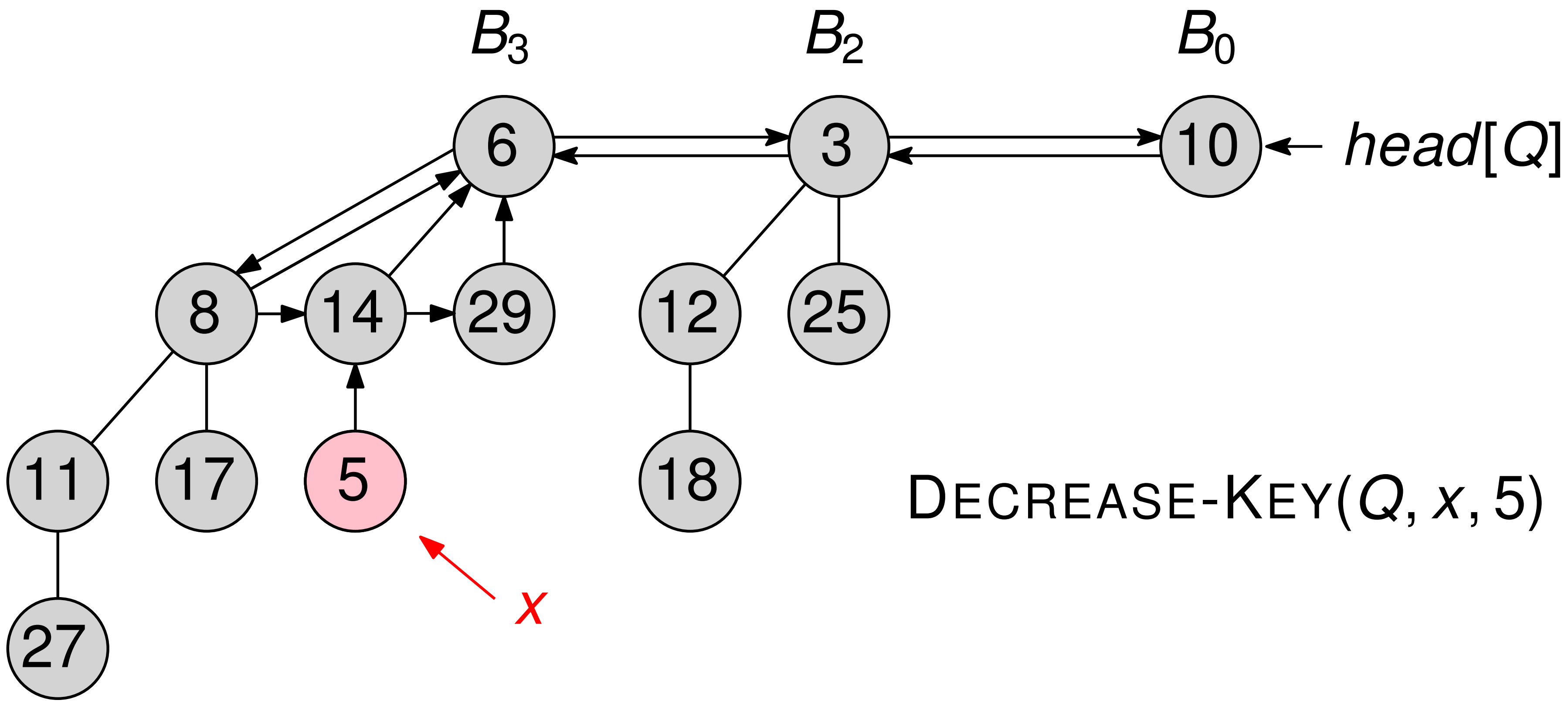

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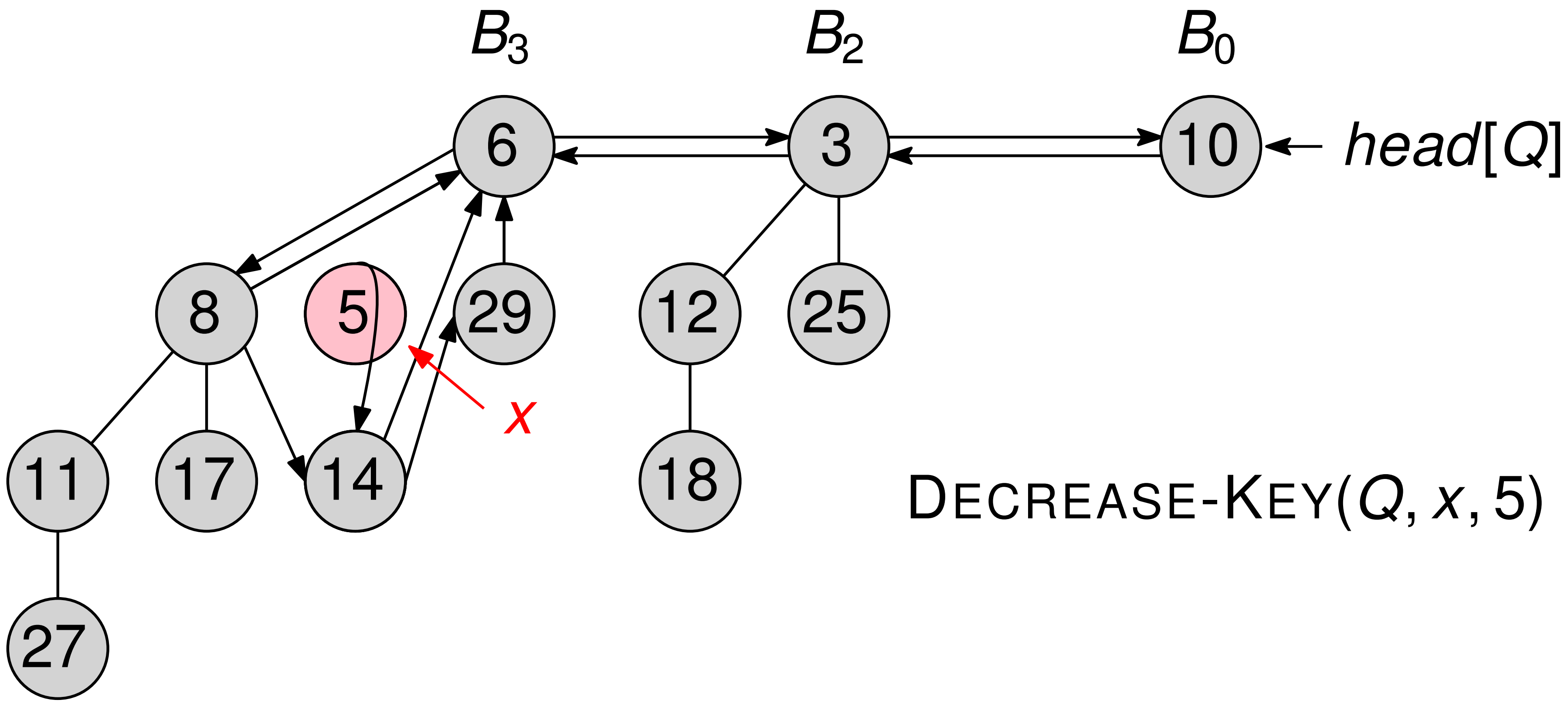

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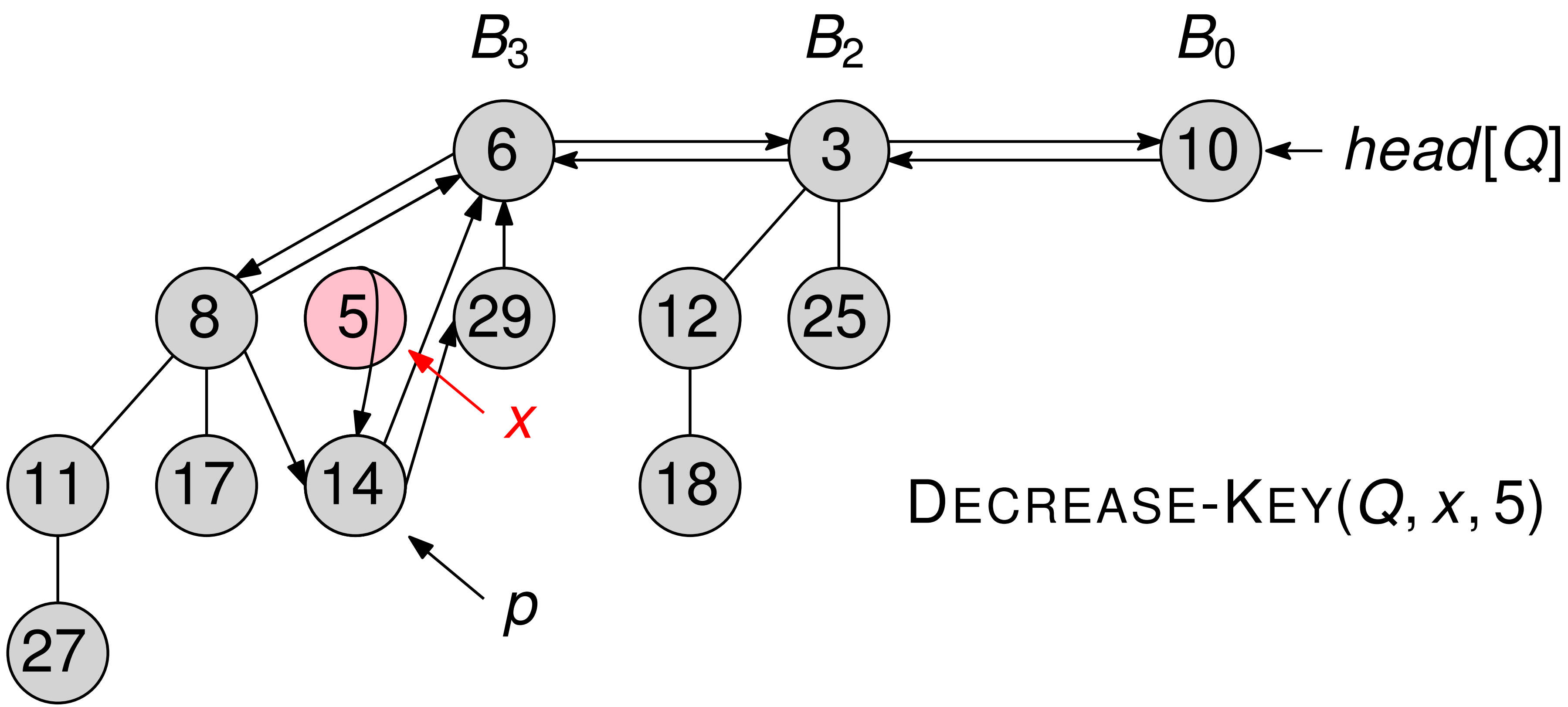
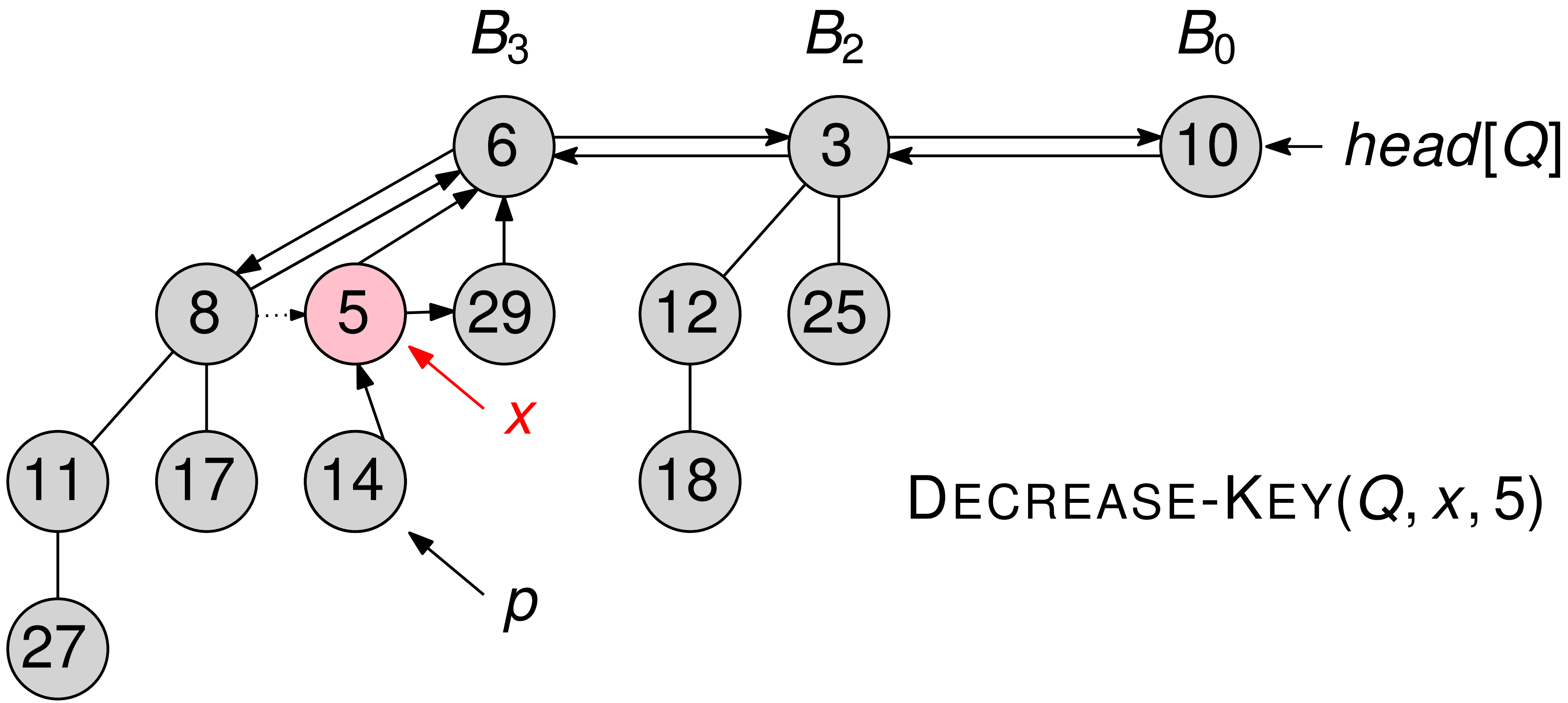


- $p = x.parent$
- $x.parent = p.parent$
- $p.parent = x$
- $x.sibling = p.sibling$
- $???.sibling = x$

DECREASEKEY(Q, x, k)



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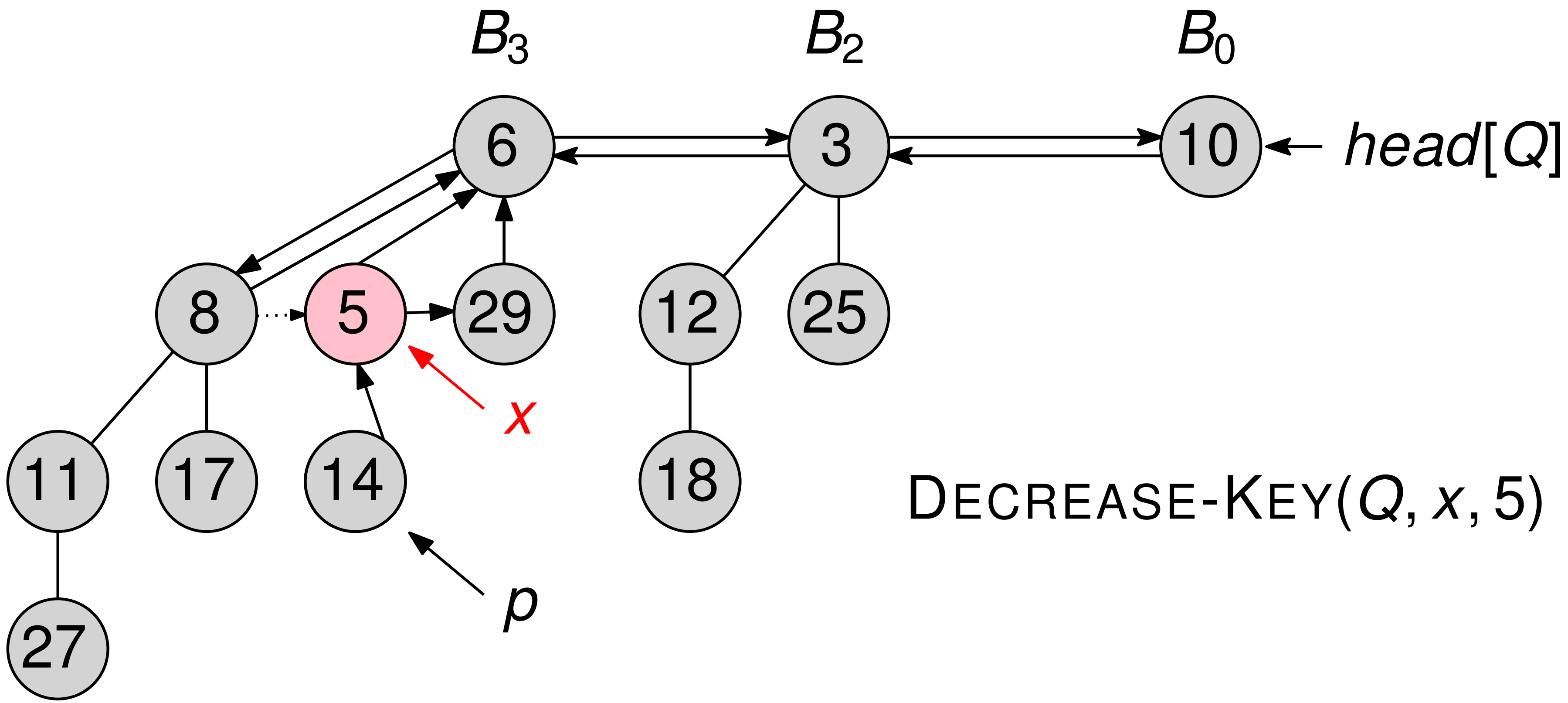

Decrease-Key(Q, x, 5)

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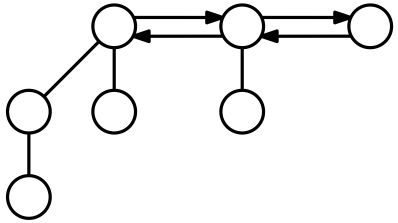
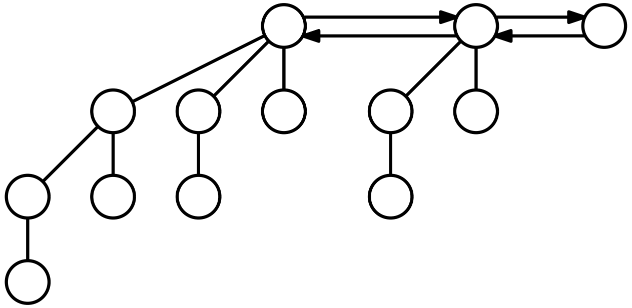
NEED:
doubly-linked sibling list

UNION(Q_1, Q_2)





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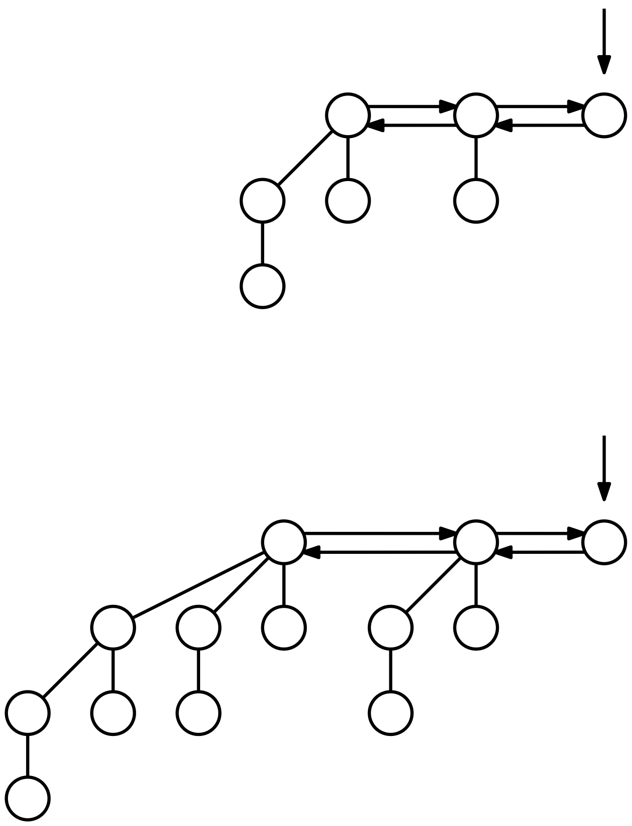

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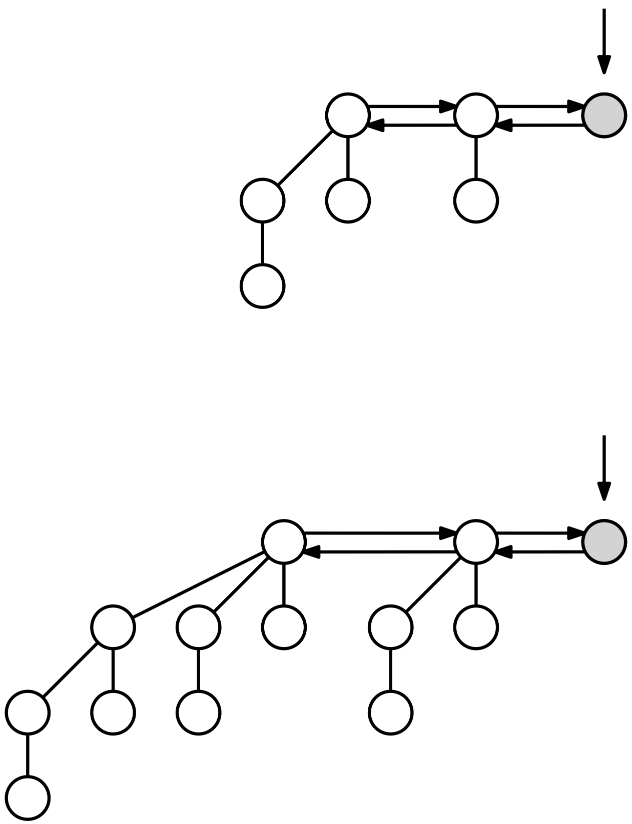

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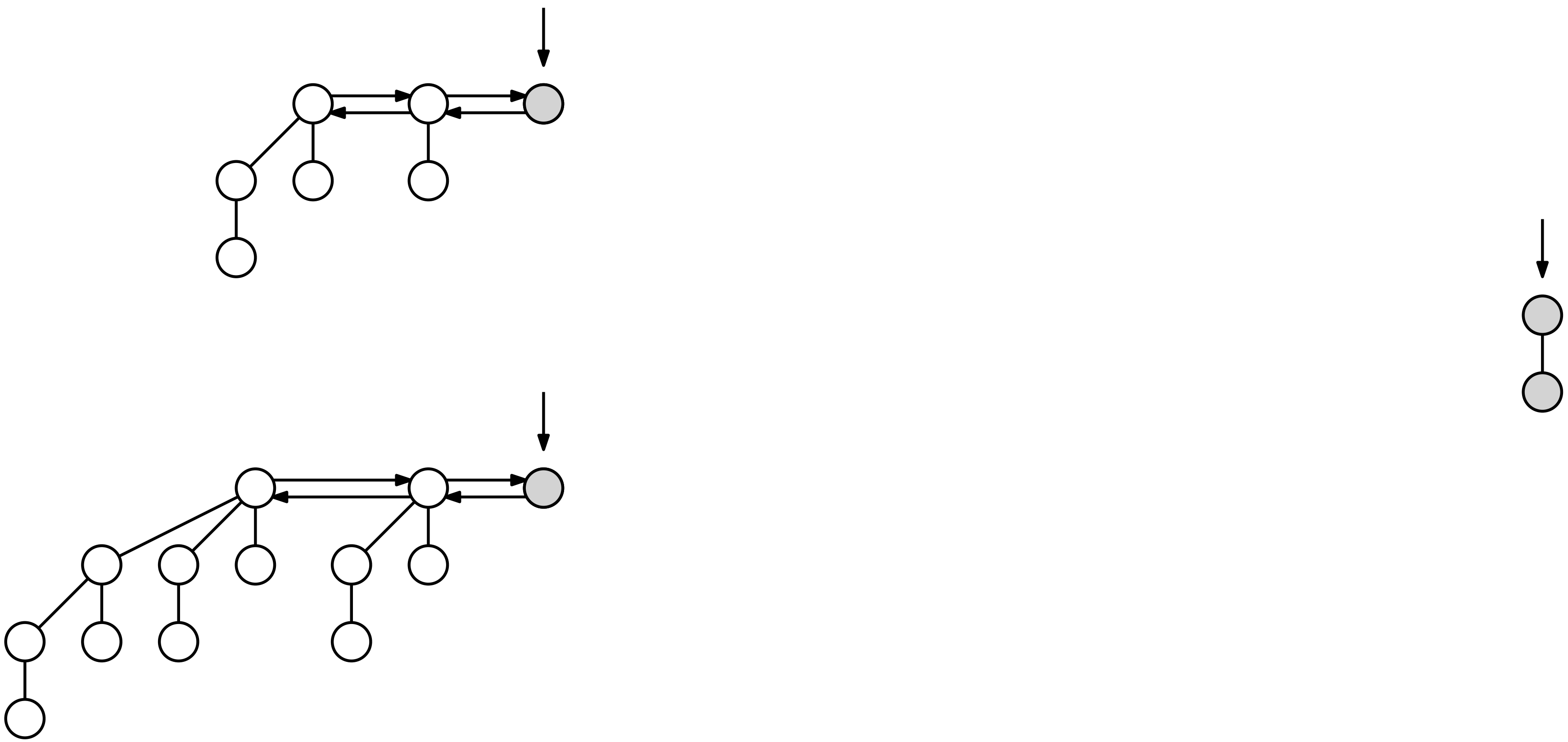

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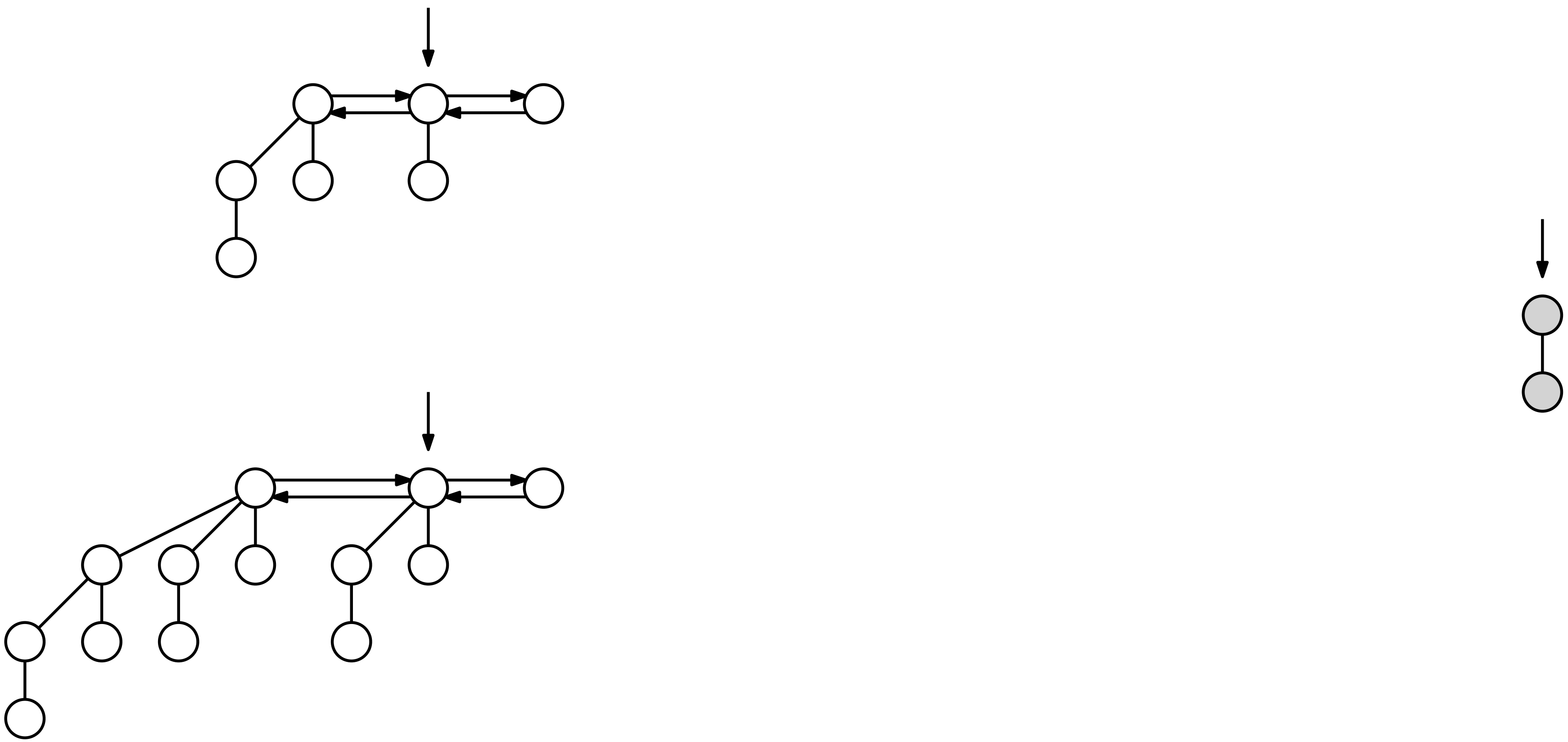

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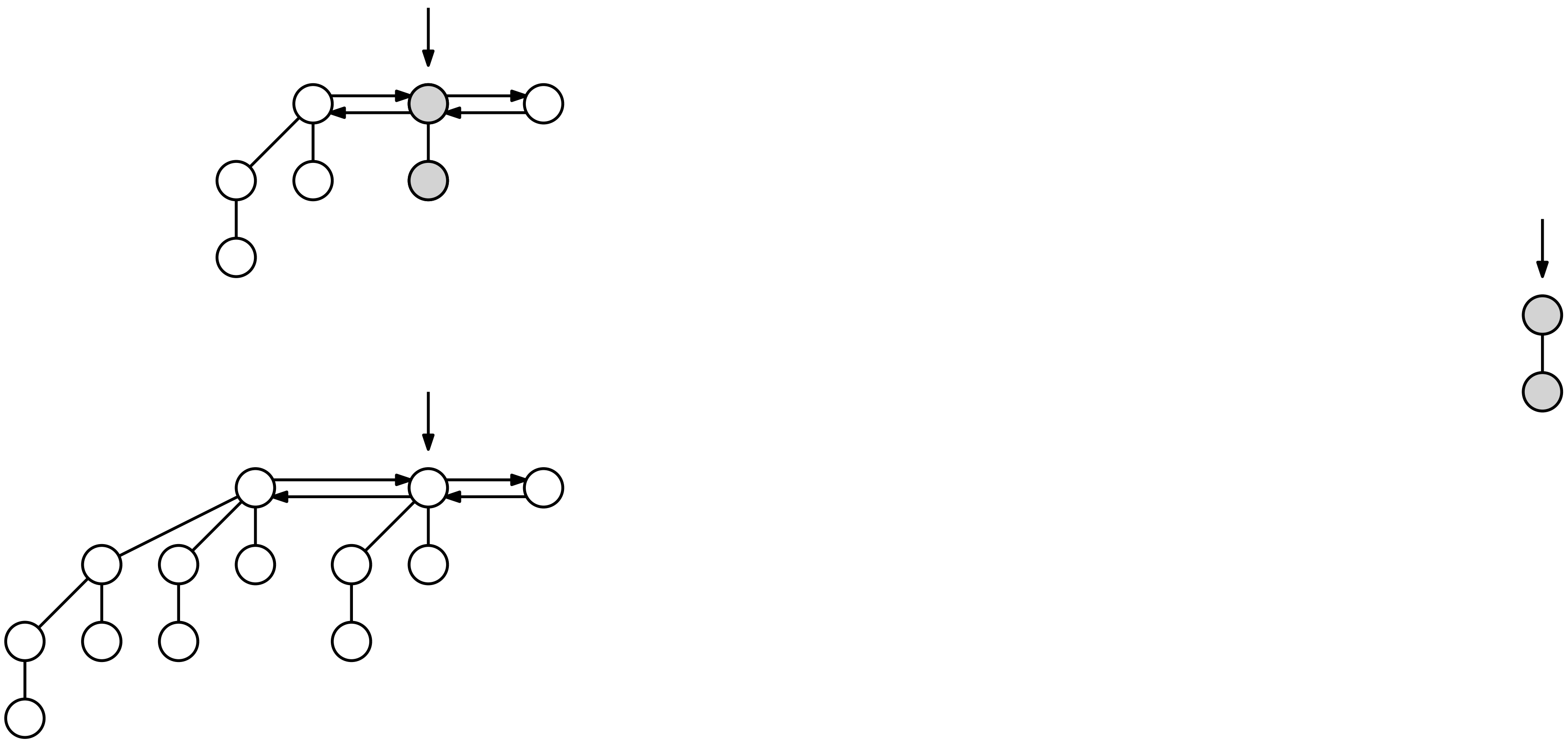

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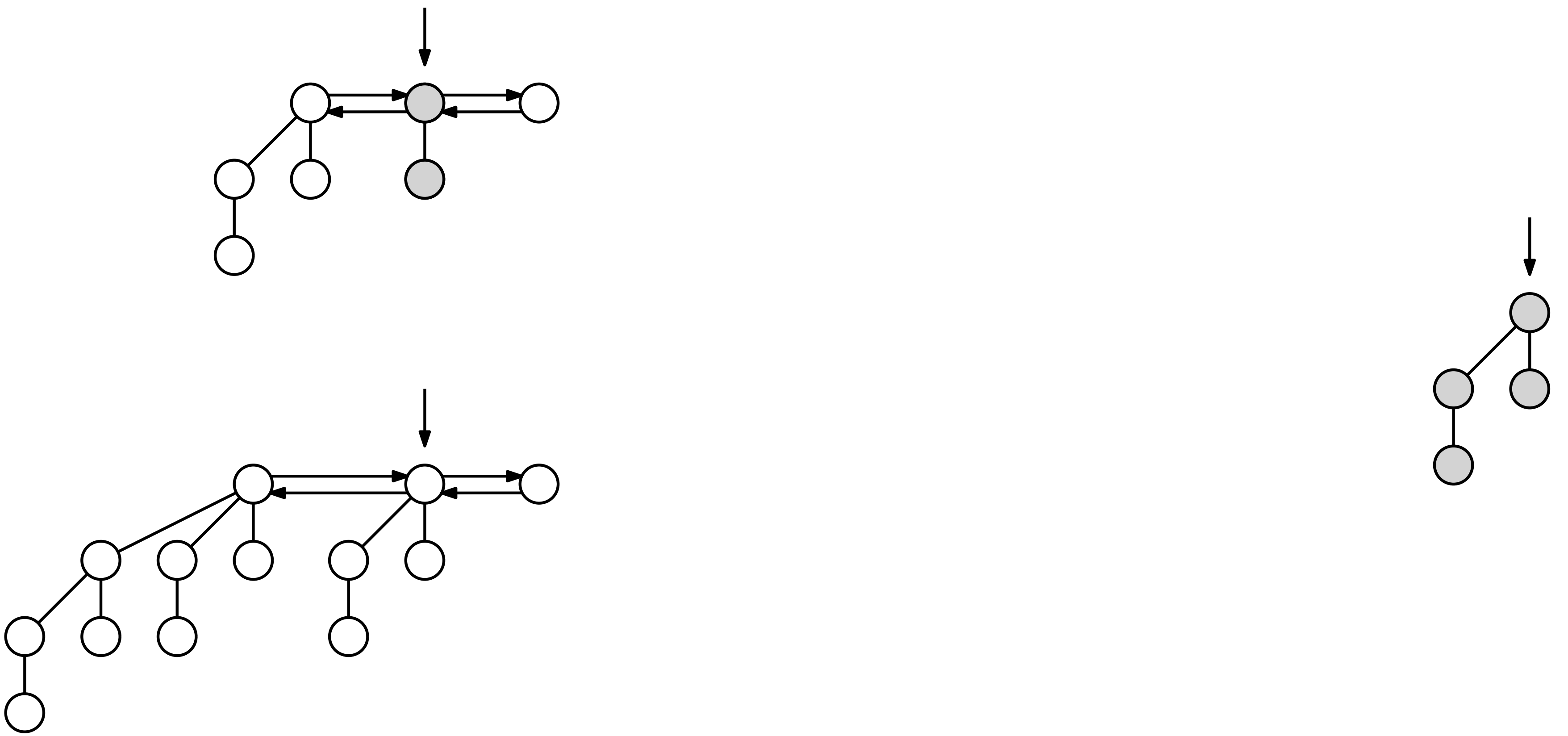

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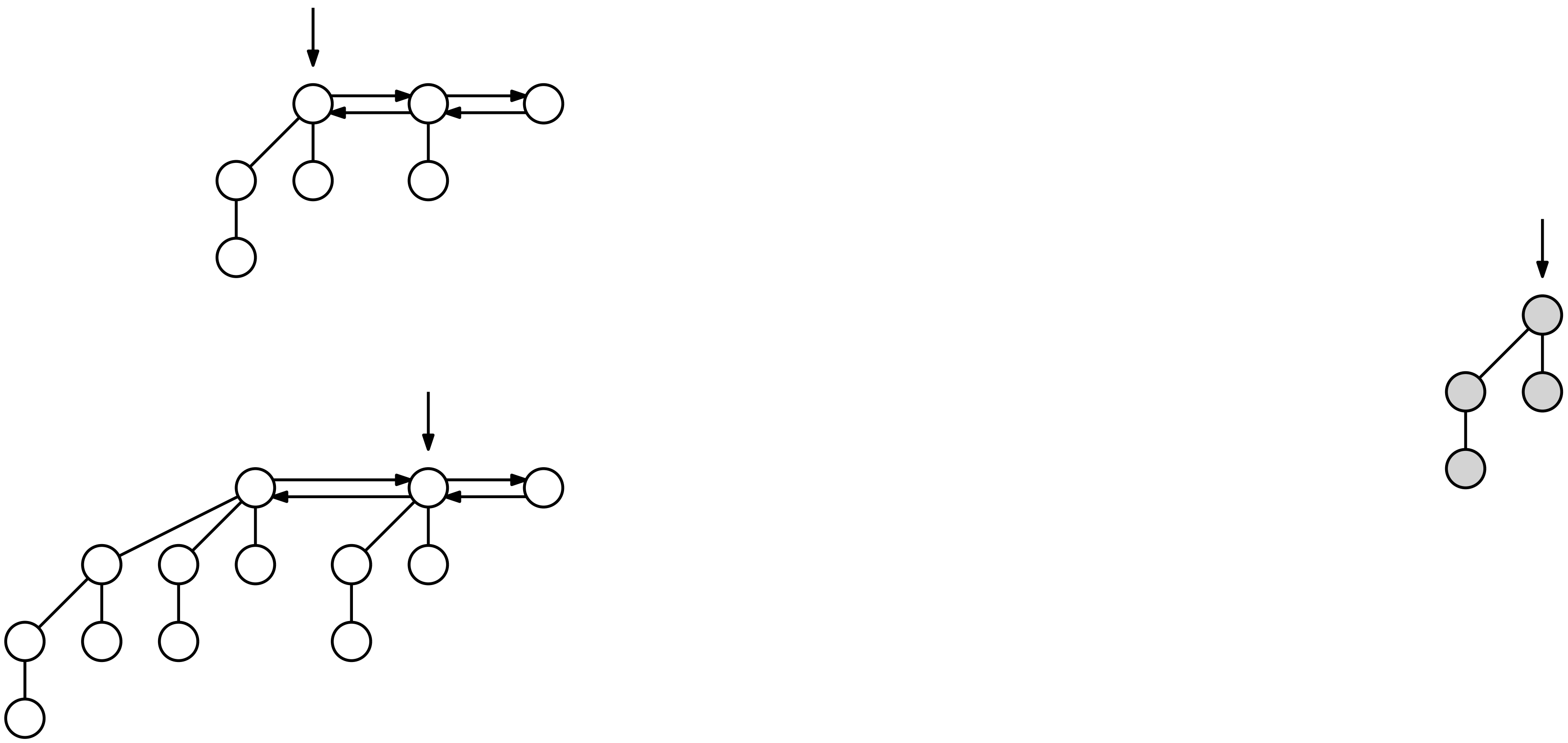

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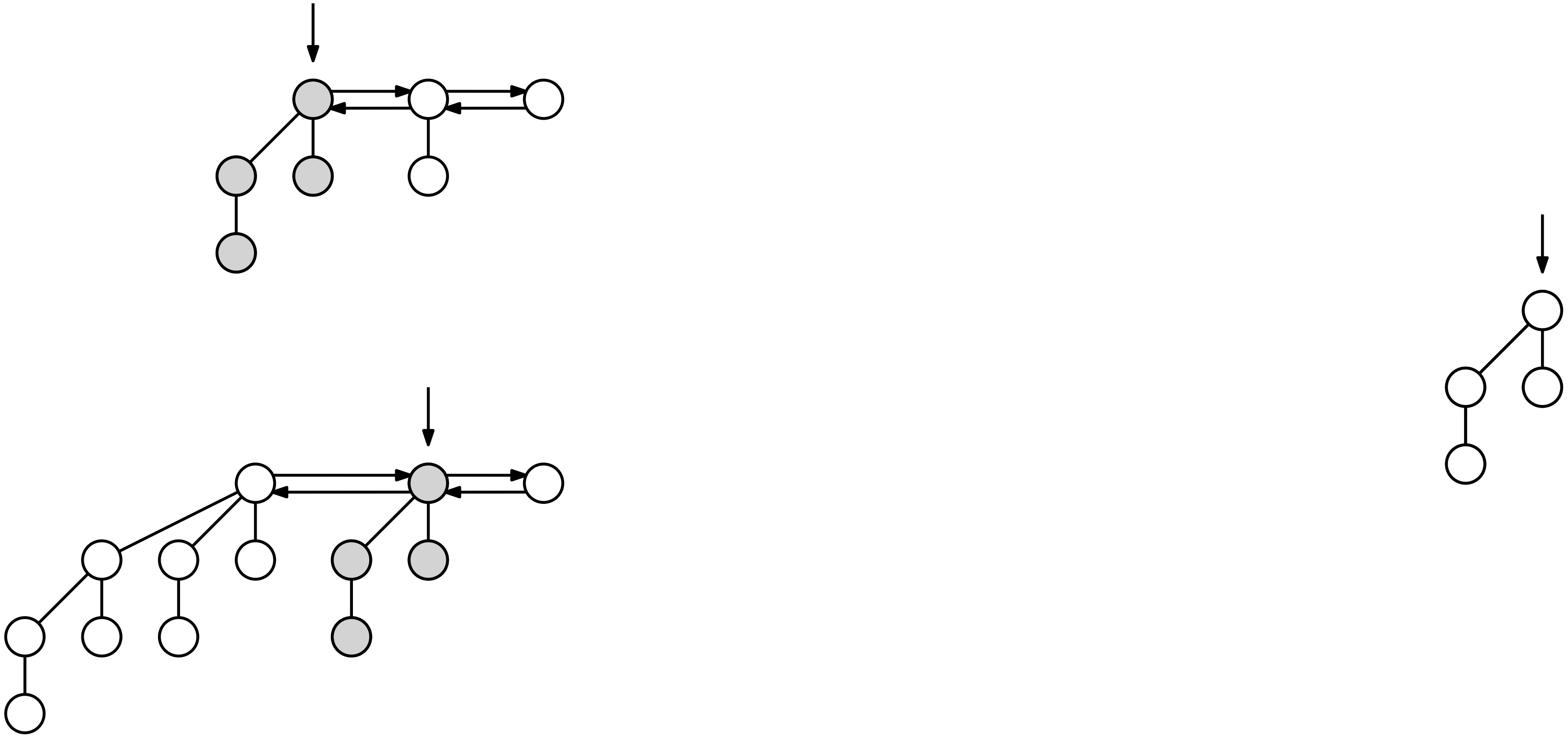
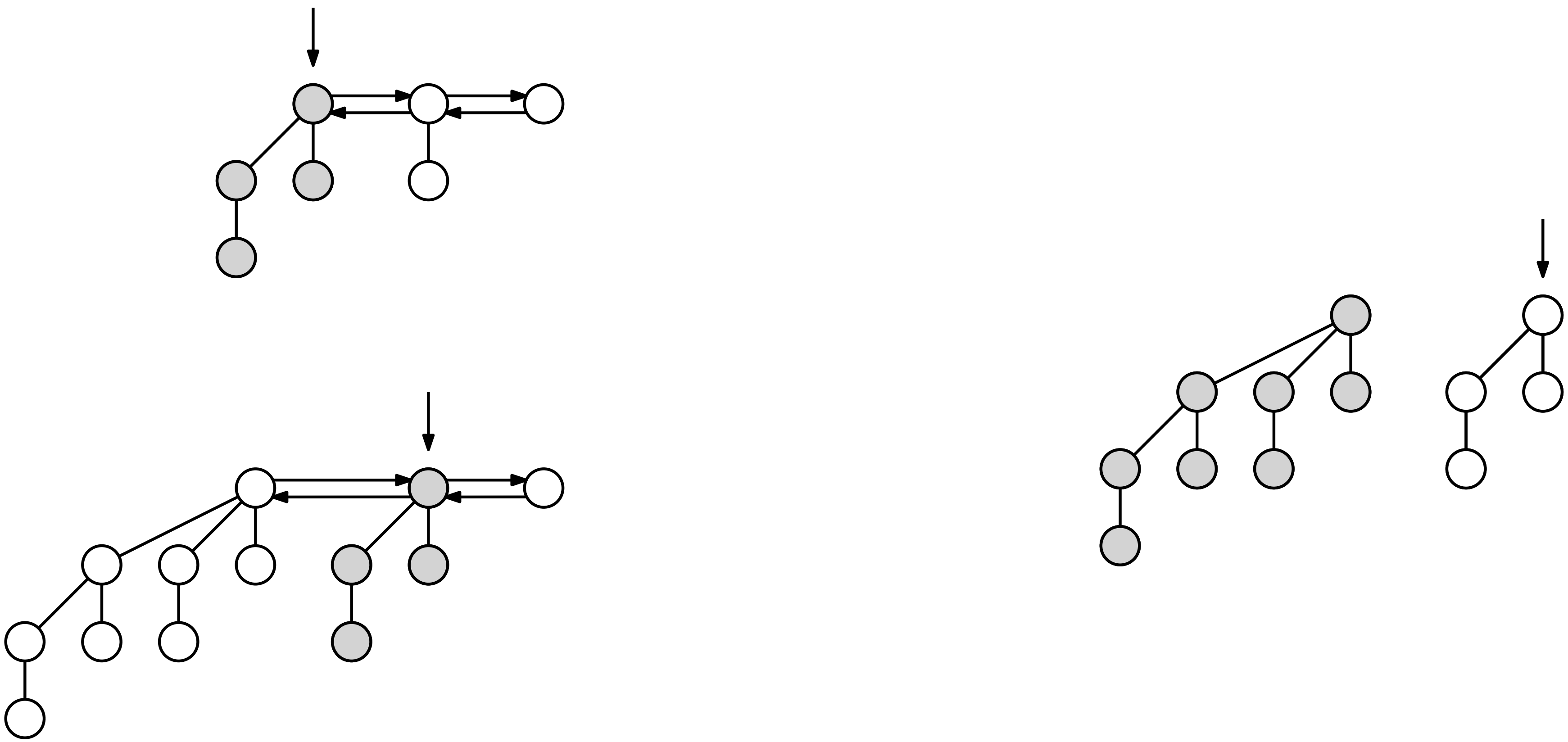


UNION(Q_1, Q_2)



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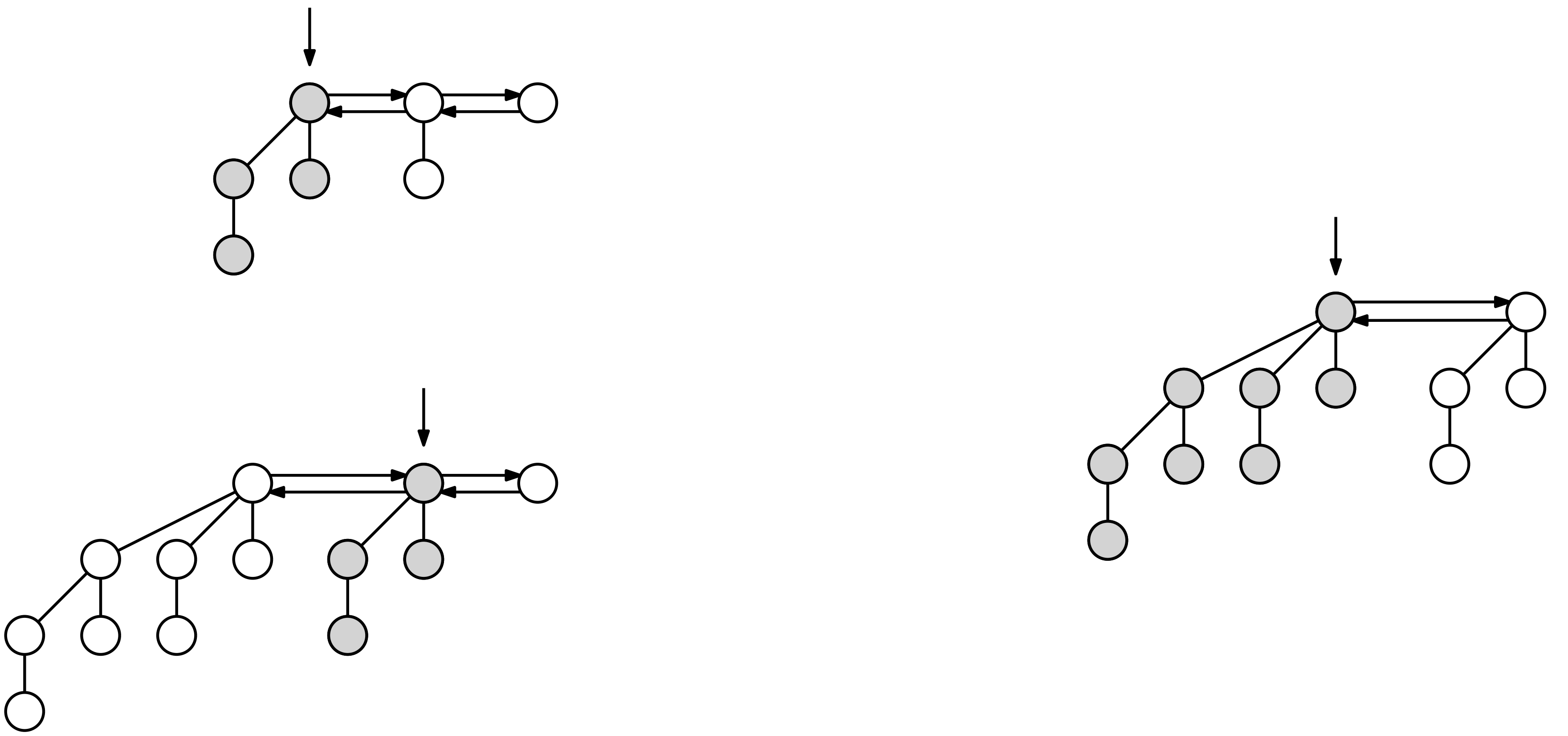

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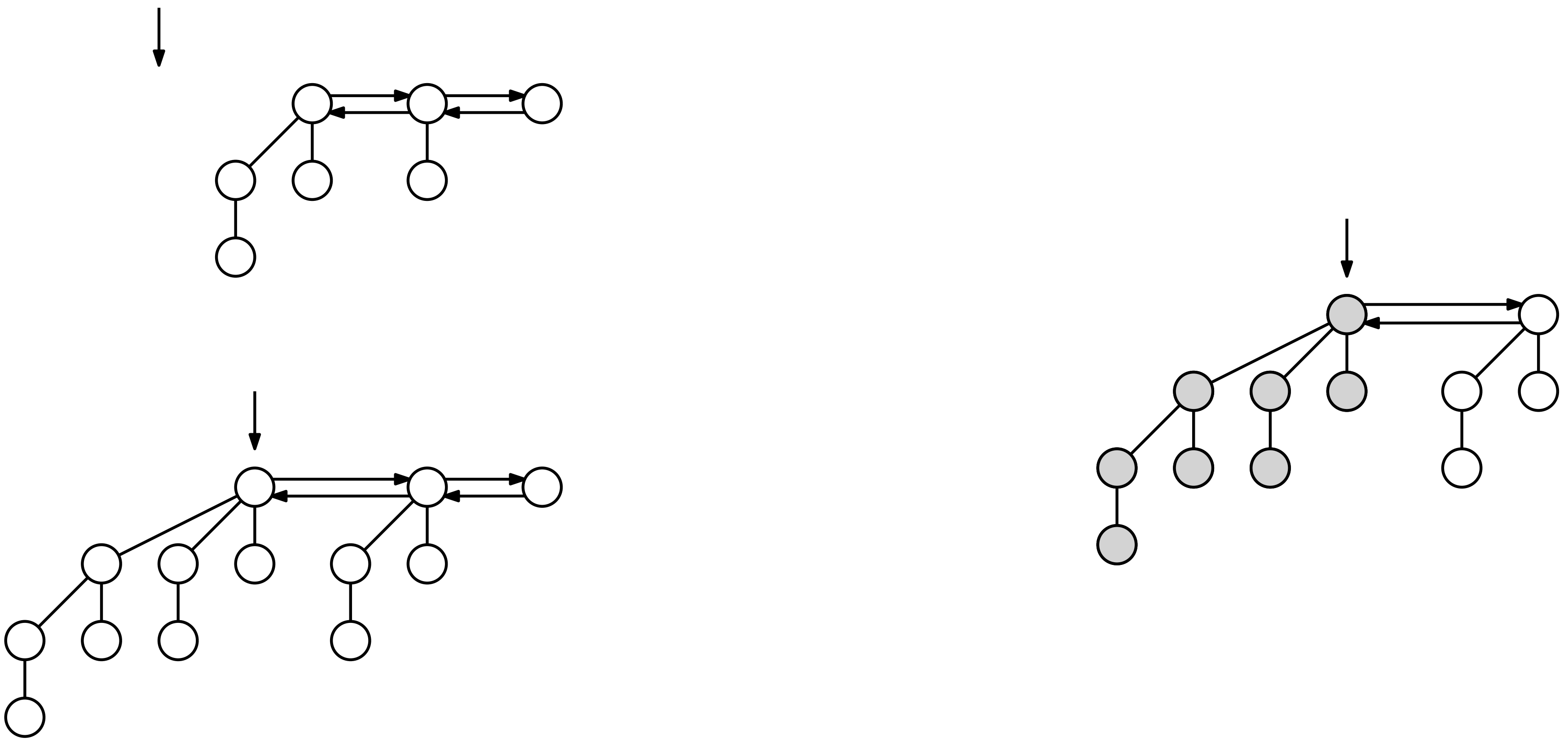

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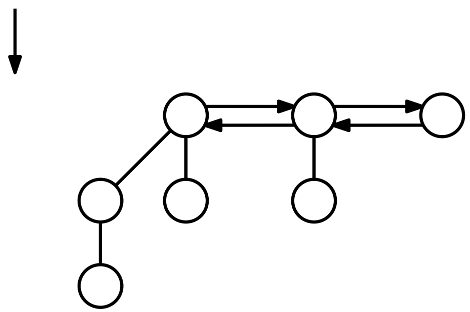

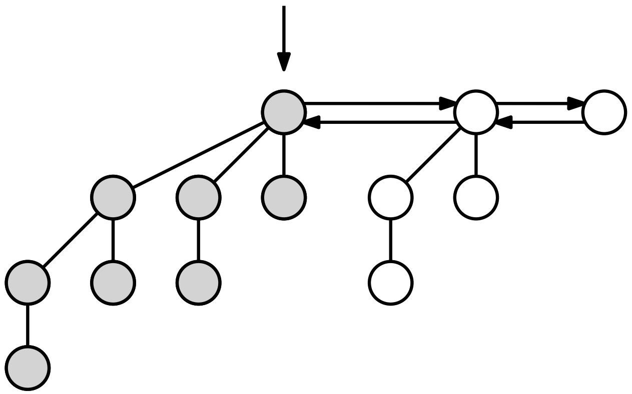

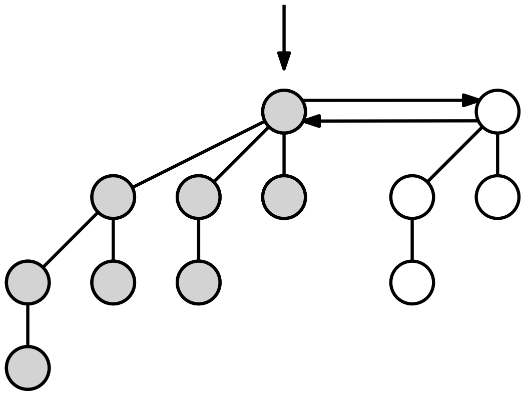

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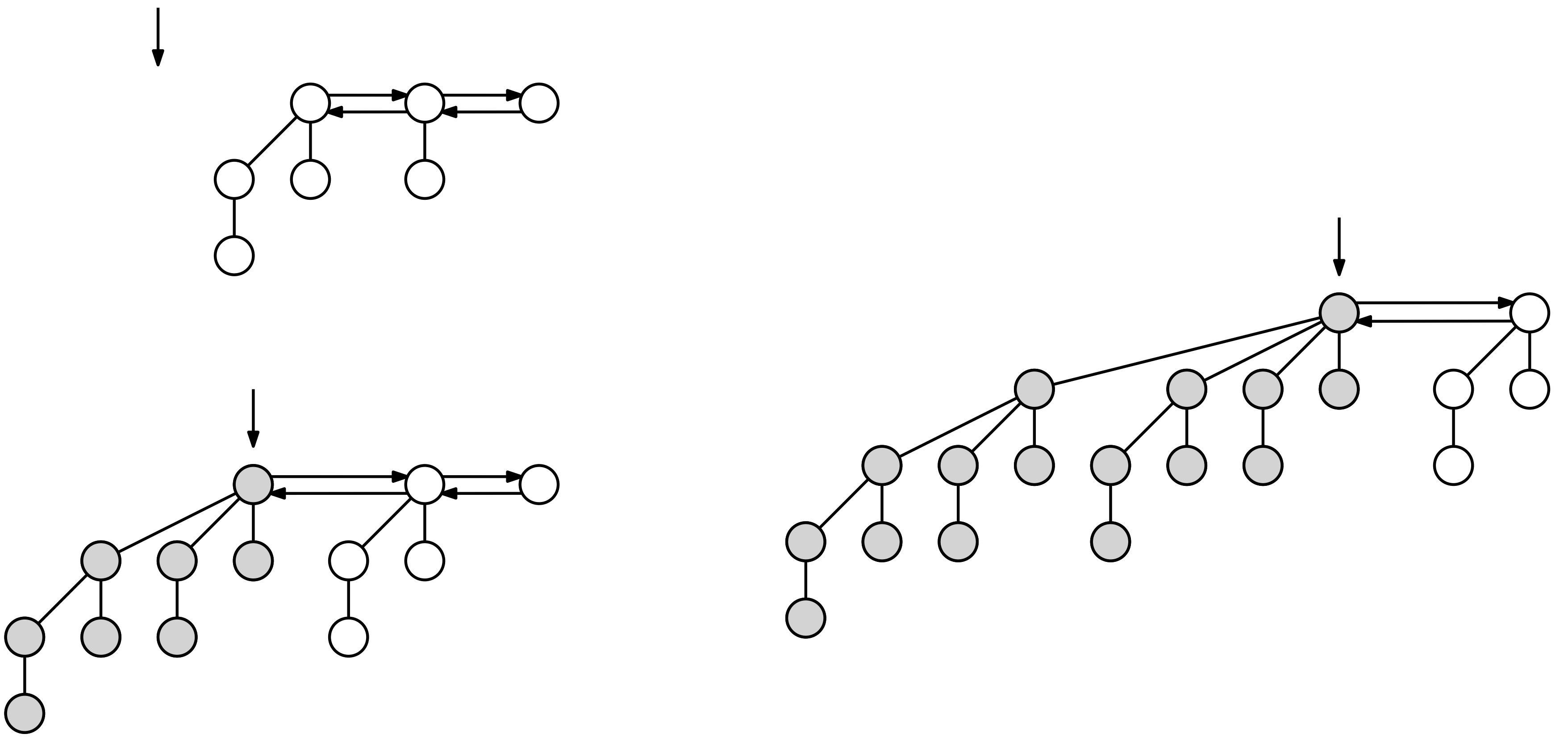

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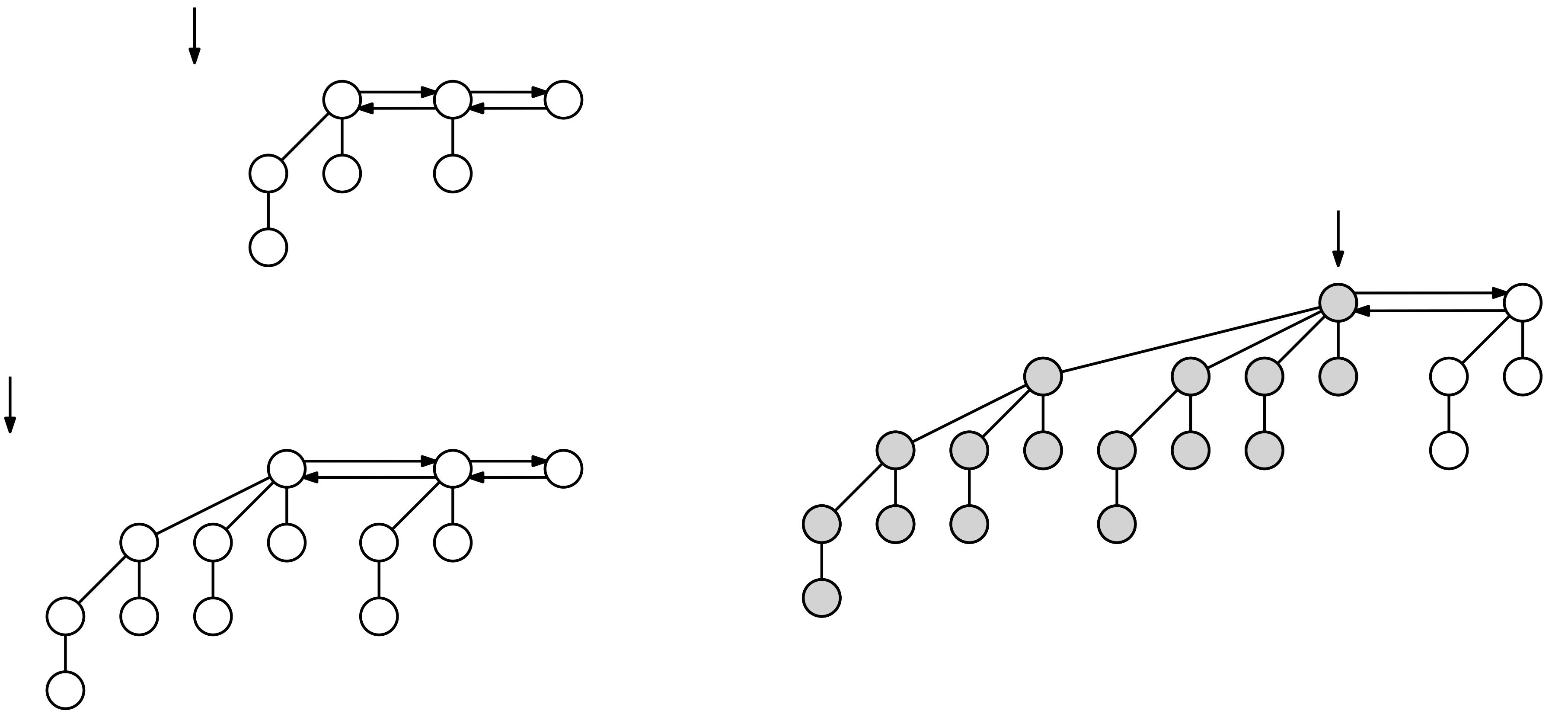

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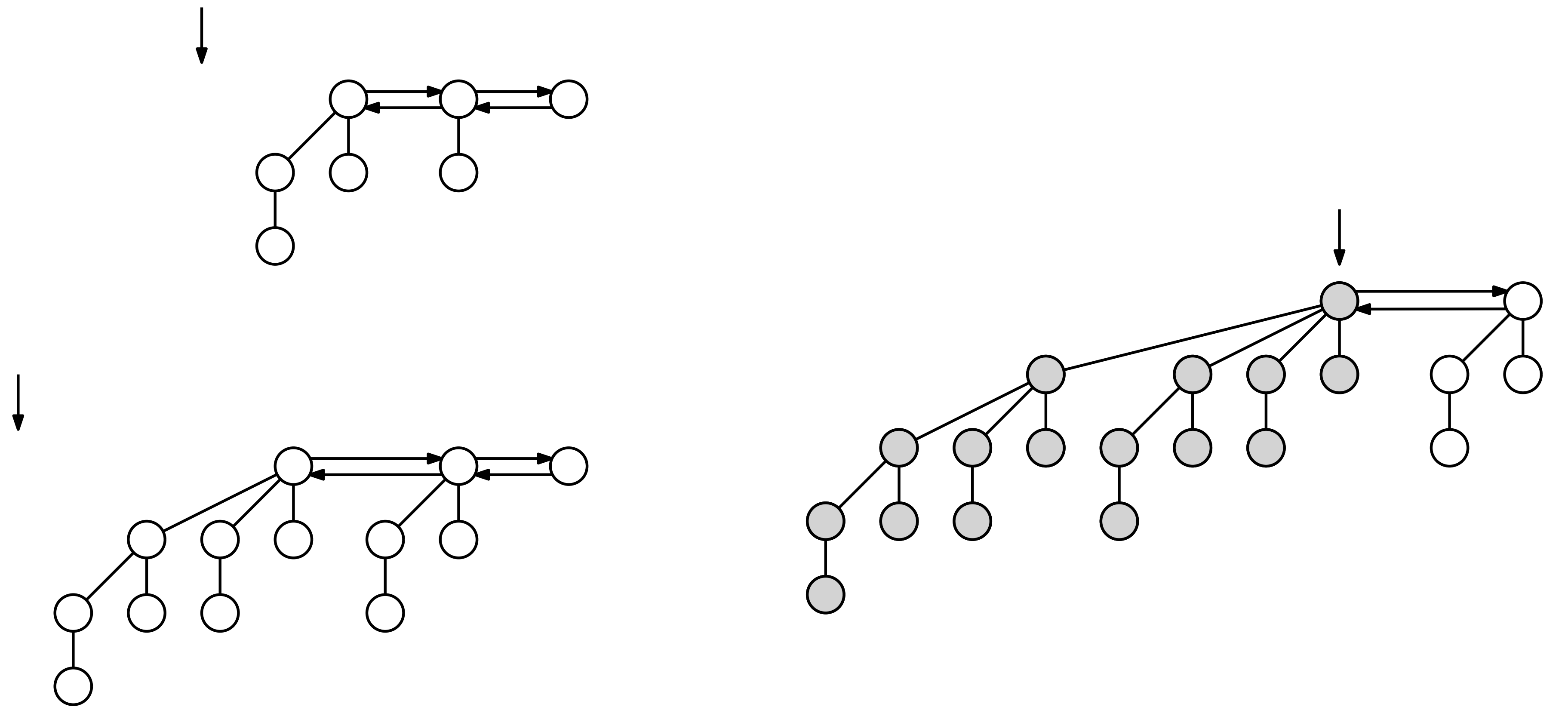
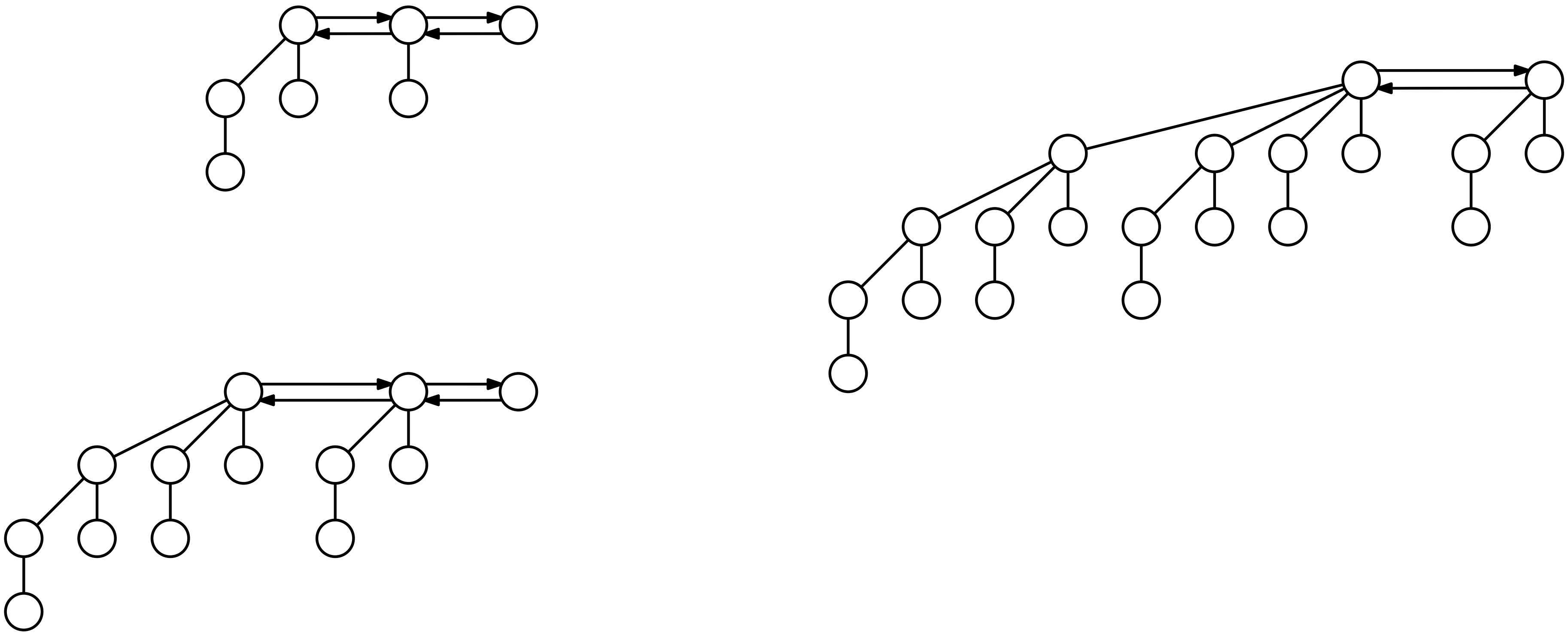


Runtime: $O(\max k) = O(\log n)$

UNION(Q_1, Q_2)



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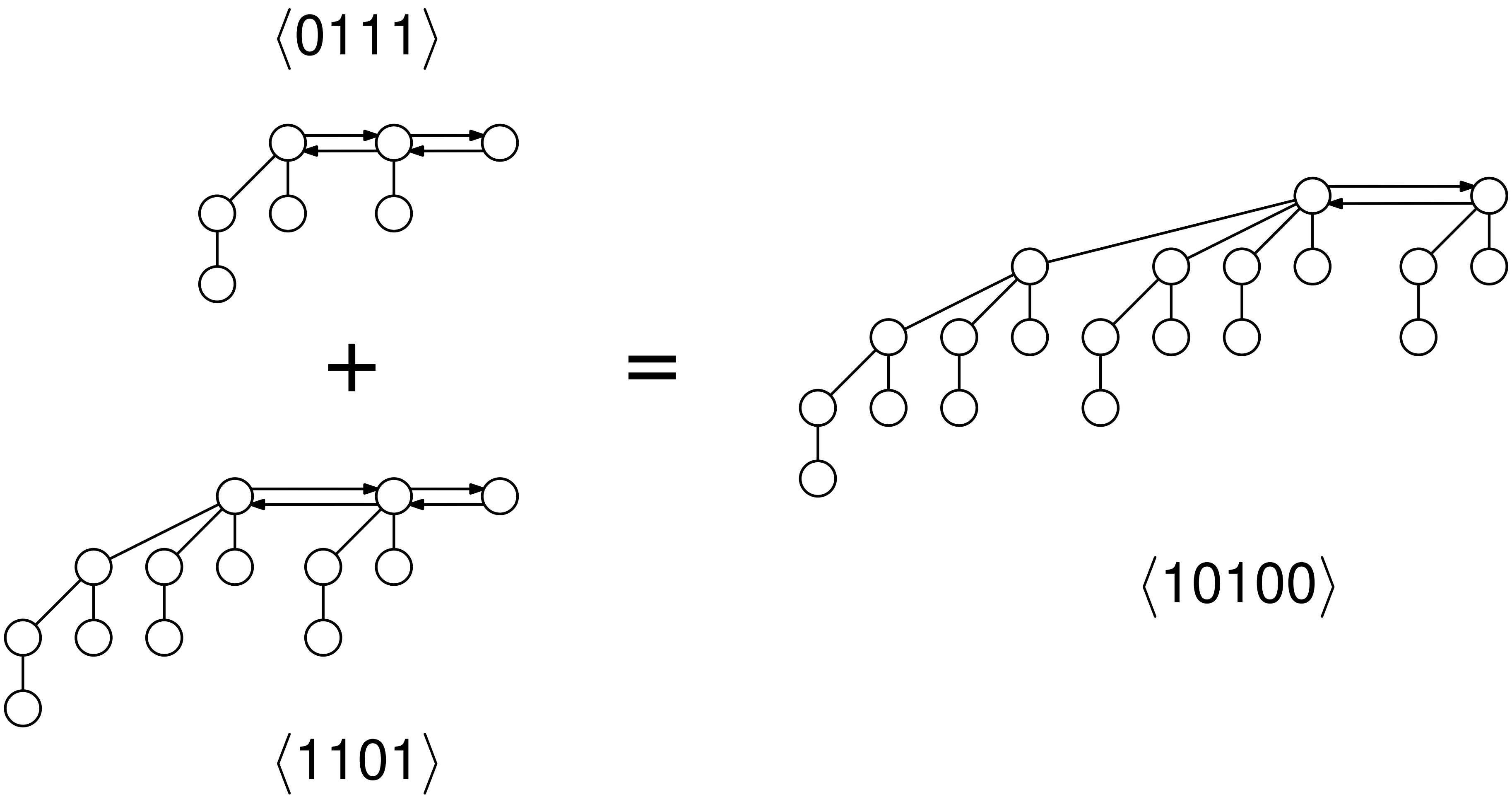
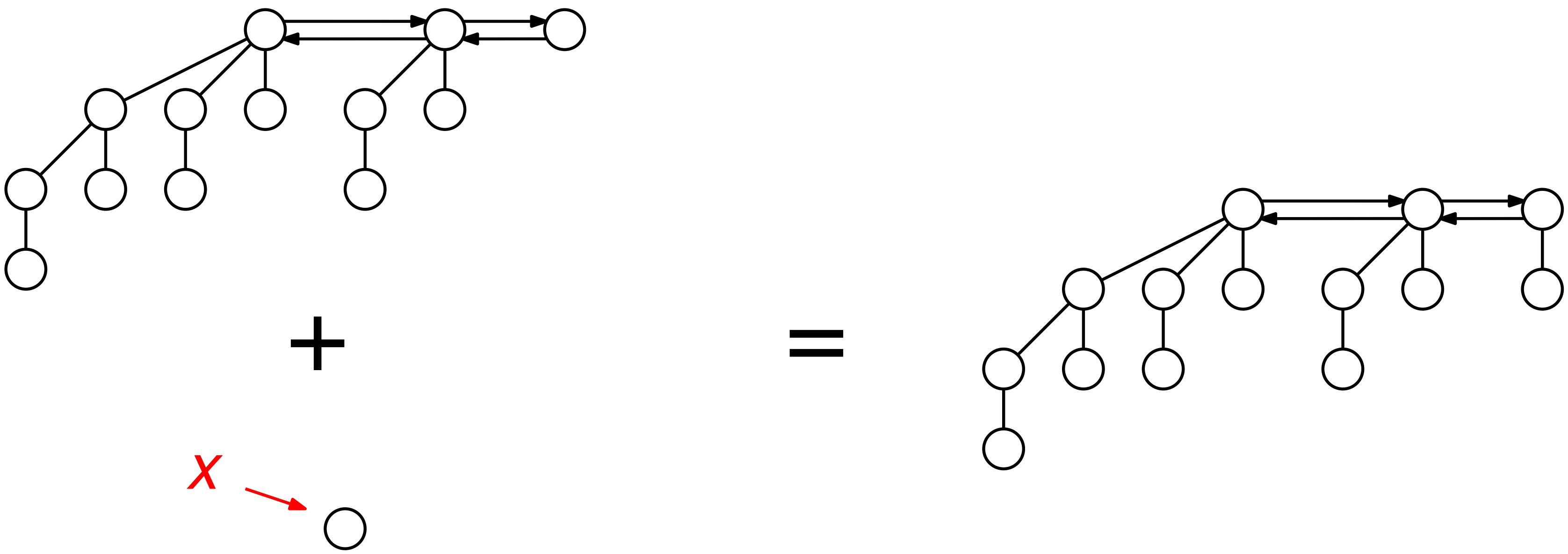


INSERT(Q, x)

Reduces to Union



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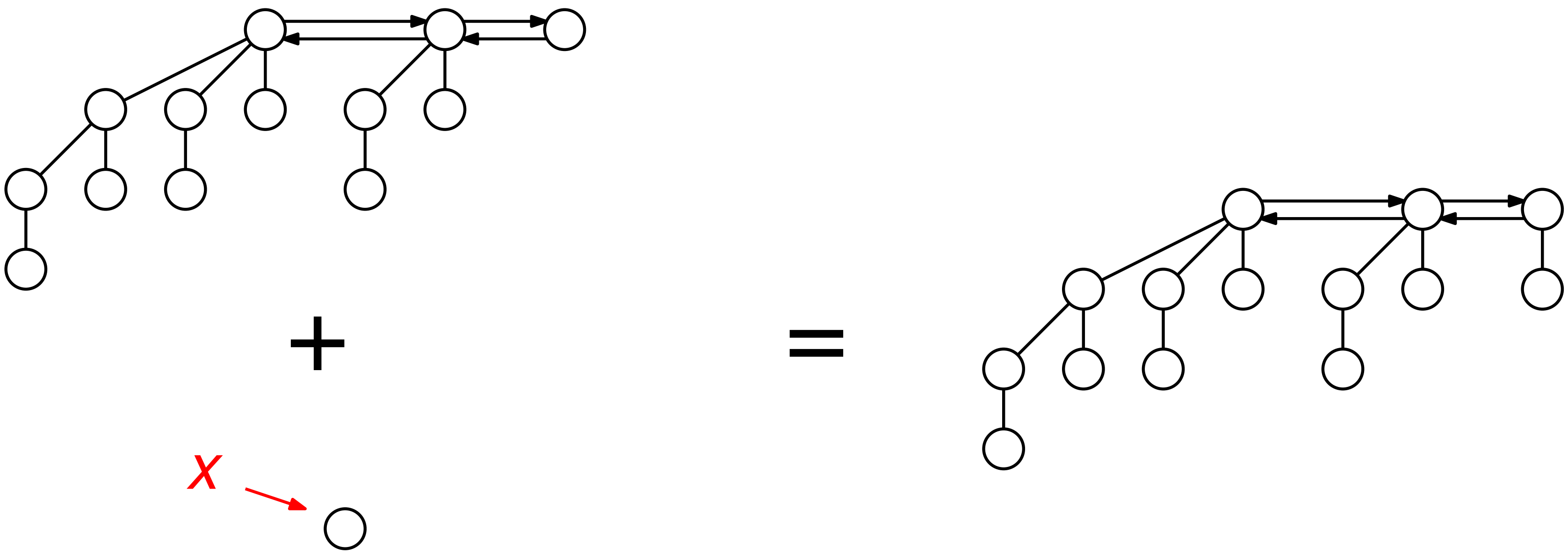
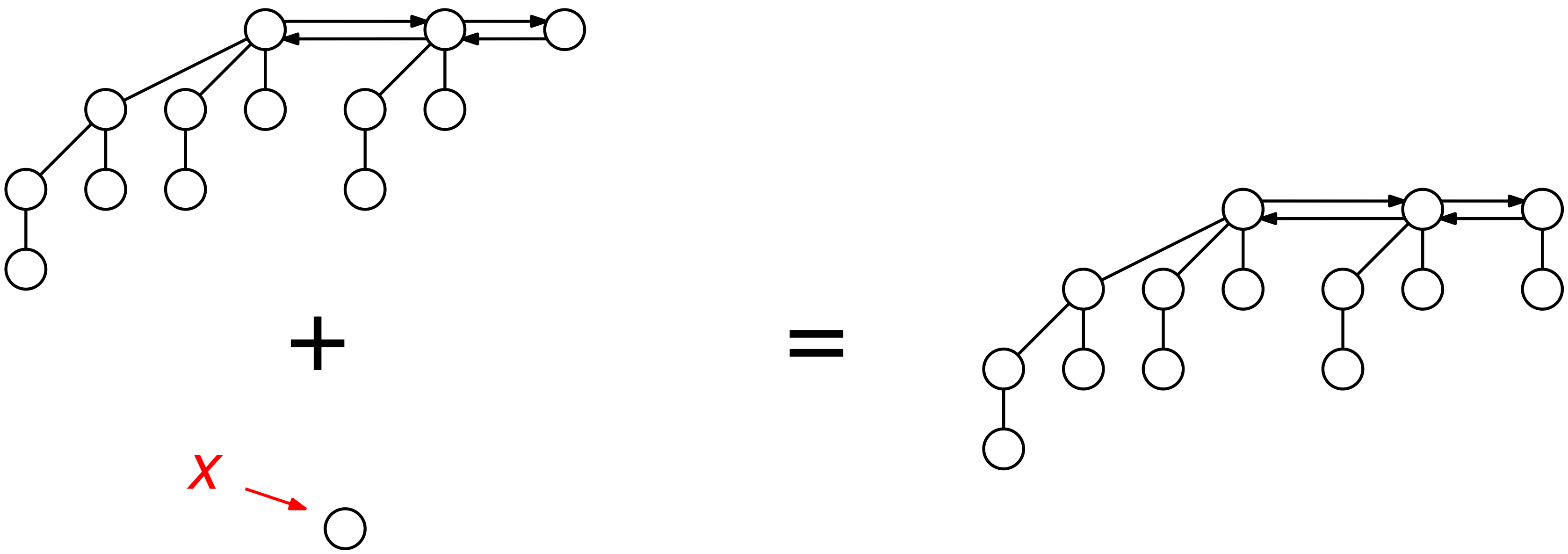


Runtime: $O(\log n)$ worst-case

INSERT(Q, x)

Reduces to Union

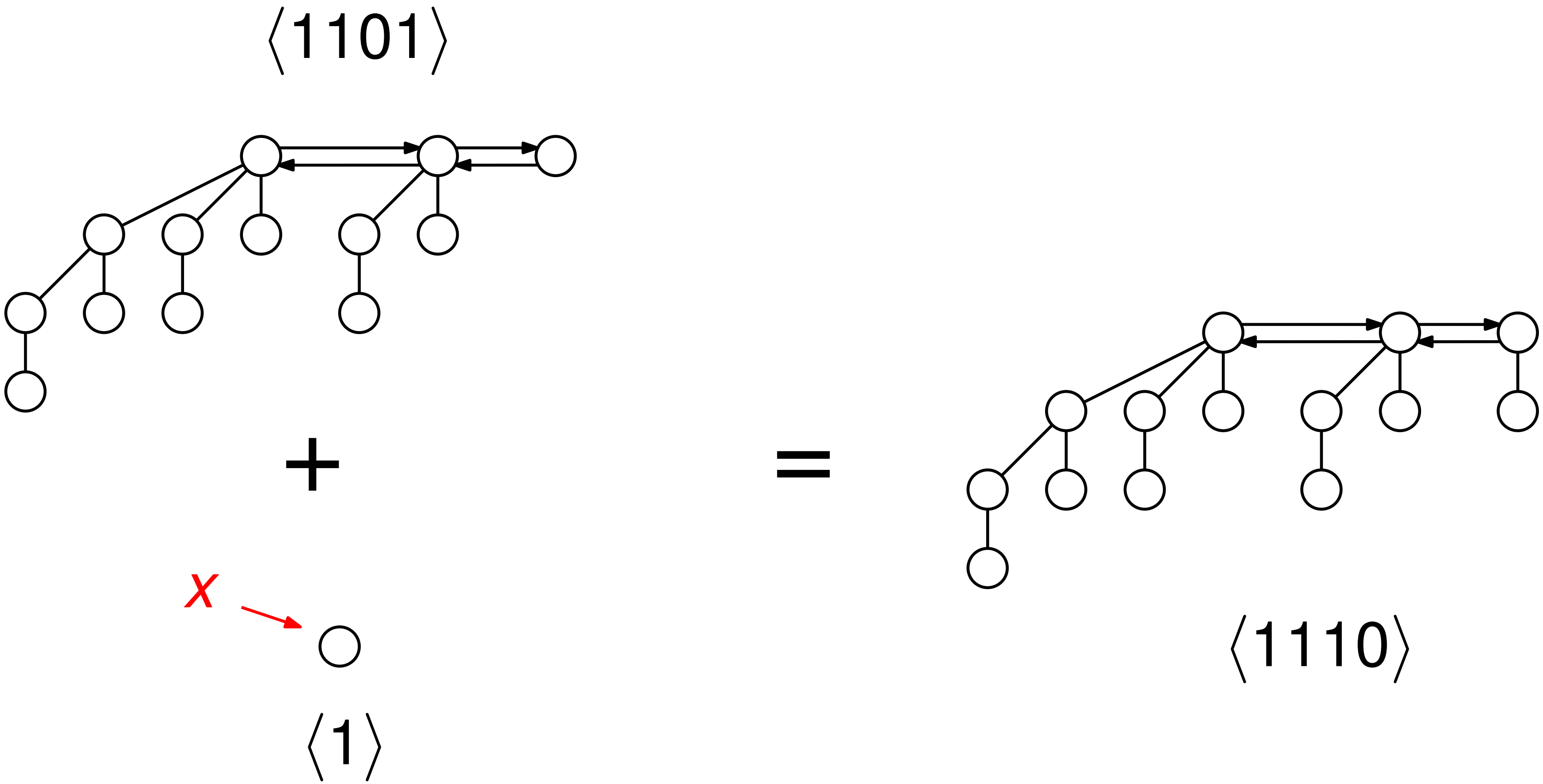


Runtime: $O(\log n)$ worst-case

Tighter analysis?

INSERT(Q, x)

Equivalent to incrementing counter



Runtime: $O(\log n)$ worst-case

Tighter analysis? $O(1)$ amortized

EXTRACT-MIN(Q)

```
function EXTRACT-MIN(Q)
    x = MINIMUM(Q)
    Q' = MAKE()
    Q'.head = x.leftchild
    LINKEDLIST-EXTRACT(x)
    for each child y of x do
        y.parent = NIL
    Q = UNION(Q, Q')
    return x
```

EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)
  x = MINIMUM(Q)
  Q' = MAKE()
  $Q'.head = x.leftchild$
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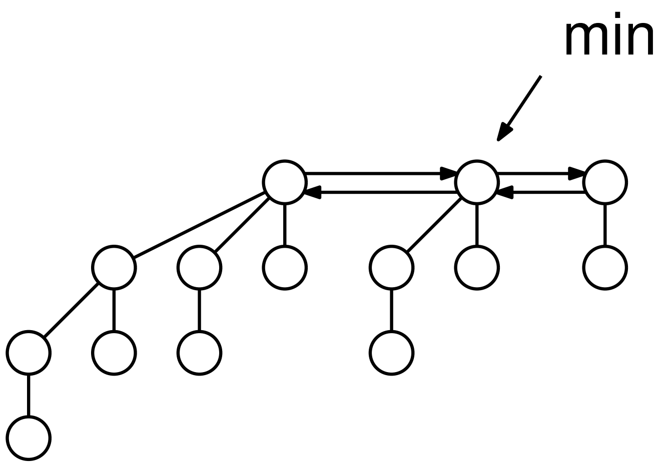

EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)

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Q' = MAKE()

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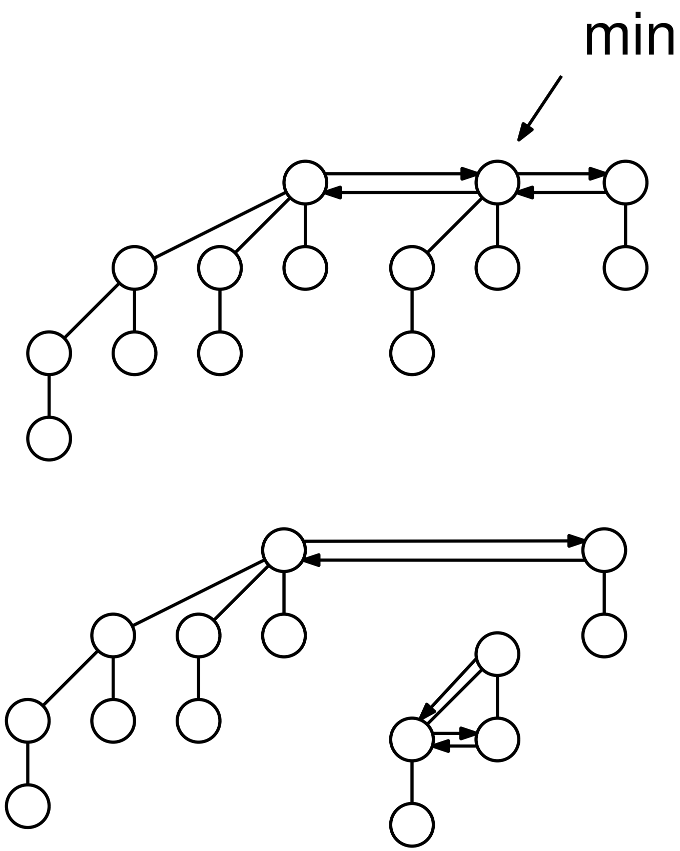

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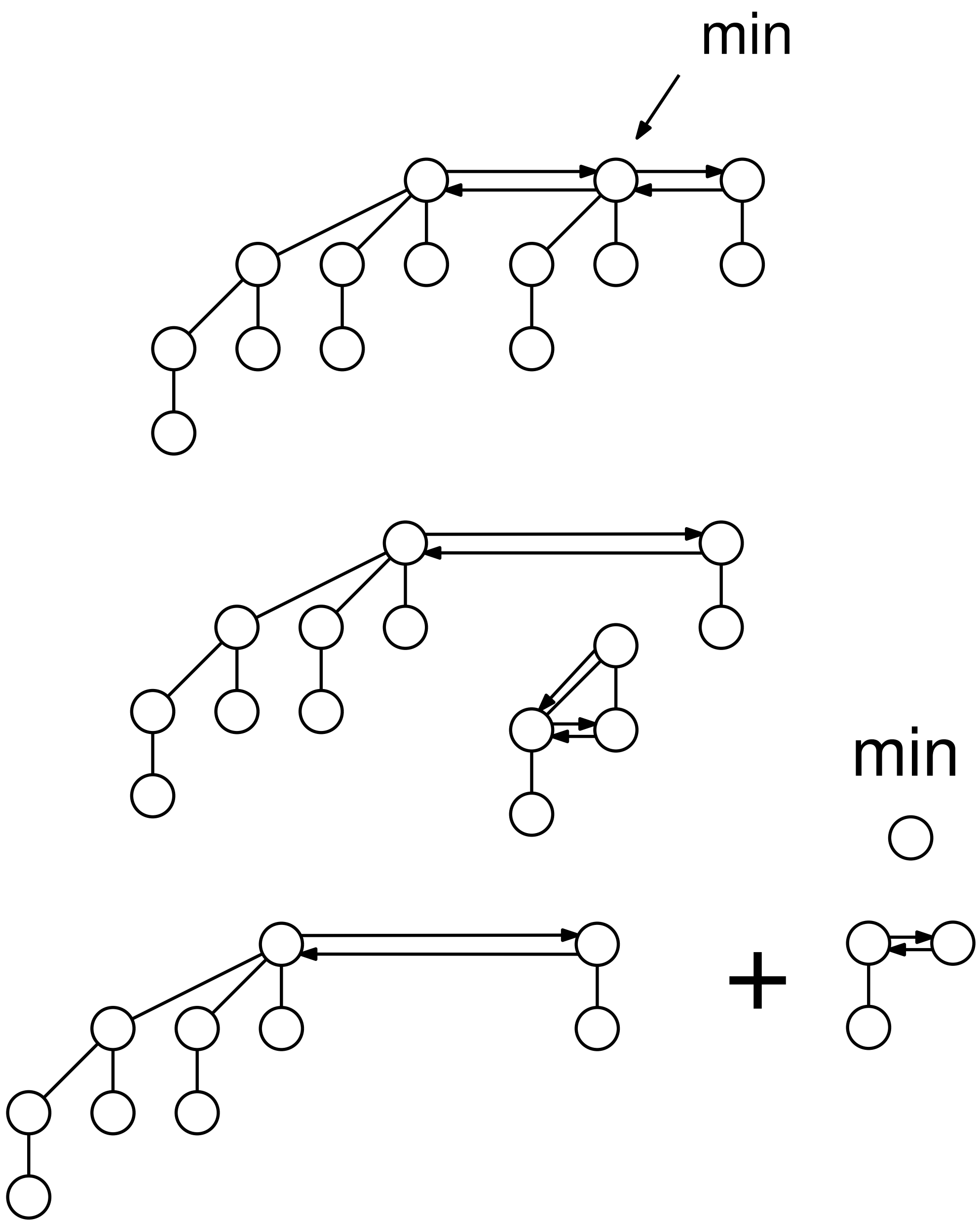
- **return** x



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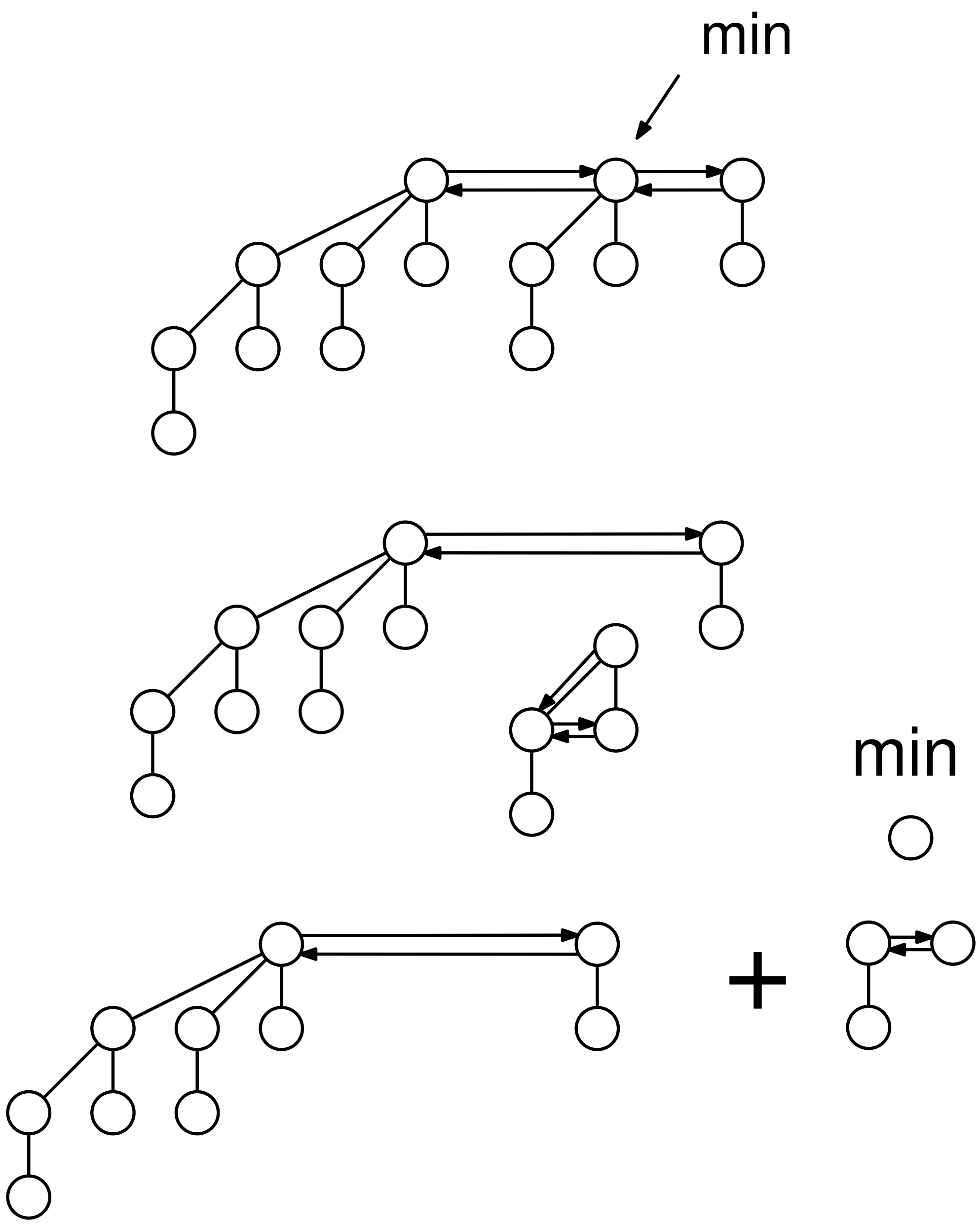


Analysis: $O(\log n)$

DELETE(x)

function DELETE(x)
  DECREASE-KEY(Q, x, $-\infty$)
  EXTRACT-MIN(Q)

Analysis: $O(\log n)$

Binomial Heap Summary

	Binary	Binomial
MAKE()	$O(1)$	$O(1)$
INSERT(Q, x)	$O(\log n)$	$O(1)^*$
MINIMUM(Q)	$O(1)$	$O(1)$
EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)$
DECREASE-KEY(Q, x, k)	$O(\log n)$	$O(\log n)$
DELETE(Q, x)	$O(\log n)$	$O(\log n)$
UNION(Q_1, Q_2)	$O(n)$	$O(\log n)$

* Amortized cost

Binomial Heap Summary

	Binary	Binomial	Lazy Binomial
■ MAKE()	$O(1)$	$O(1)$	$O(1)$
■ INSERT(Q, x)	$O(\log n)$	$O(1)^*$	$O(1)$
■ MINIMUM(Q)	$O(1)$	$O(1)$	$O(1)$
■ EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)$	$O(\log n)^*$
■ DECREASE-KEY(Q, x, k)	$O(\log n)$	$O(\log n)$	$O(\log n)$
■ DELETE(Q, x)	$O(\log n)$	$O(\log n)$	$O(\log n)^*$
■ UNION(Q_1, Q_2)	$O(n)$	$O(\log n)$	$O(1)$

* Amortized cost

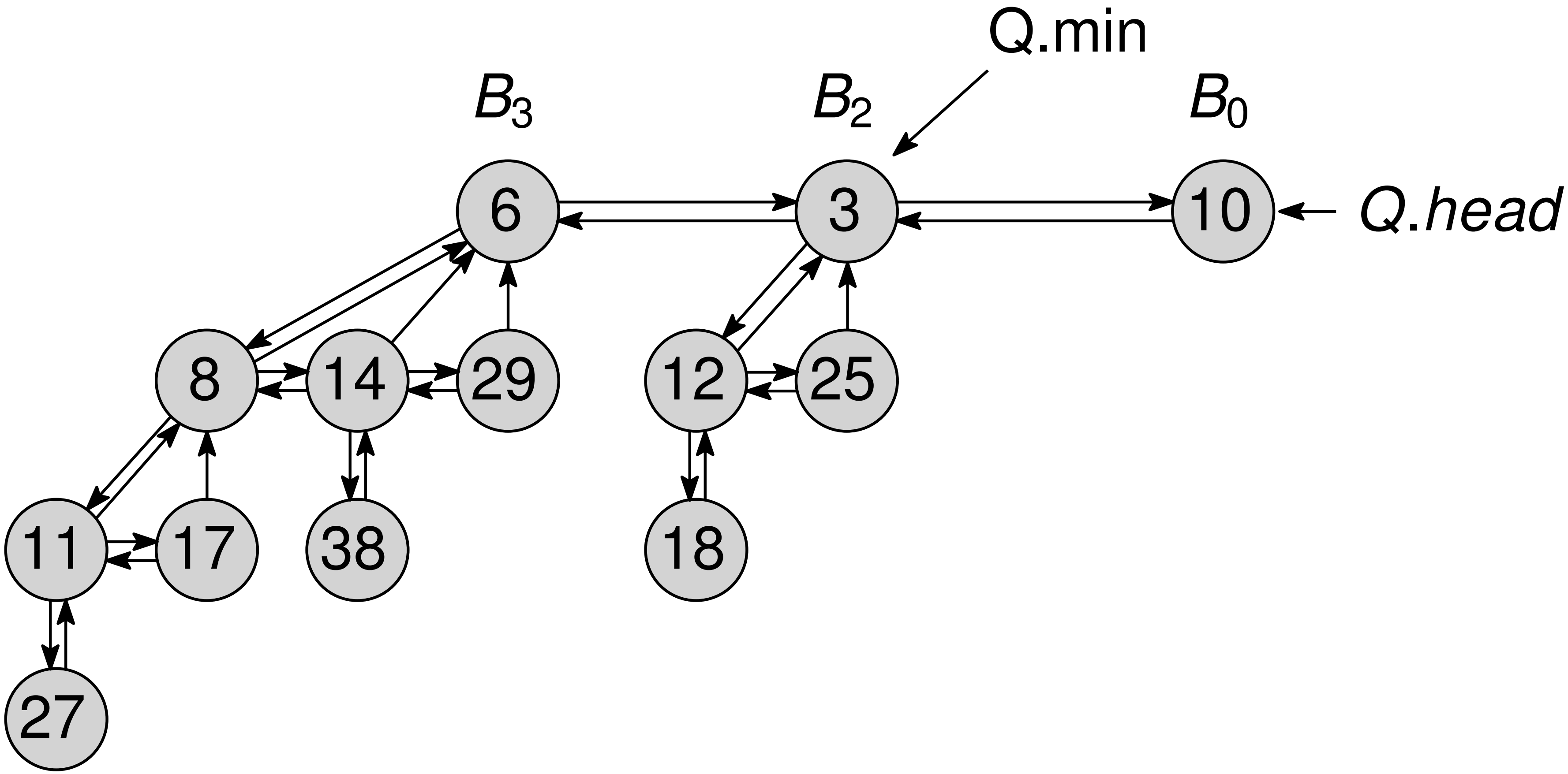
Lazy Binomial Heaps (Homework)

- Don't consolidate trees during UNION
- Consolidate during EXTRACT-MIN(Q)

Binomial Heaps

Collection of heap-ordered binomial trees:

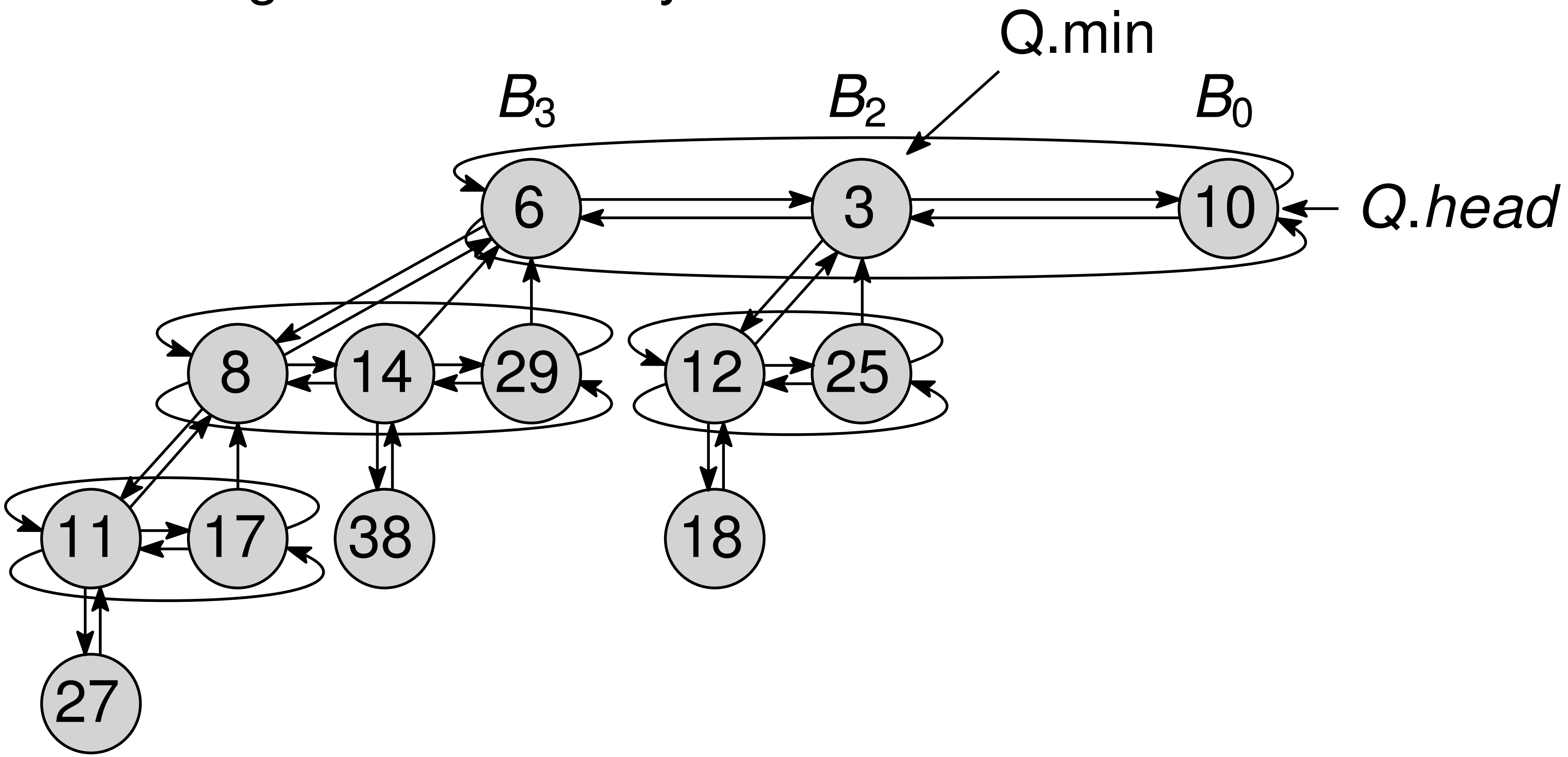
- Each tree is heap-ordered
- At most **one** tree B_k, for $k = 0, 1, 2, \ldots \lfloor \log n \rfloor$



Lazy Binomial Heaps

Collection of heap-ordered binomial trees:

- Each tree is heap-ordered
- *Arbitrary* number of trees in the root list
- Sibling lists are doubly-linked *circular* lists



Lazy Binomial Heaps

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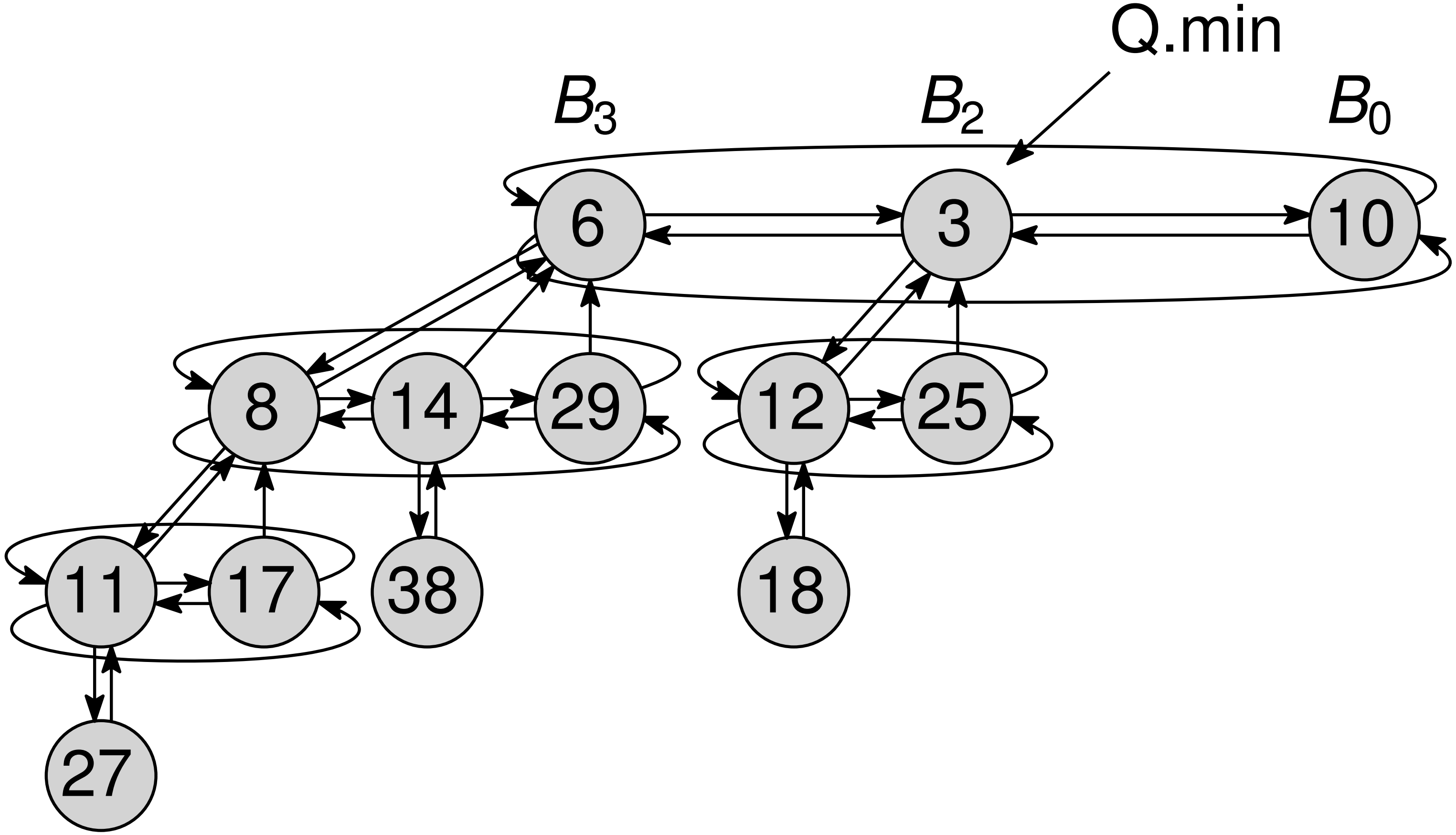
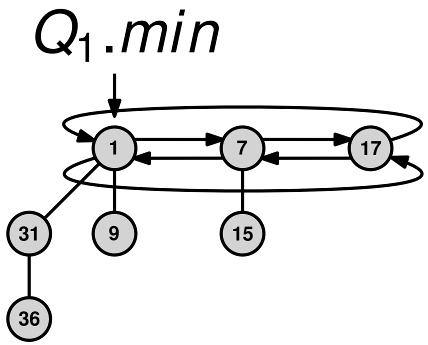
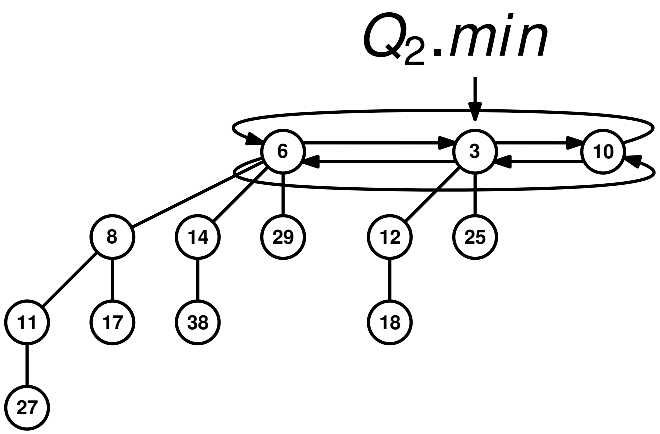


Lazy Union(Q_1, Q_2)





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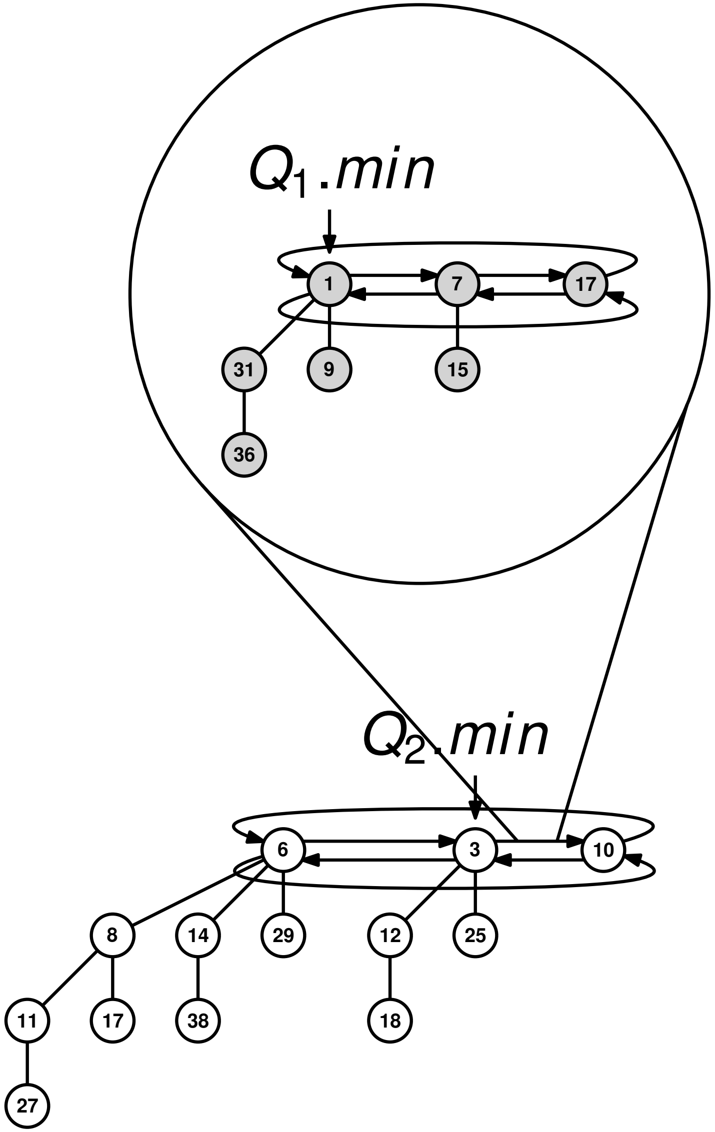
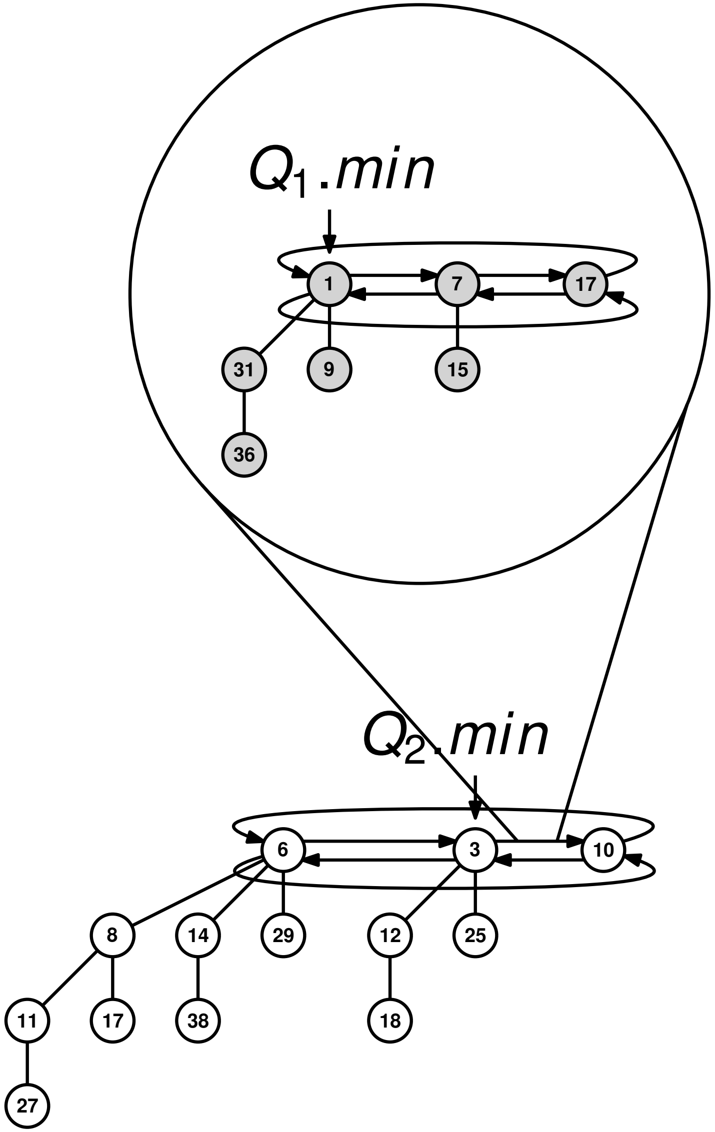
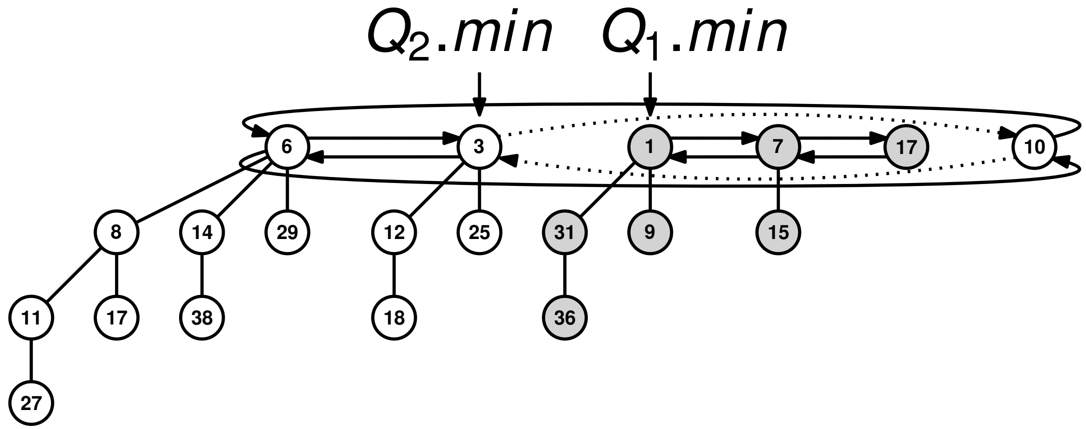
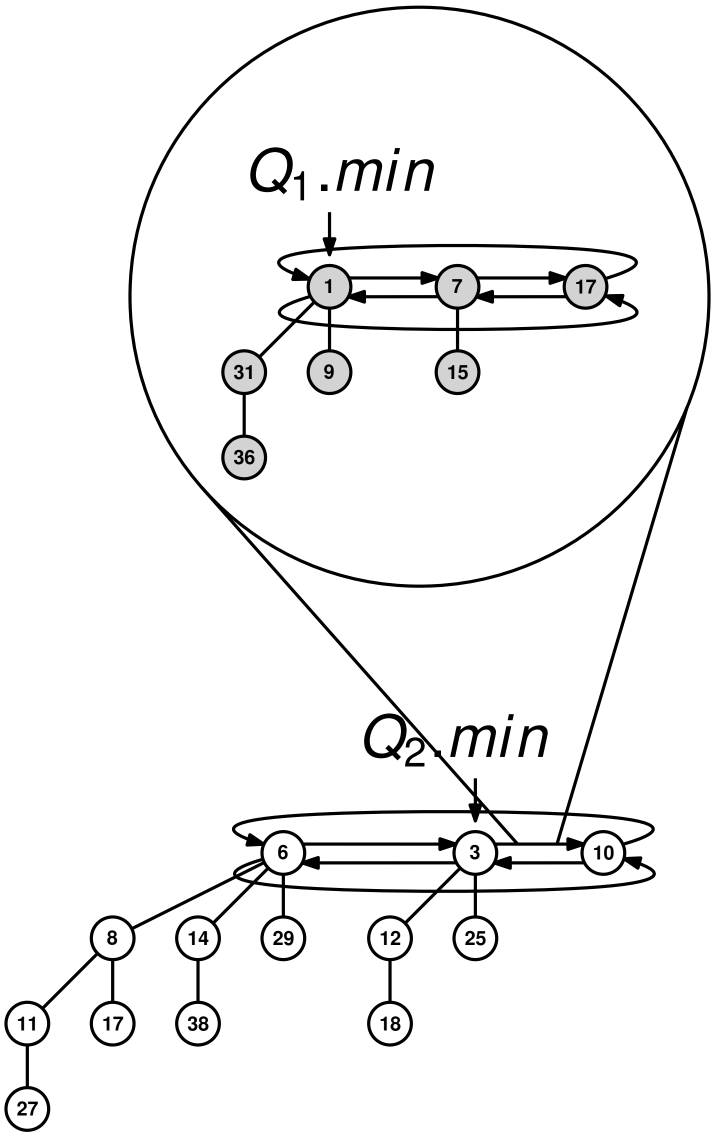


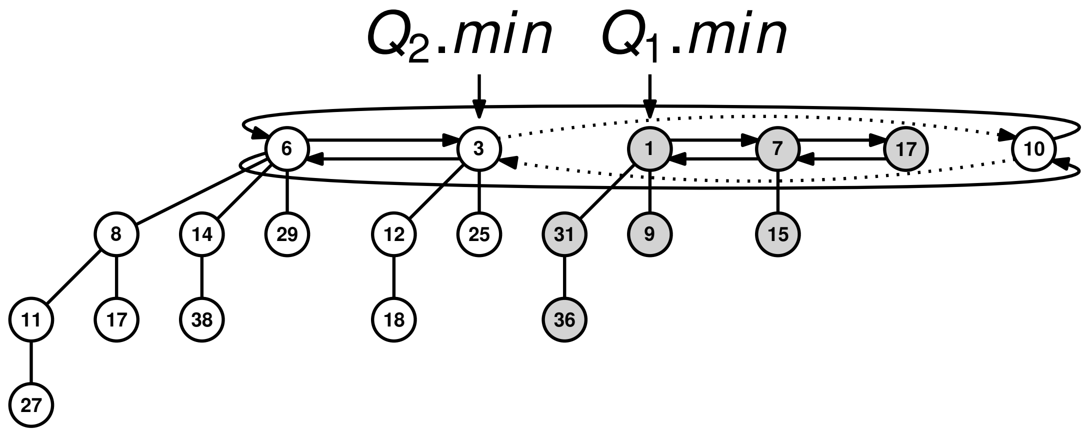
Lazy $\textsc{Union}(Q_1, Q_2)$





Lazy $\textsc{Union}(Q_1, Q_2)$





function $\textsc{Union}(Q_1, Q_2)$

$L_1 \leftarrow Q_1.min.left$

$R_2 \leftarrow Q_2.min.right$

$L_1.right \leftarrow R_2$

$R_2.left \leftarrow L_1$

$Q_2.min.right \leftarrow Q_1.min$

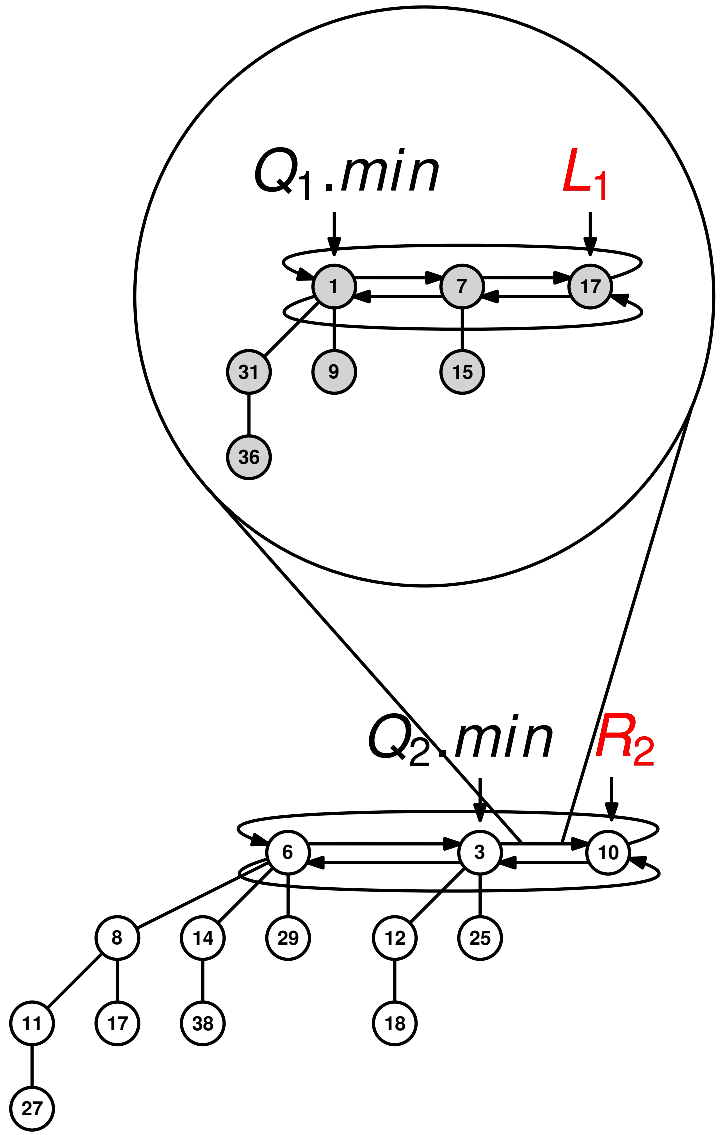
$Q_1.min.left \leftarrow Q_2.min$

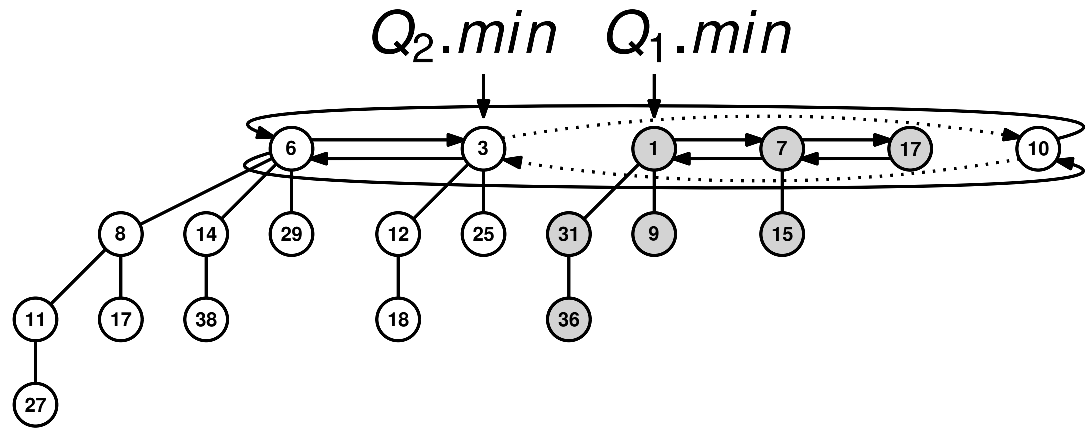
if $Q_1.min.key < Q_2.min.key$ **then**

$Q_2.min \leftarrow Q_1.min$

return Q_2

Lazy UNION(Q_1, Q_2)





function UNION(Q_1, Q_2)

→ $L_1 \leftarrow Q_1.min.left$

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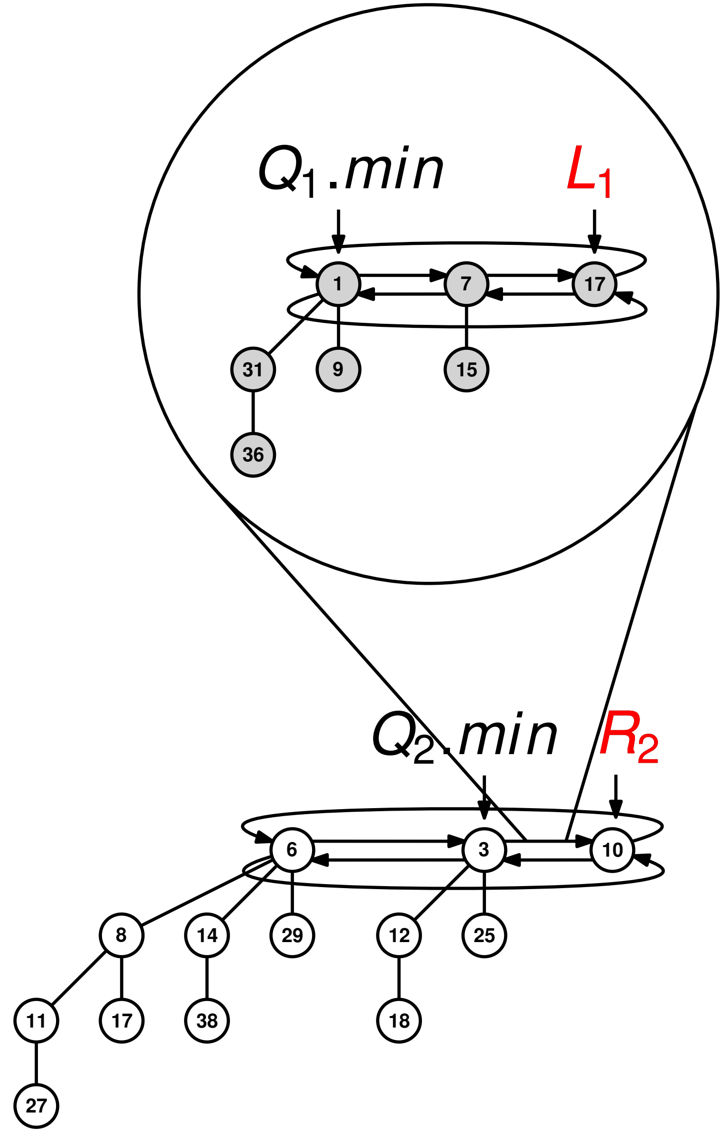

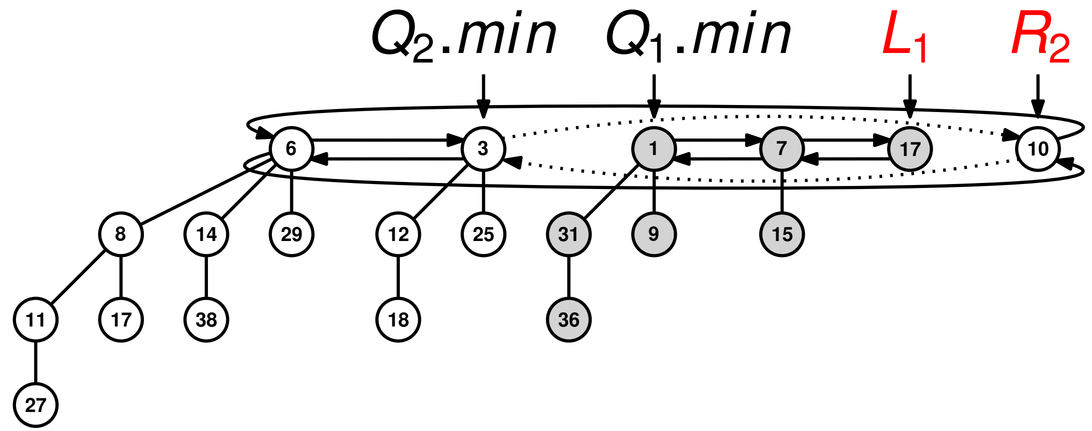

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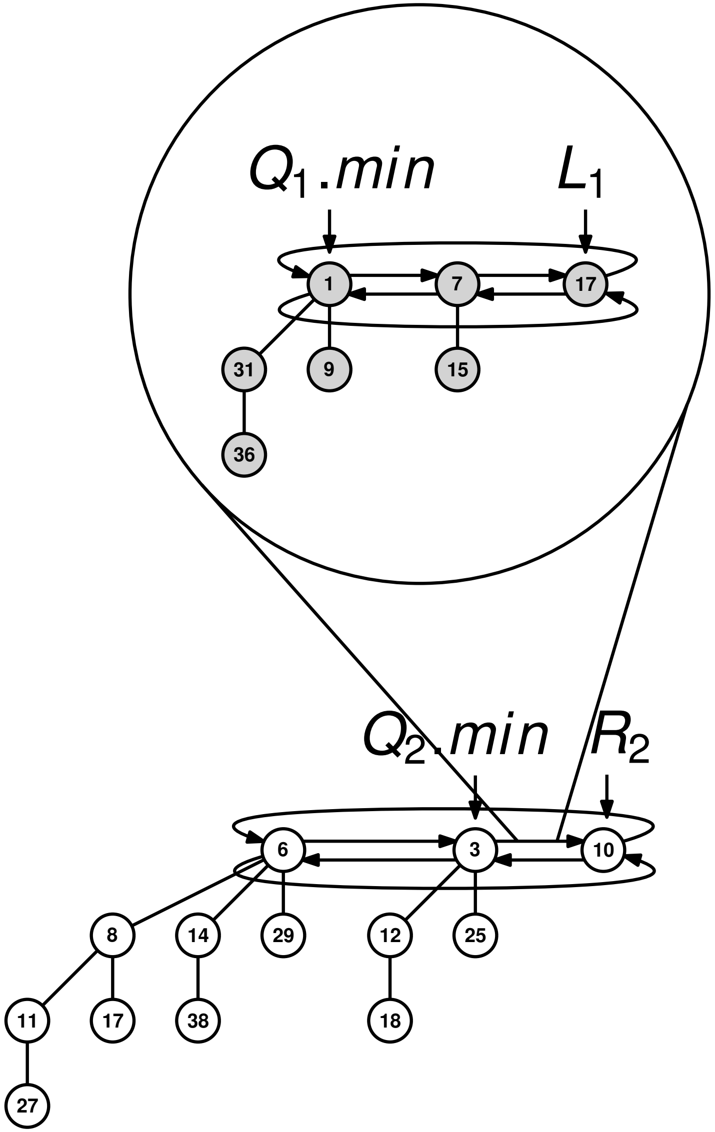
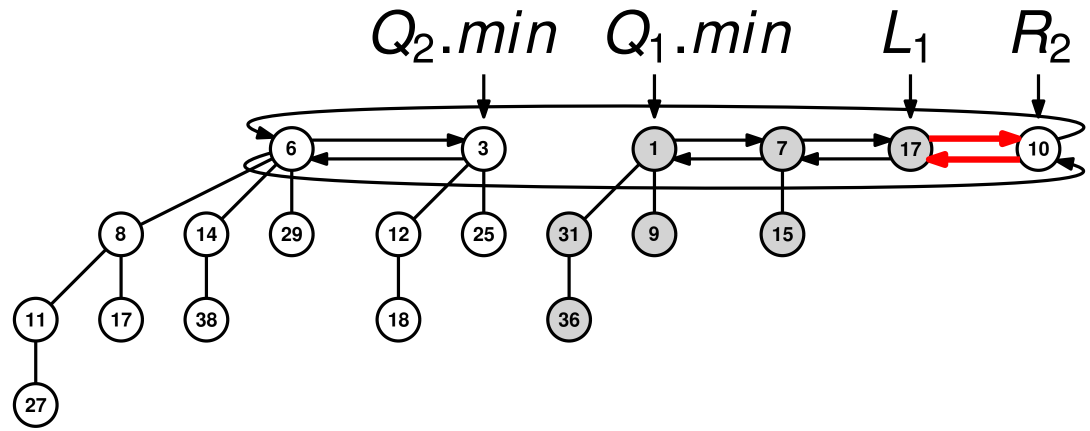
Lazy $\textsc{Union}(Q_1, Q_2)$





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$L_1 \leftarrow Q_1.min.left$

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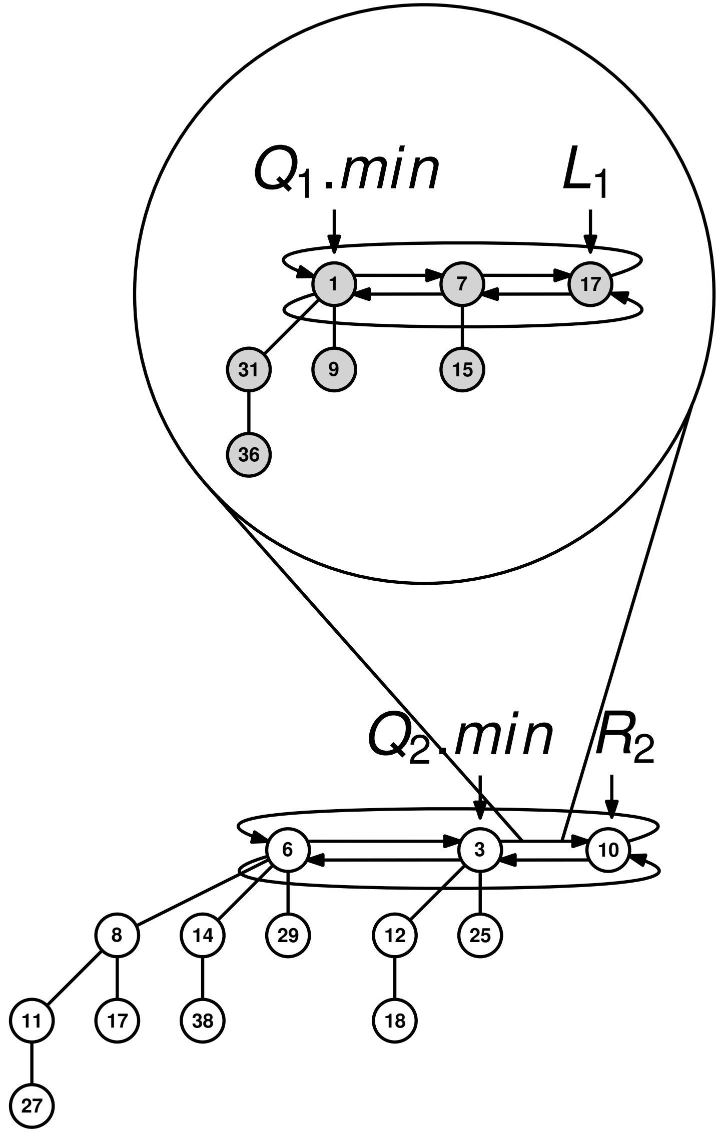
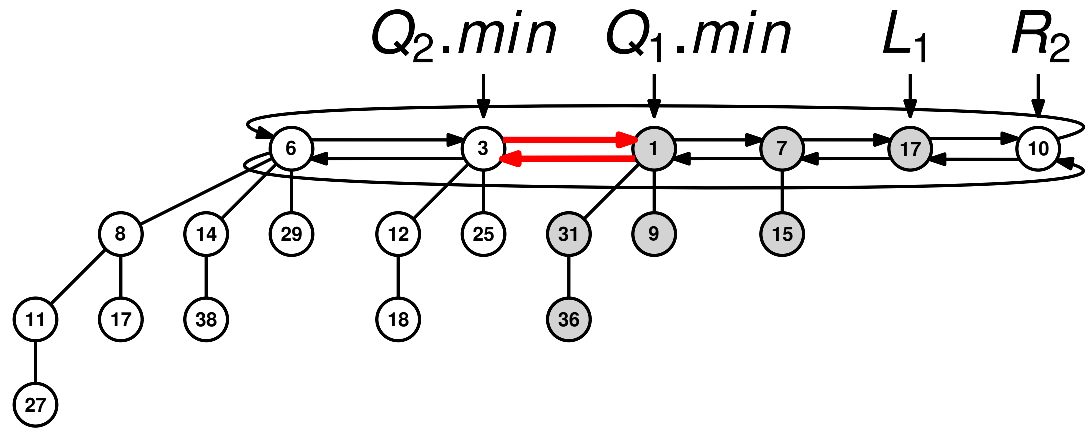
Lazy UNION(Q_1, Q_2)





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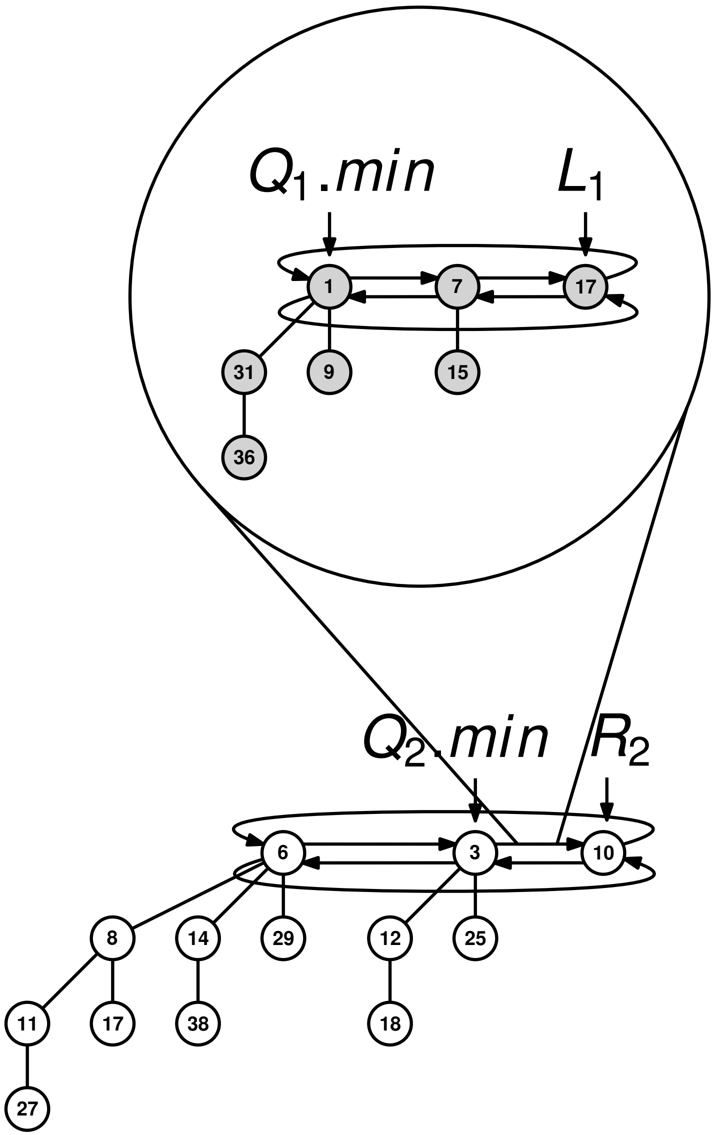
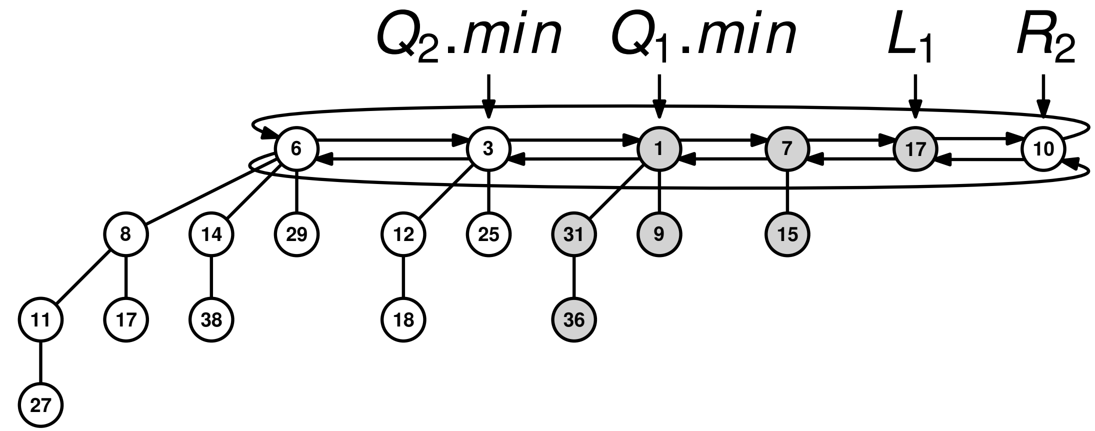
Lazy UNION(Q_1, Q_2)





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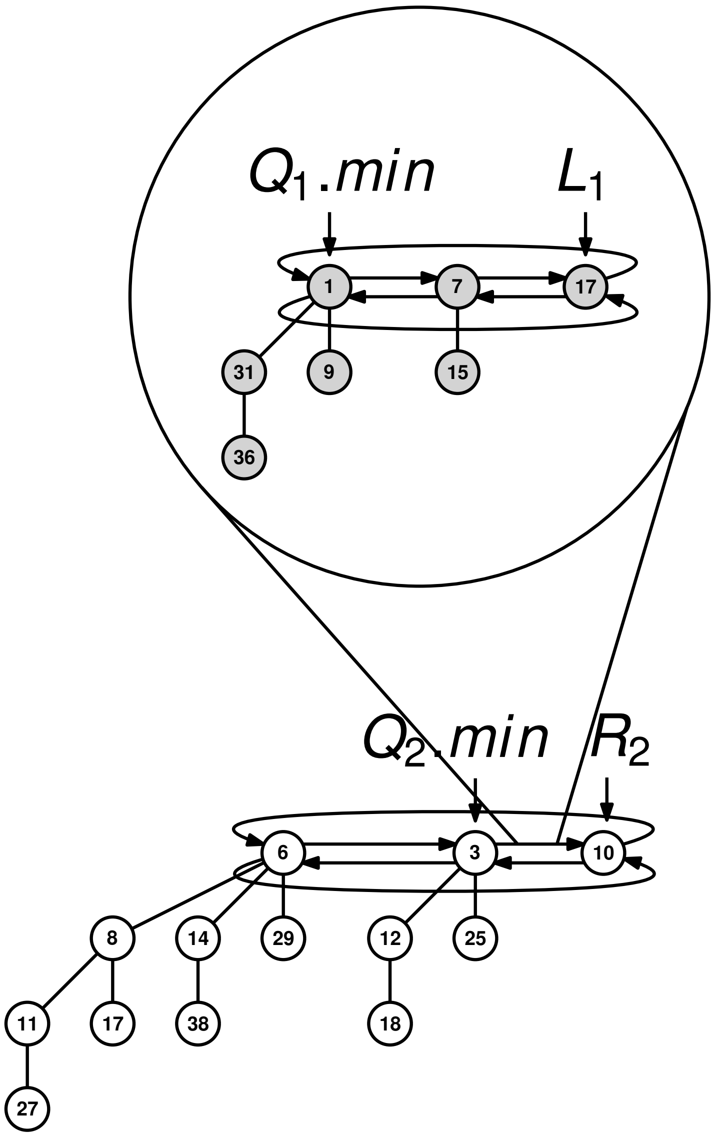
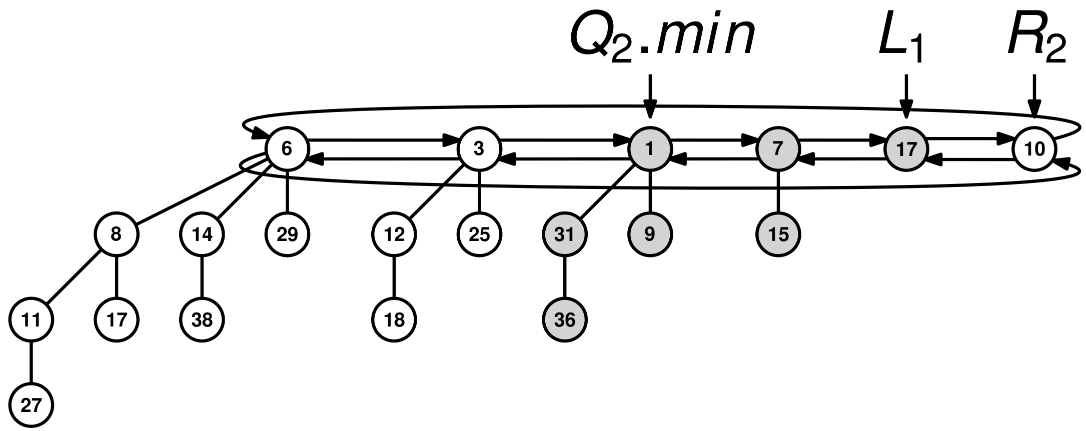
Lazy UNION(Q_1, Q_2)





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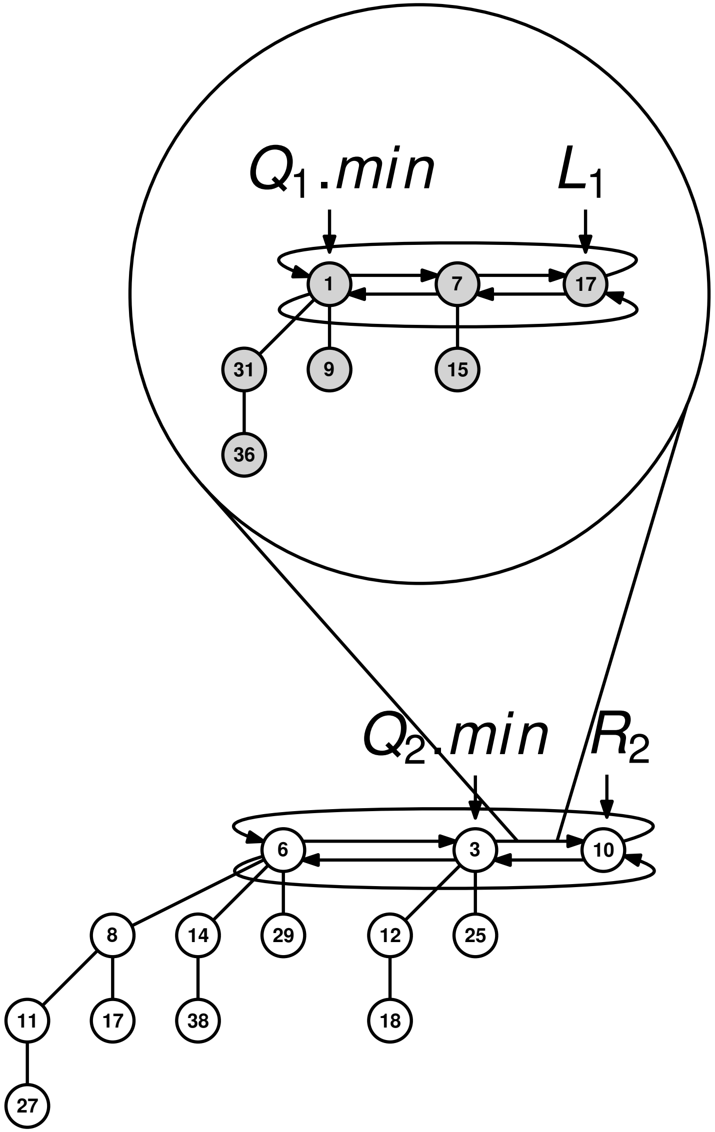

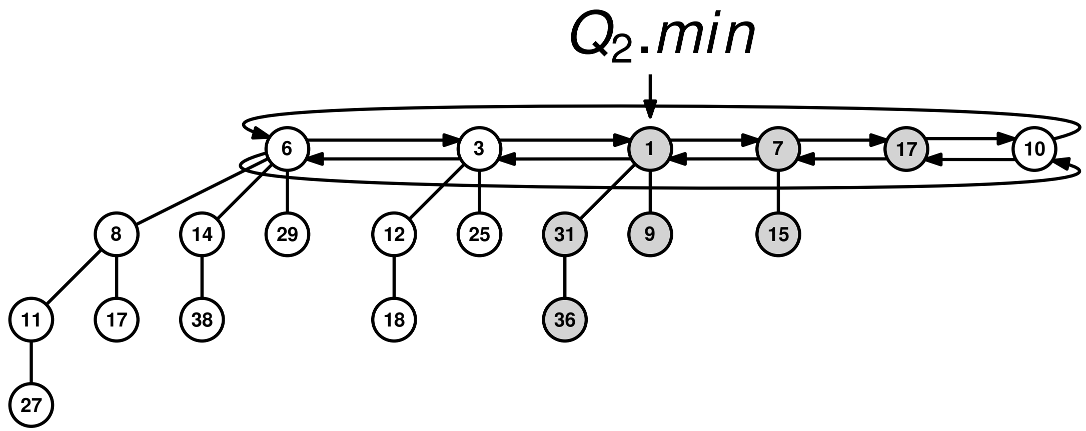

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$O(1)$ time worst-case

Extract-Min