ICS 621: Analysis of Algorithms	Spring 2024
Problem Set 3	
Prof. Nodari Sitchinava	*Due: Friday, November 1, 2024 at 1:30pm*

You may discuss the problems with your classmates, however **you must write up the solutions on your own** and **list the names** of every person with whom you discussed each problem.

1 Tail Bounds Practice (40 pts)

Consider a sequence of independent indicator random variables $Z_2, Z_3, \ldots Z_n$, such that $\Pr[Z_i = 1] = \frac{1}{i}$, and let $Z = \sum_{i=2}^{n} Z_i$. Prove that $\Pr[Z \geq 4 \ln n] \leq \frac{1}{n^2}$.

2 Two-dimensional maxima (60 pts)

https://xkcd.com/309/

You are working on your final project for ICS 621 and decide to buy a new computer to test and run your code. You would like to buy a fast computer to make your code compile and run quicker. Unfortunately, fast computers are expensive and you are on a student budget. You can't decide what is more important for you: the speed or the price. What you definitely don't want is a bad deal: a computer that is slower and more expensive than another one.

There are too many options out there. To figure out which computer to buy, you decide to create a two-dimensional scatter plot of all your options: each point represents a computer you can buy, where the x coordinate represents the speed of the computer and the y coordinate represents its value (the amount of money left in your bank account after the purchase). Looking at the points of the scatter plot (see Figure 1), it becomes obvious that you are only interested in the computers that correspond to points that have no other points in the quandrant to the right and above them (the red points). These types of points are known as *2D maxima*.

Imagine you are given n points, chosen uniformly at random on a $2,000 \times 2,000$ grid, chosen independently of each other, with no two points having the same x or y coordinate. Prove that the number of 2D maxima points in such setting is $O(\log n)$ with high probability. *Hint: Consider points in the decreasing order of their x coordinates and think about which points form the 2D maxima.*

3 Boosting High Probability Bounds (OPTIONAL - 0 pts)

Let H be the random variable that denotes the height of a treap on n elements. In lecture we have proven that $\Pr[H \geq 8 \ln n] \leq \frac{2}{n}$.

(a) Prove that $\Pr[H \geq 2c \ln n] \leq \frac{2}{n^{c \ln c - c}}$ for any $c \geq 4$.

(b) Use the answer in part (a) to argue that the height of the treap is $O(\log n)$ with probability at least $1 - \frac{2}{n^c}$ for any choice of $c \geq 1$, i.e., you can boost the probability that the height of a treap is $O(\log n)$ to be as large as you like, as long as you don't mind increasing the constant hidden in the big-O notation.

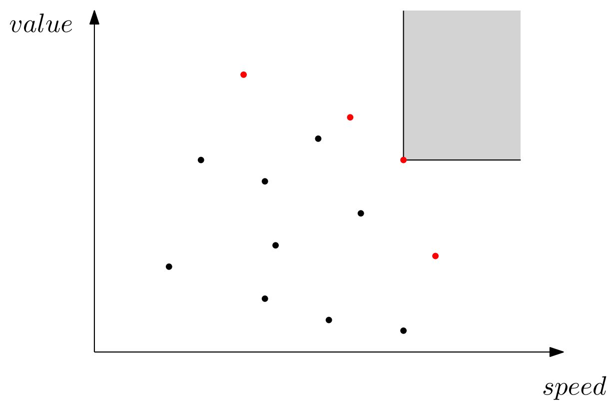

Figure 1: A scatter plot of computer options by speed and value (larger is better). Computers represented by the red points form 2D maxima: no other computer has both better speed **and** better value. Equivalently, every red point contains no other point in the quadrant to the right and above it.