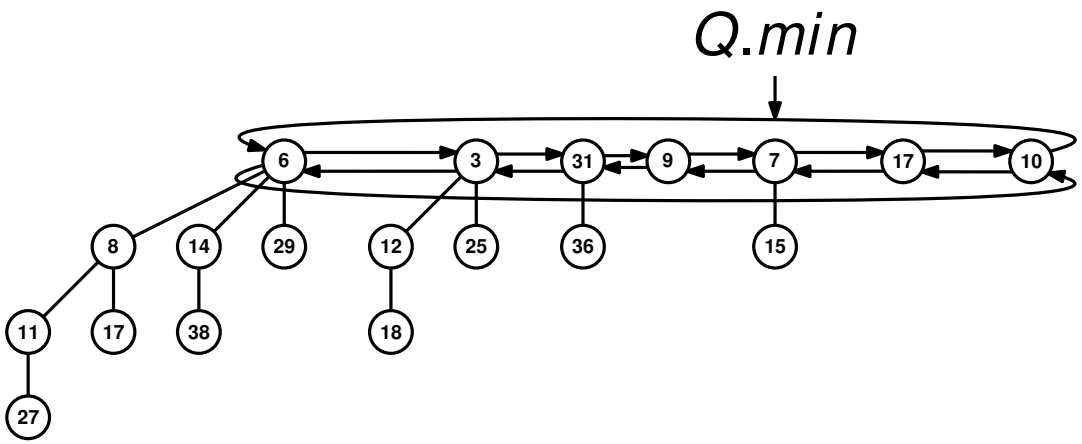
ICS 621: Analysis of Algorithms

Prof. Nodari Sitchinava



Mergeable Priority Queues: Fibonacci Heaps

Heaps

	Binomial
▪ MAKE()	$O(1)$
▪ INSERT(Q, x)	$O(1)^*$
▪ MINIMUM(Q)	$O(1)$
▪ EXTRACT-MIN(Q)	$O(\log n)$
▪ DECREASE-KEY(Q, x, k)	$O(\log n)$
▪ DELETE(Q, x)	$O(\log n)$
▪ UNION(Q_1, Q_2)	$O(\log n)$

* Amortized cost

Heaps

	Binomial	Lazy Binomial
MAKE()	$O(1)$	$O(1)$
INSERT(Q, x)	$O(1)^*$	$O(1)$
MINIMUM(Q)	$O(1)$	$O(1)$
EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)^*$
DECREASE-KEY(Q, x, k)	$O(\log n)$	$O(\log n)$
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Heaps

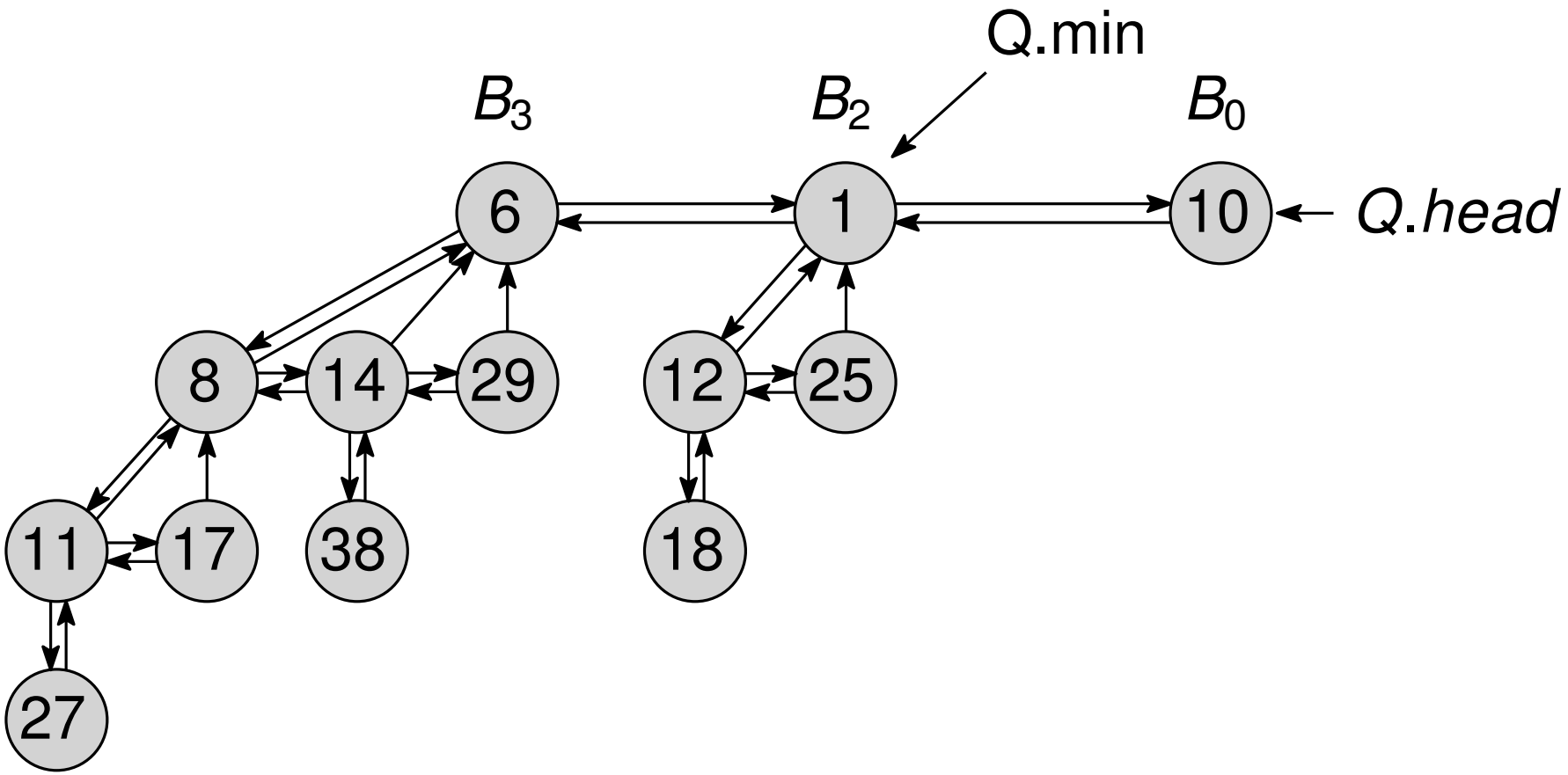
	Binomial	Lazy Binomial	Fibonacci
MAKE()	$O(1)$	$O(1)$	$O(1)$
INSERT(Q, x)	$O(1)^*$	$O(1)$	$O(1)$
MINIMUM(Q)	$O(1)$	$O(1)$	$O(1)$
EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)^*$	$O(\log n)^*$
DECREASE-KEY(Q, x, k)	$O(\log n)$	$O(\log n)$	$O(1)^*$
DELETE(Q, x)	$O(\log n)$	$O(\log n)^*$	$O(\log n)^*$
UNION(Q_1, Q_2)	$O(\log n)$	$O(1)$	$O(1)$

* Amortized cost

Reminder: Binomial Heaps

Collection of heap-ordered binomial trees:

- Each tree is heap-ordered
- At most **one** tree B_k, for $k = 0, 1, 2, \ldots \lfloor \log n \rfloor$



Reminder: Lazy Binomial Heaps

Collection of heap-ordered binomial trees:

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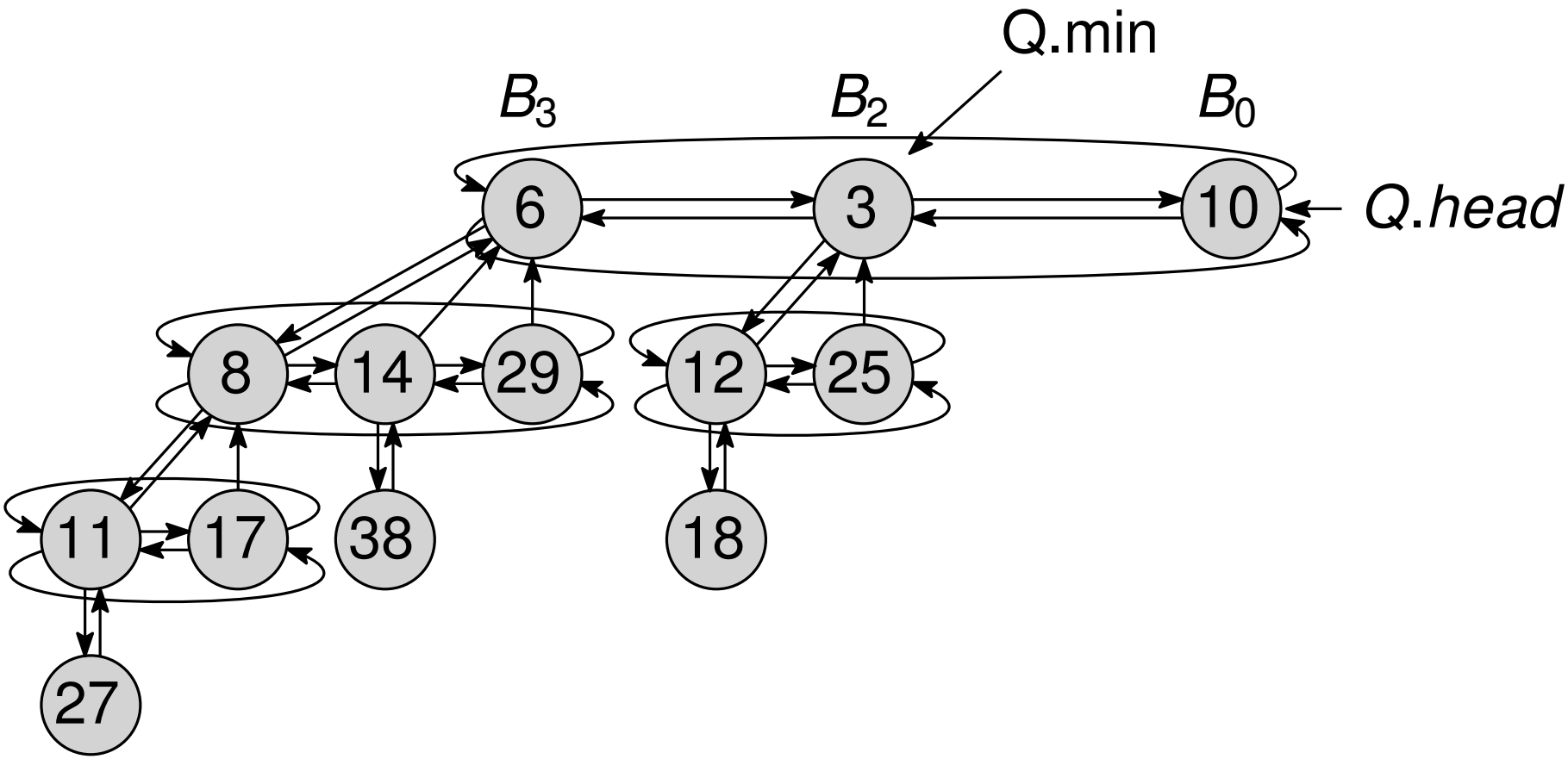
- *Arbitrary* number of trees in the root list
- Sibling lists are doubly-linked *circular* lists



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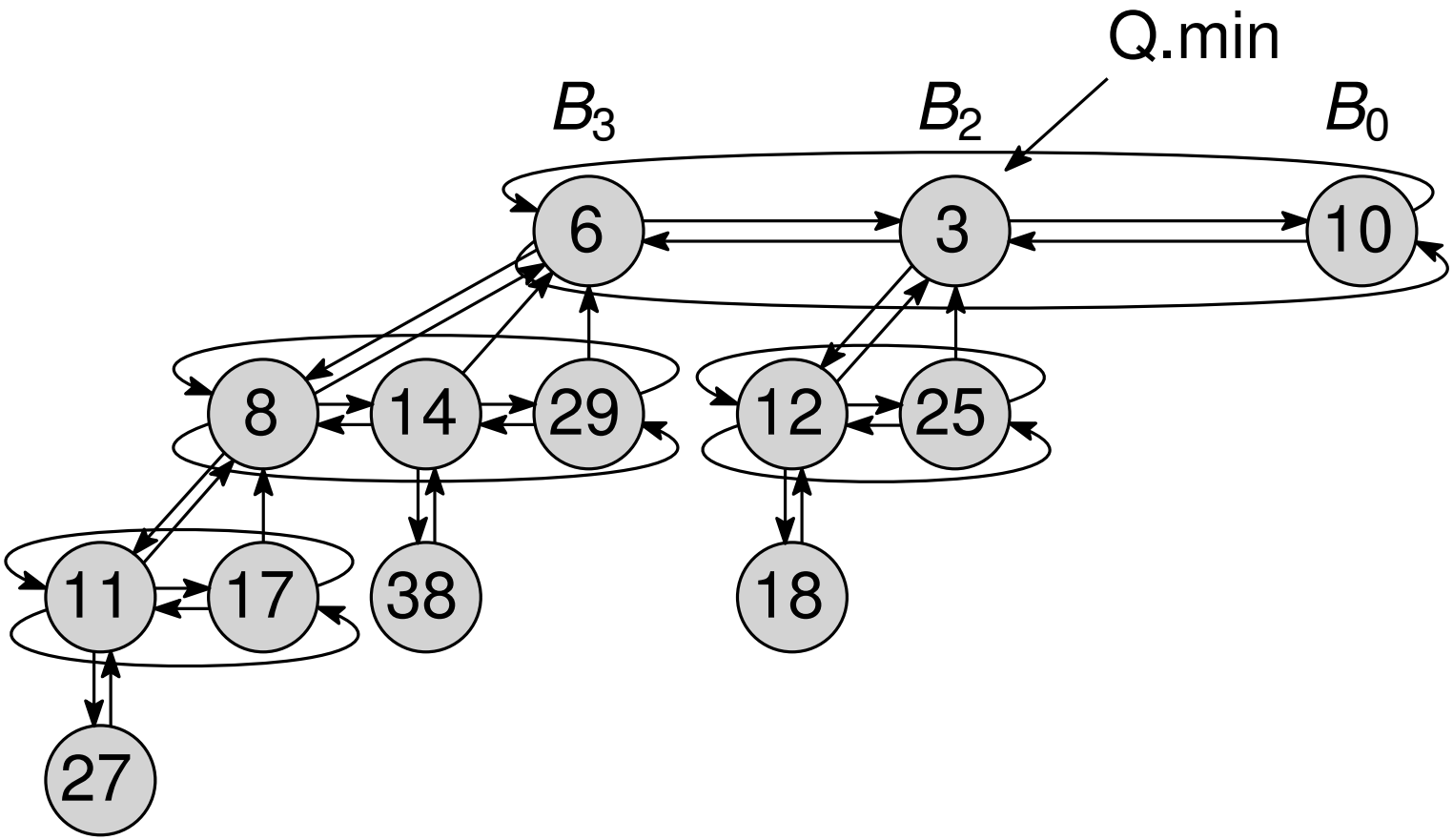
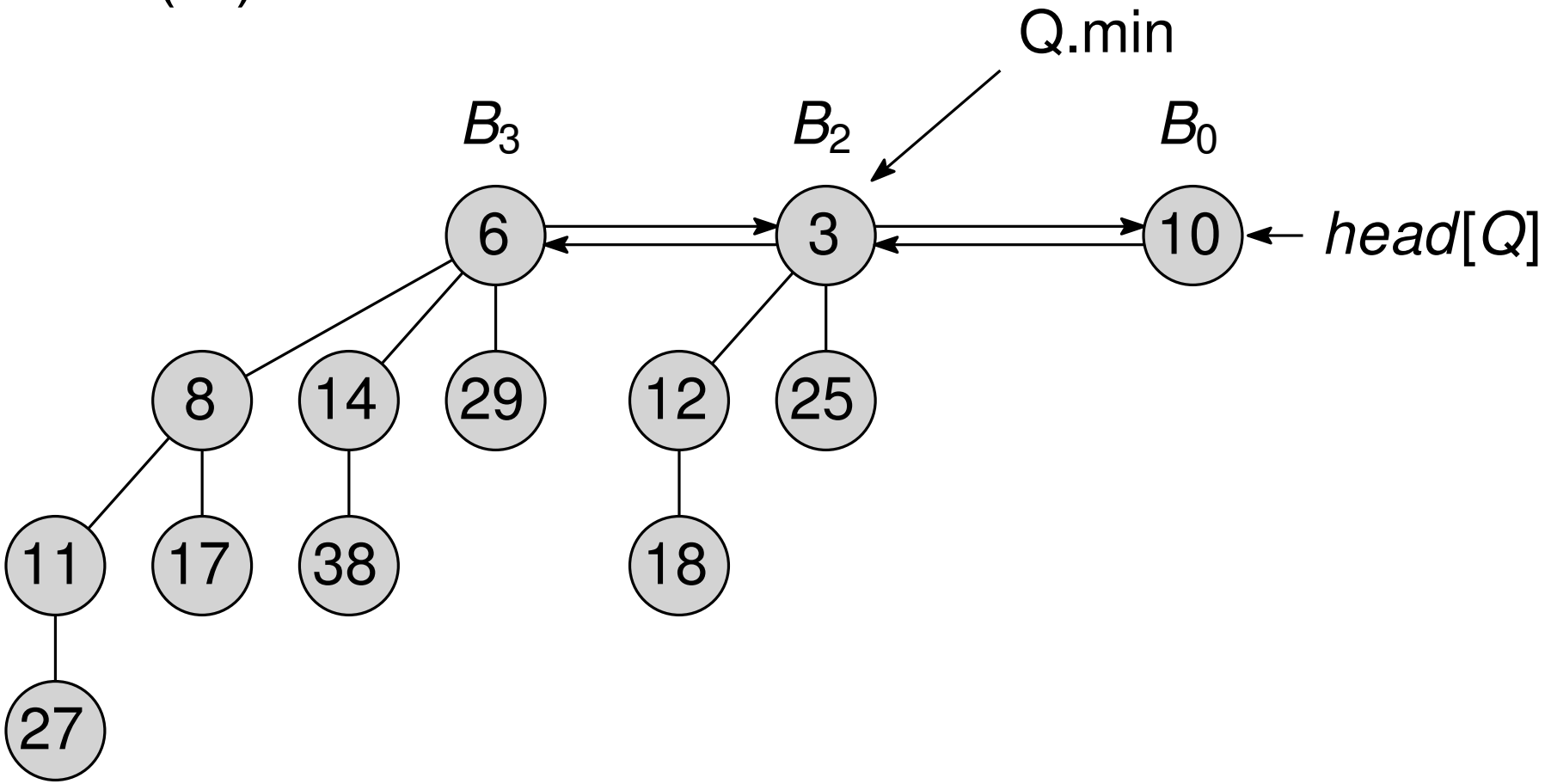


MINIMUM(Q)



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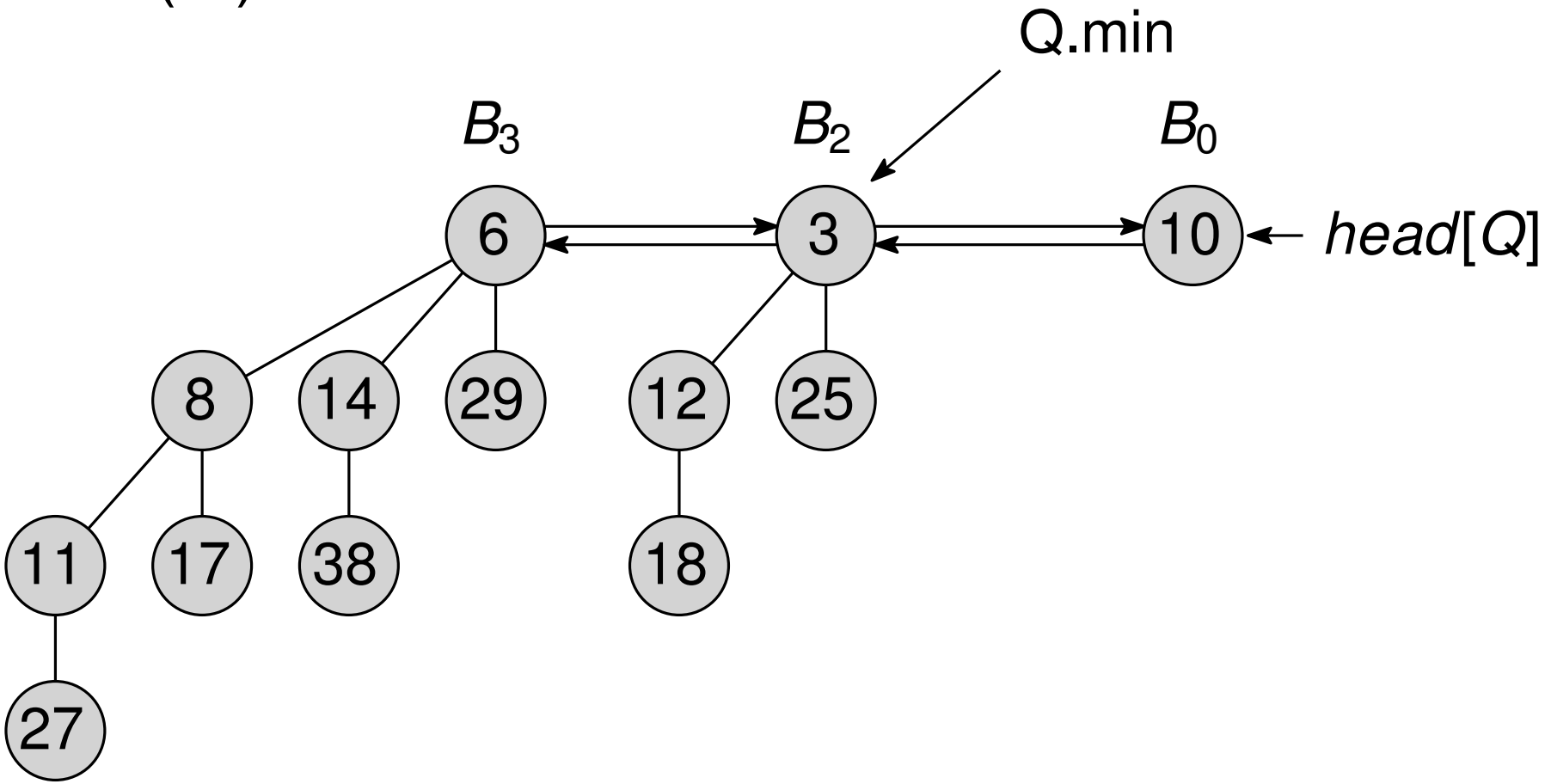
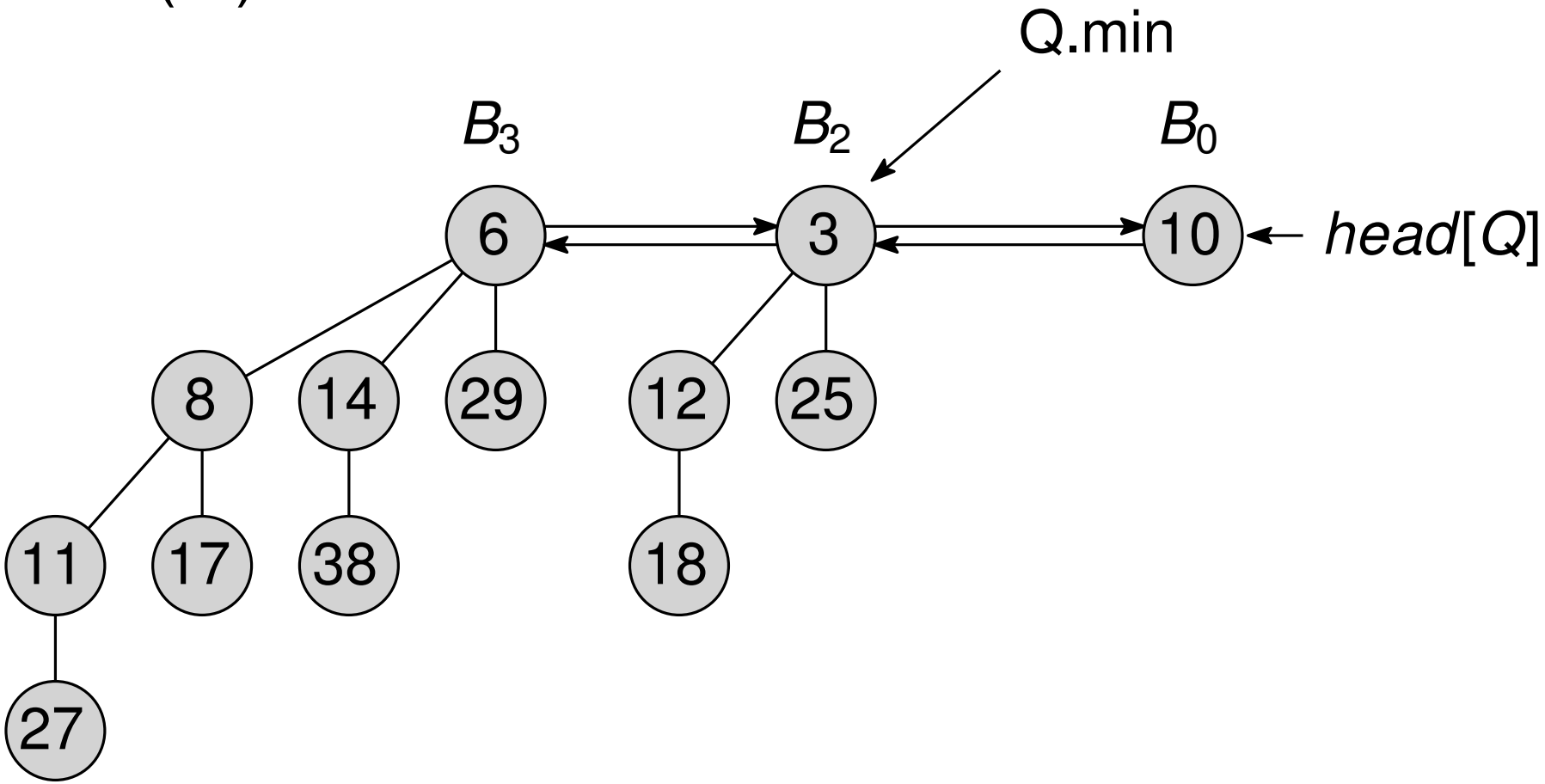


function MINIMUM(Q)
return $Q.min$

MINIMUM(Q)



function MINIMUM(Q)
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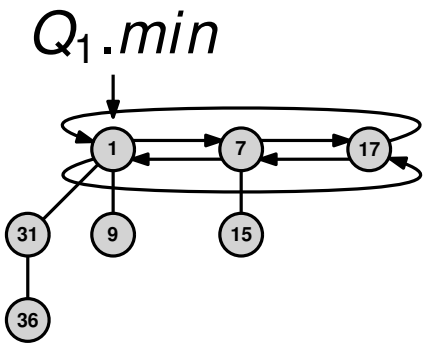
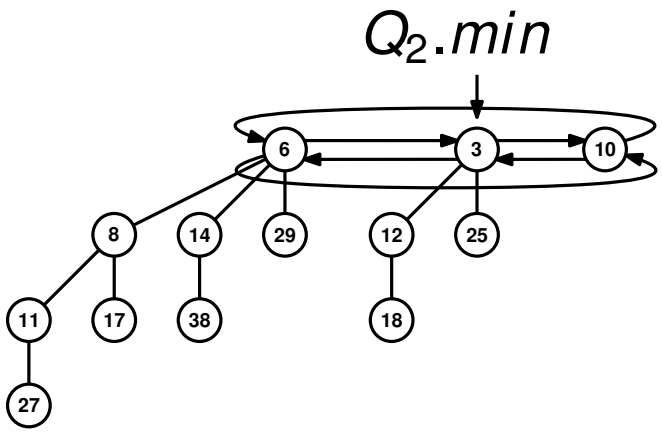
$O(1)$ time worst-case

Lazy UNION(Q_1, Q_2)





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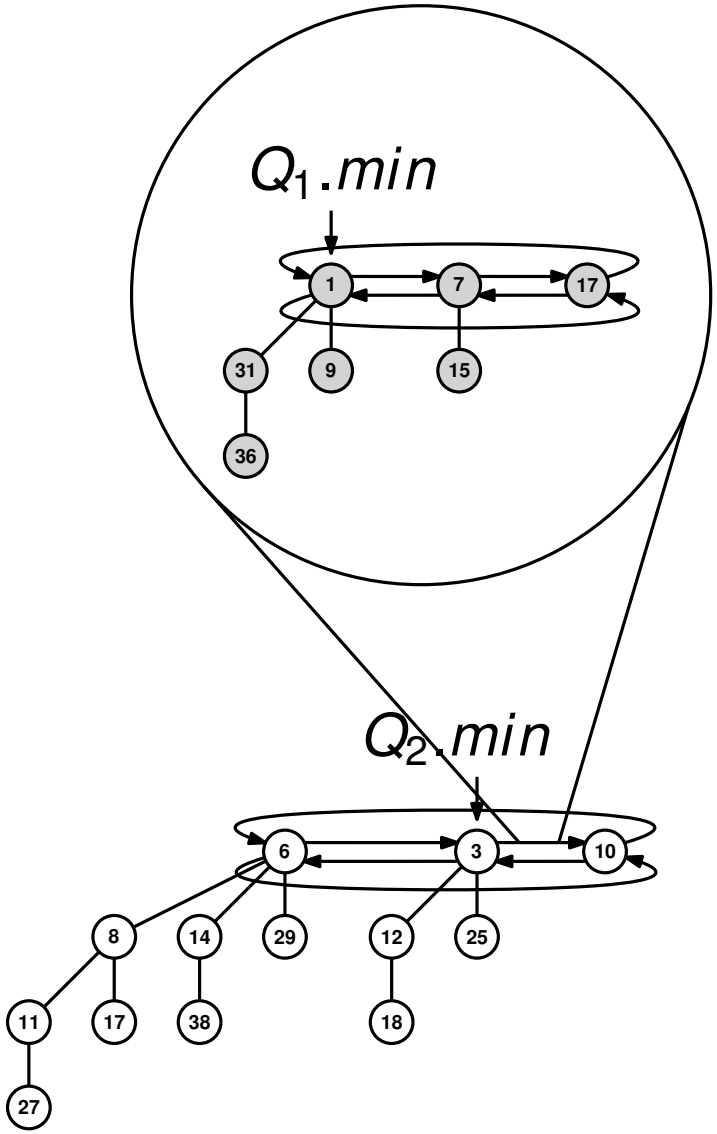

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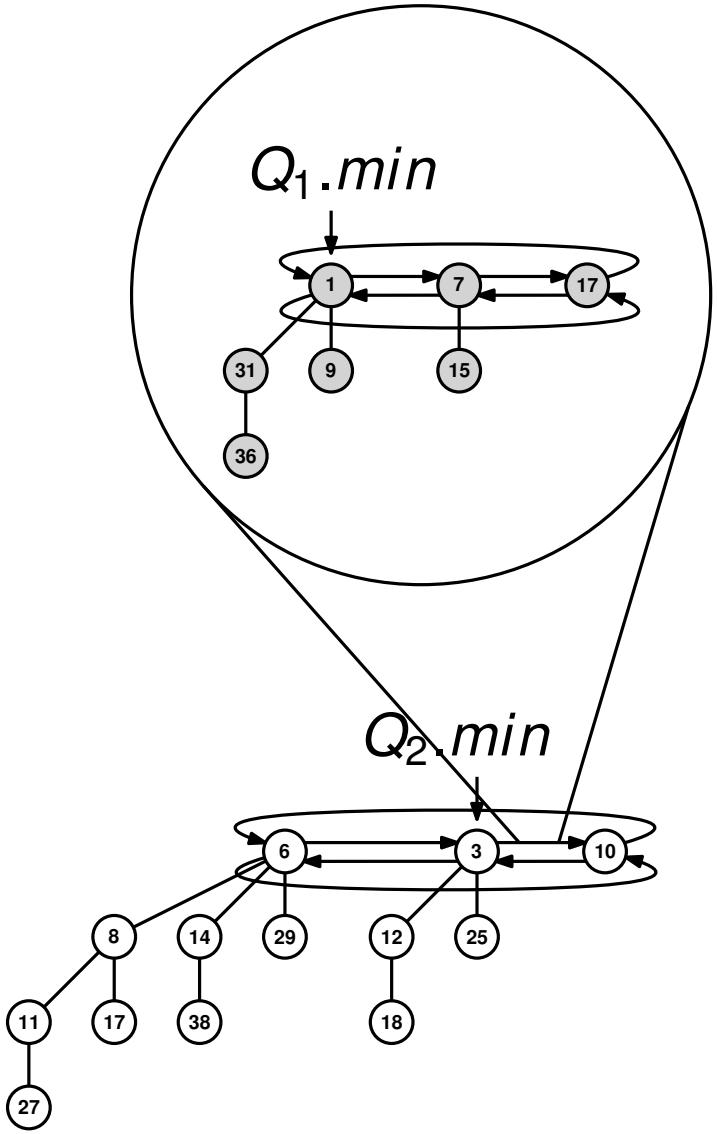

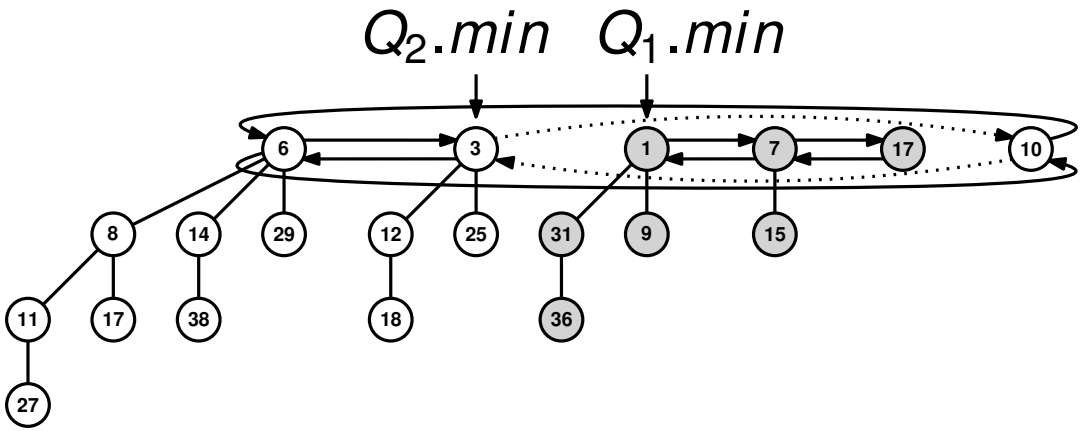

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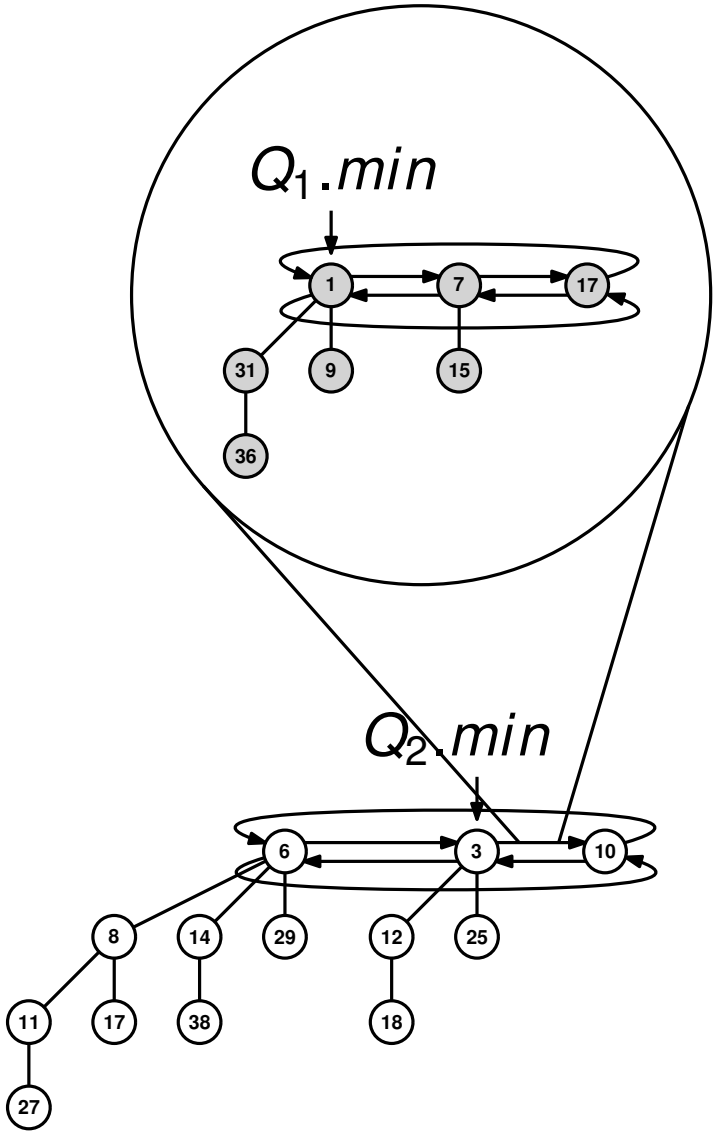

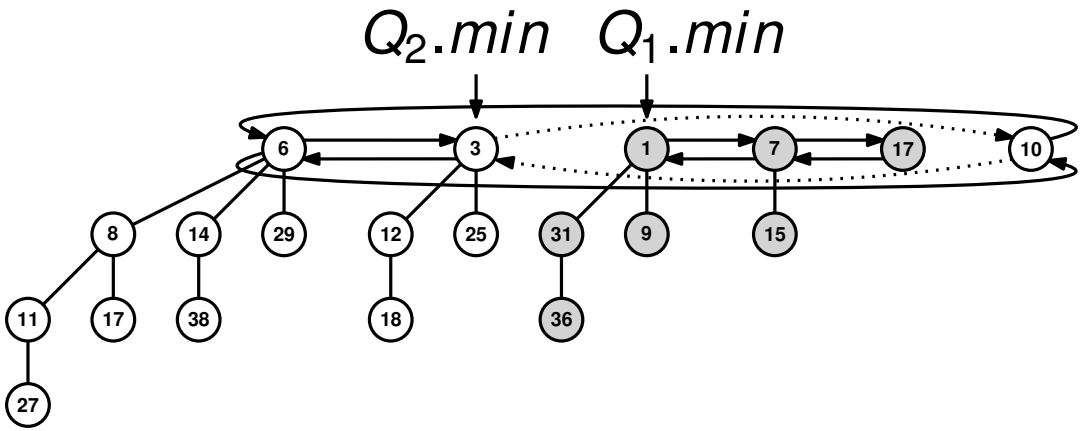

function UNION(Q_1, Q_2)

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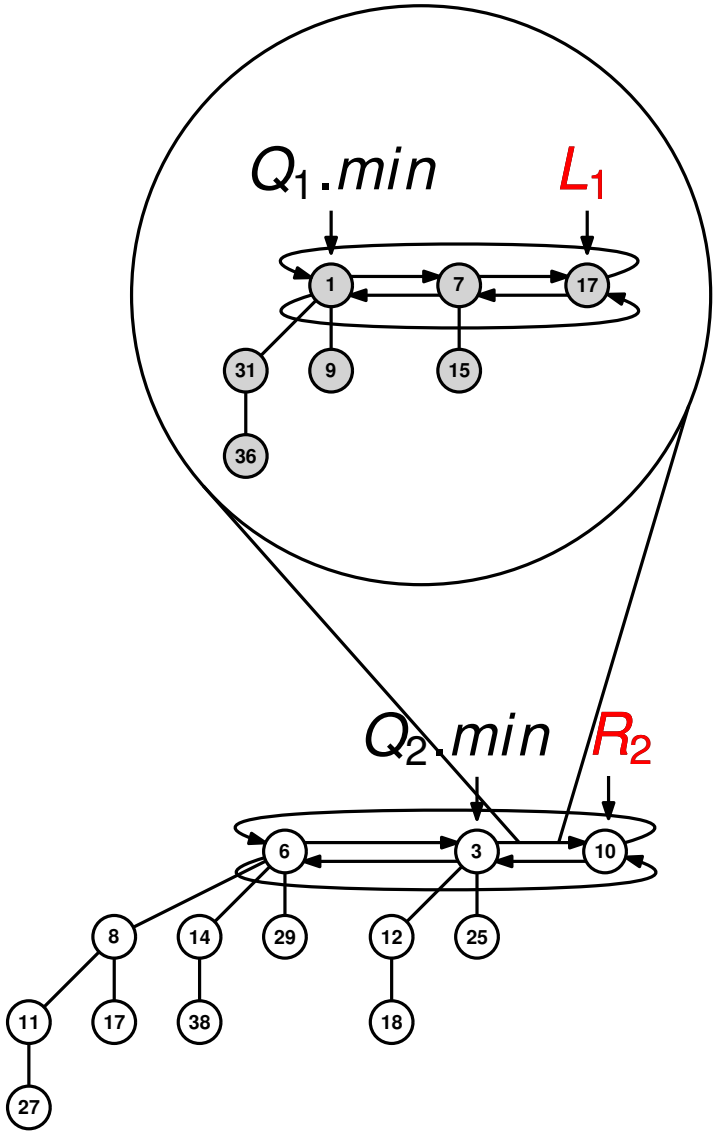

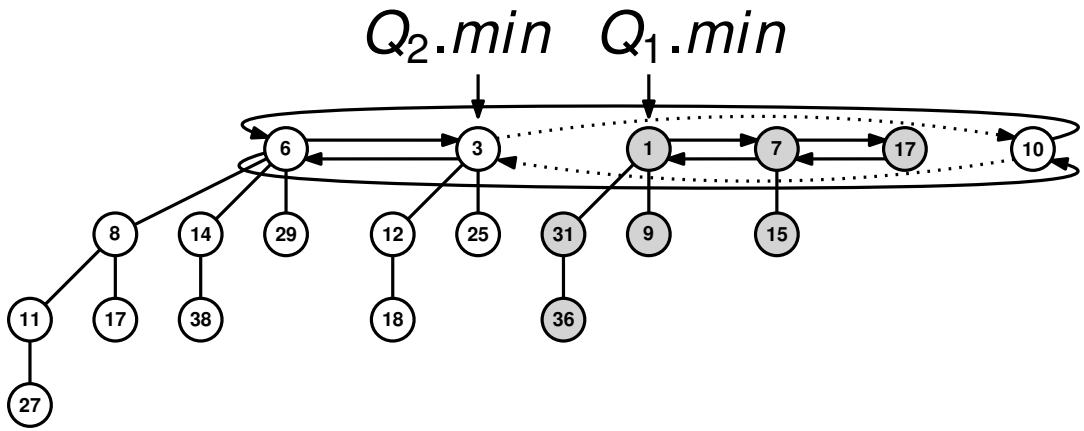

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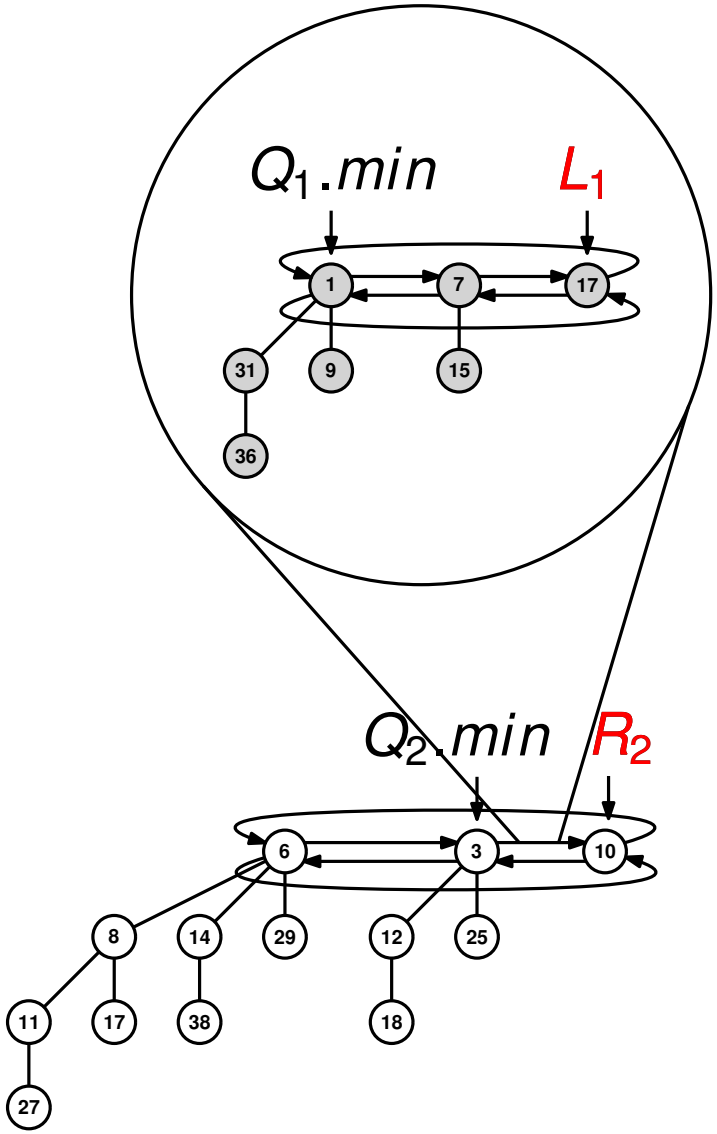

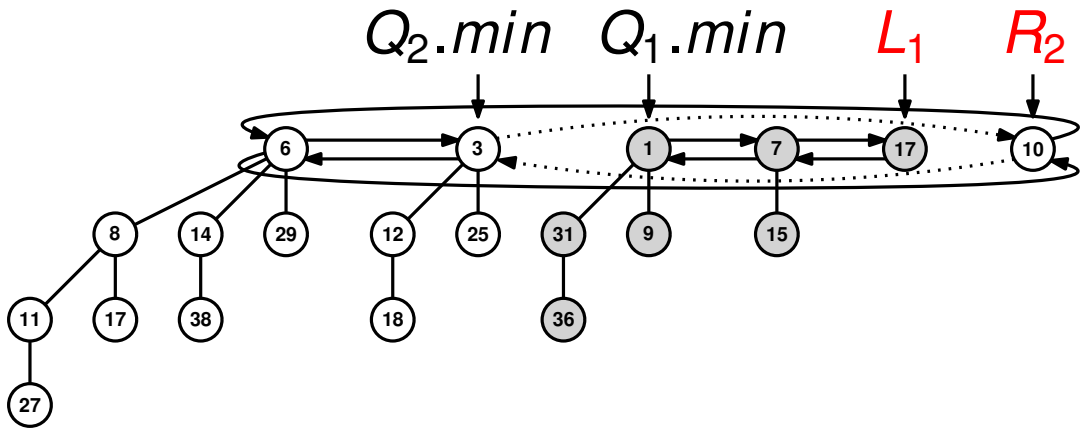

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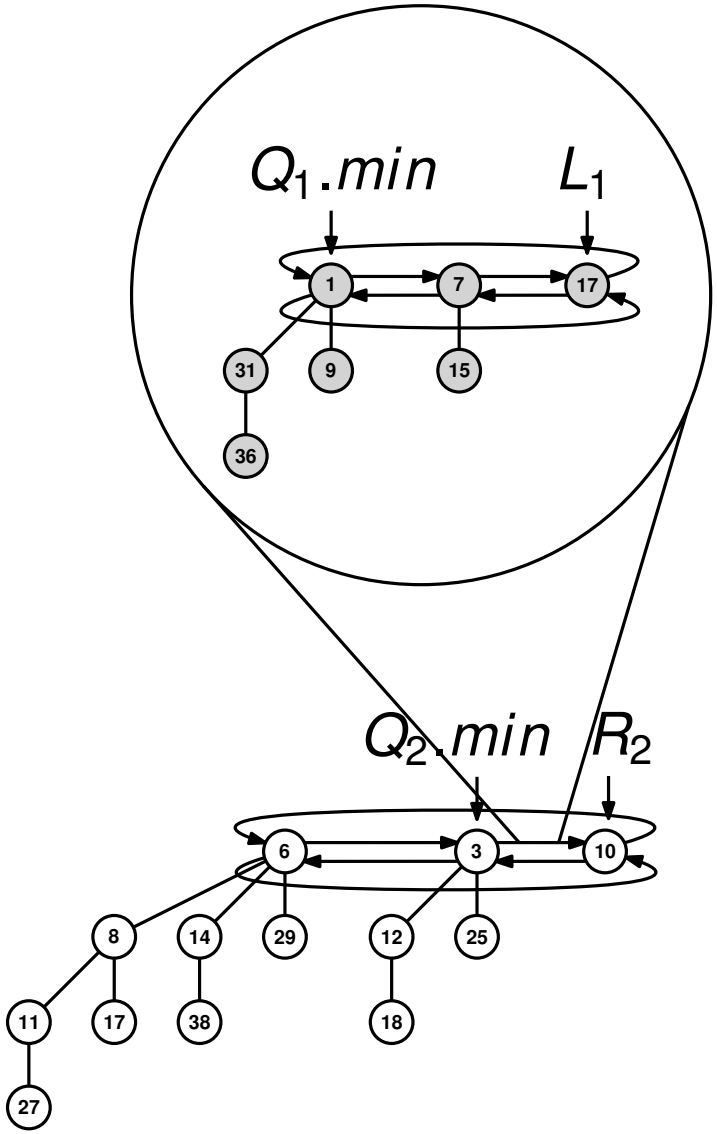

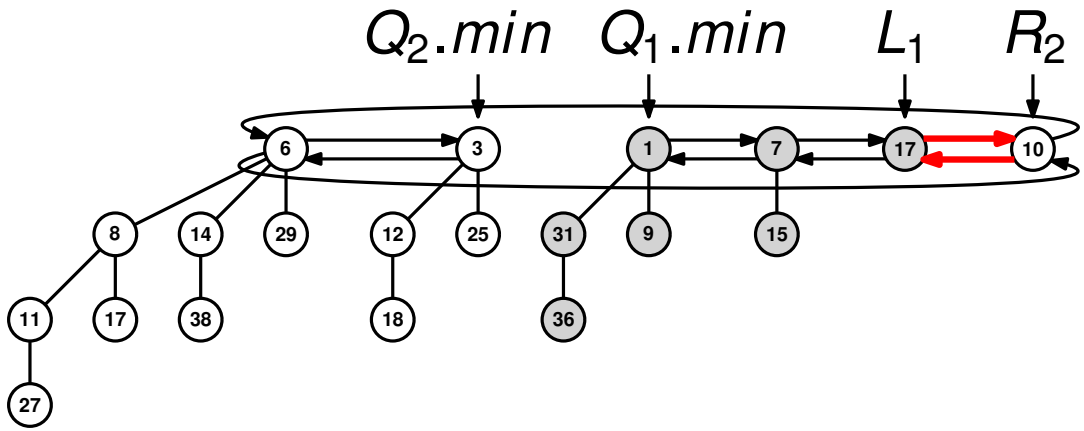

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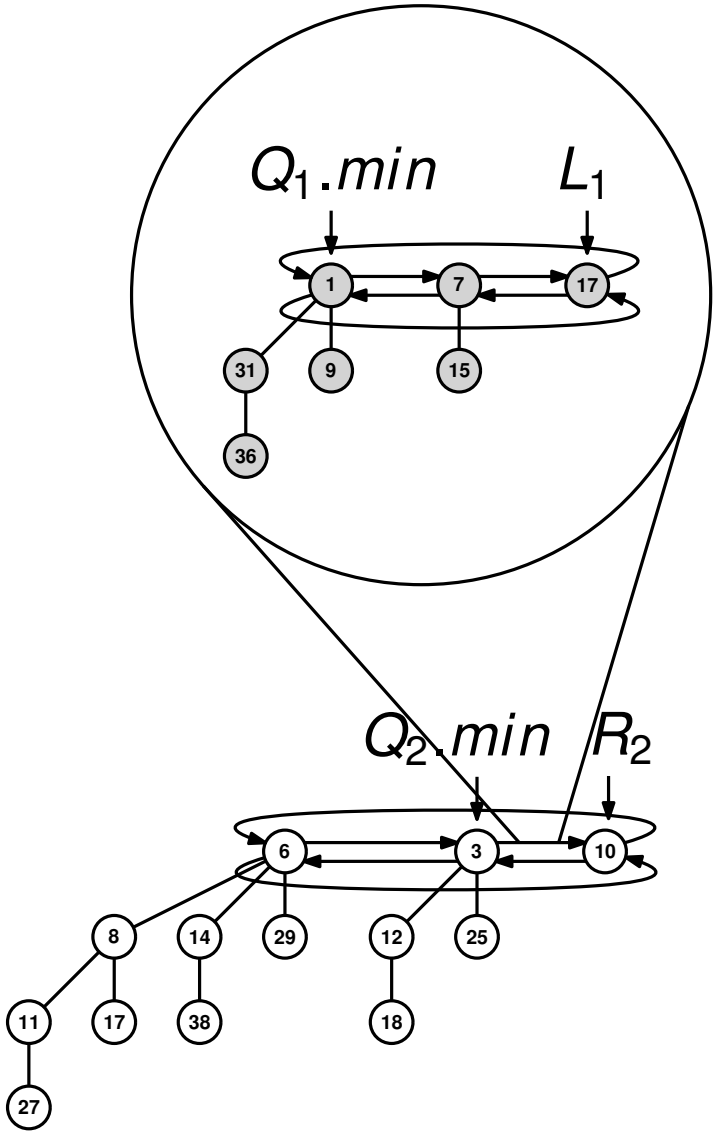

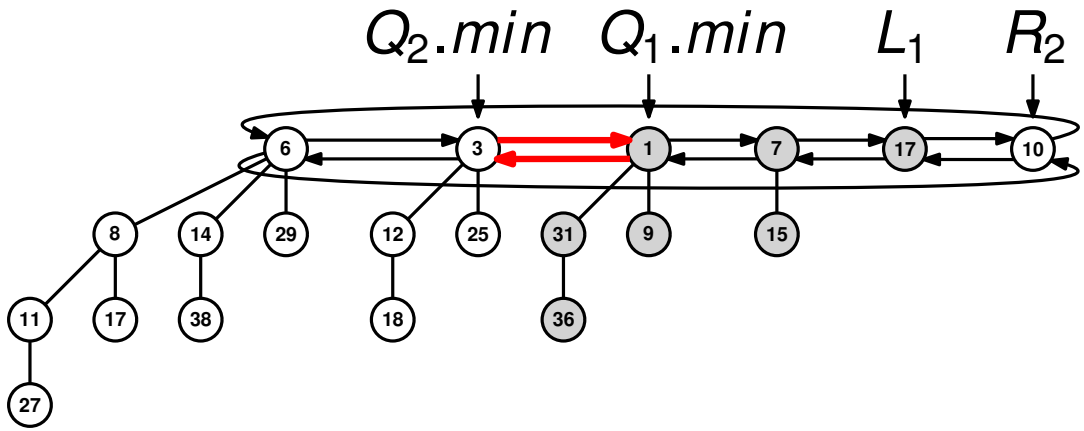

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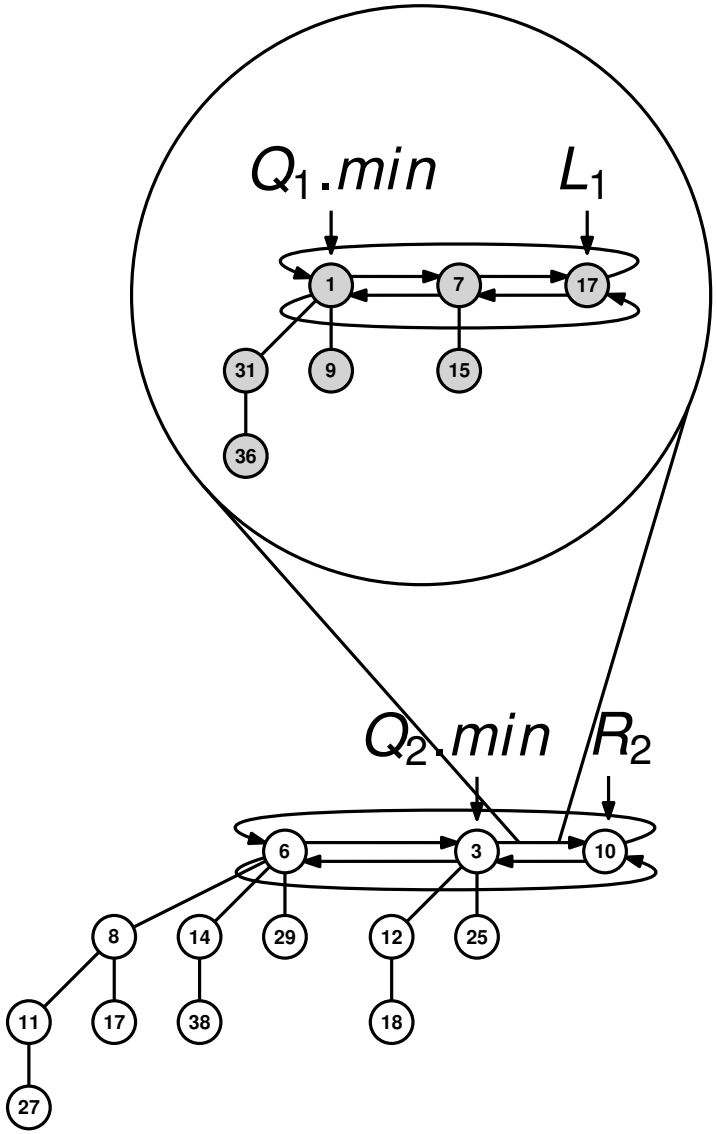

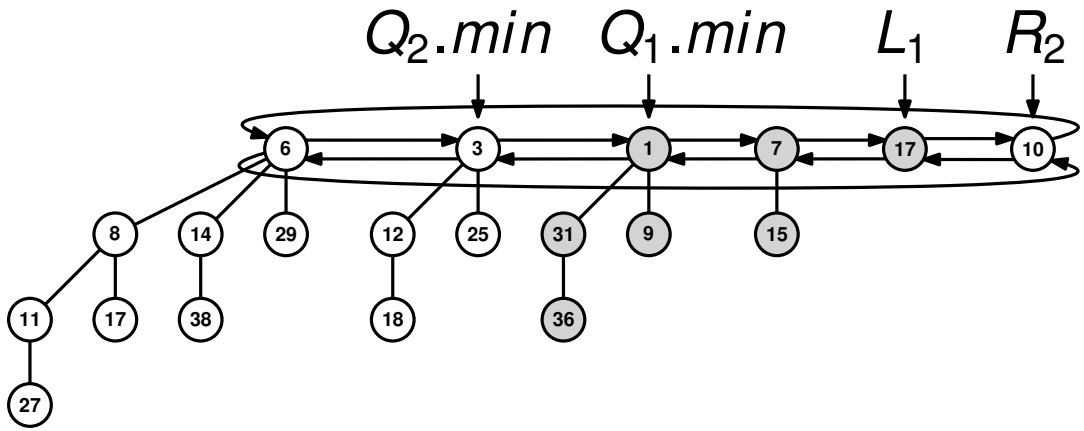

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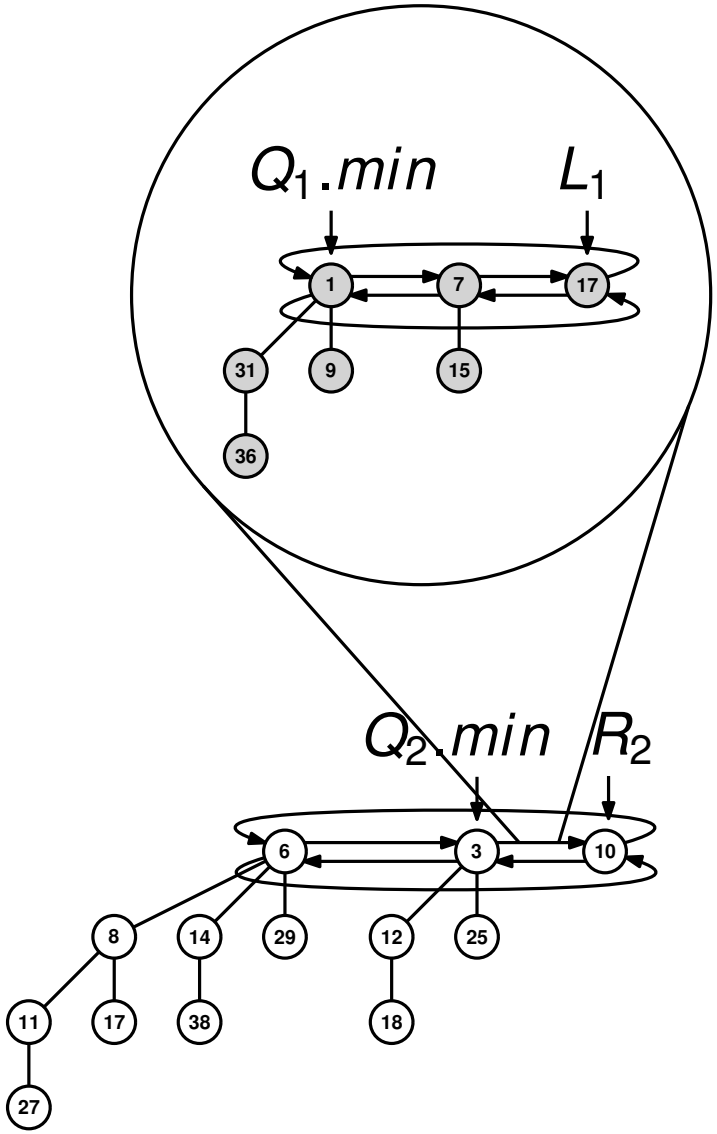

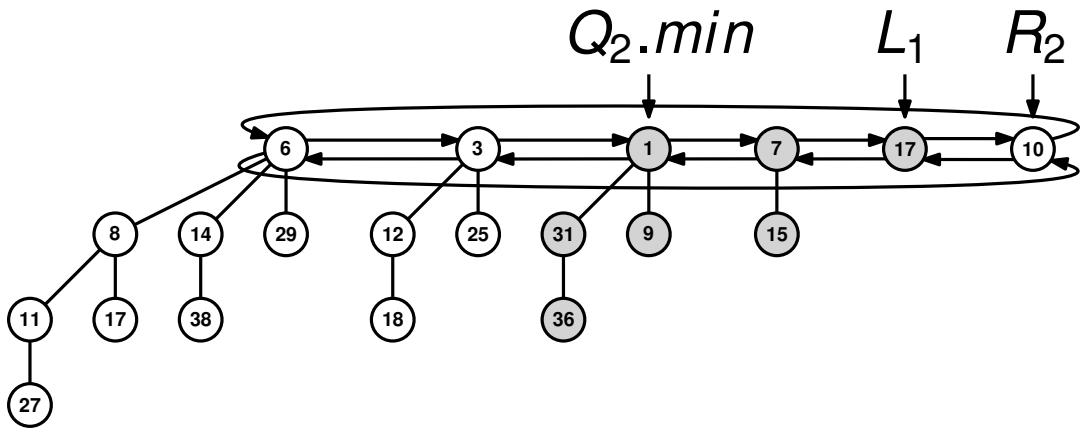

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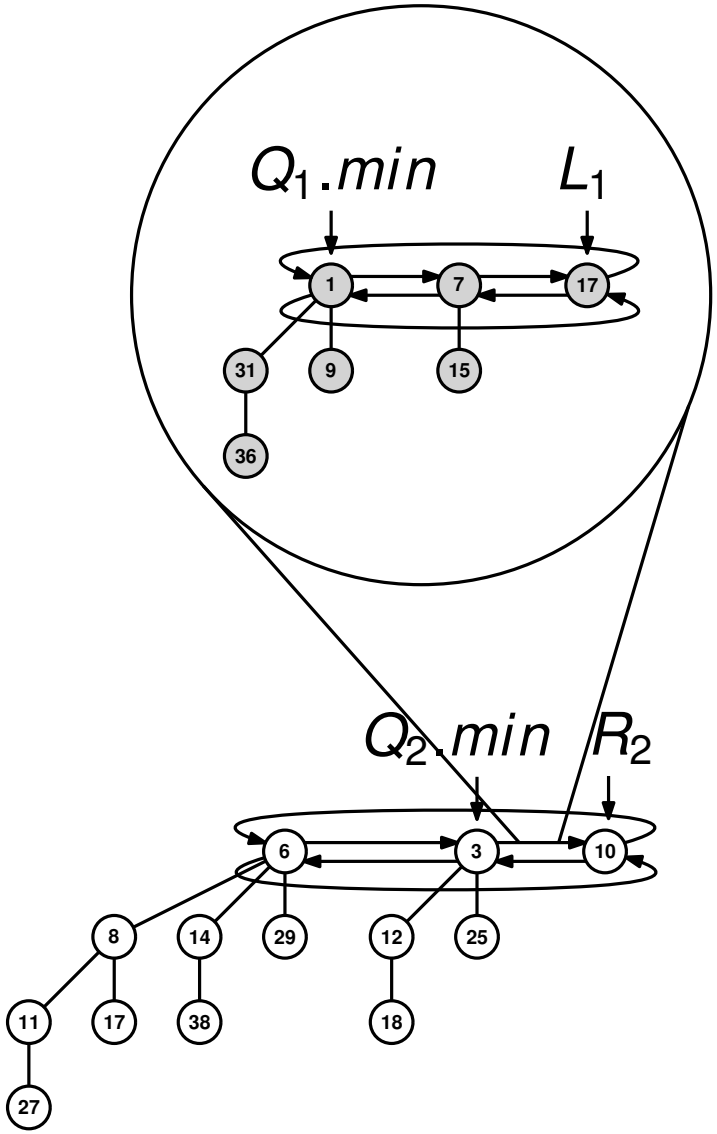

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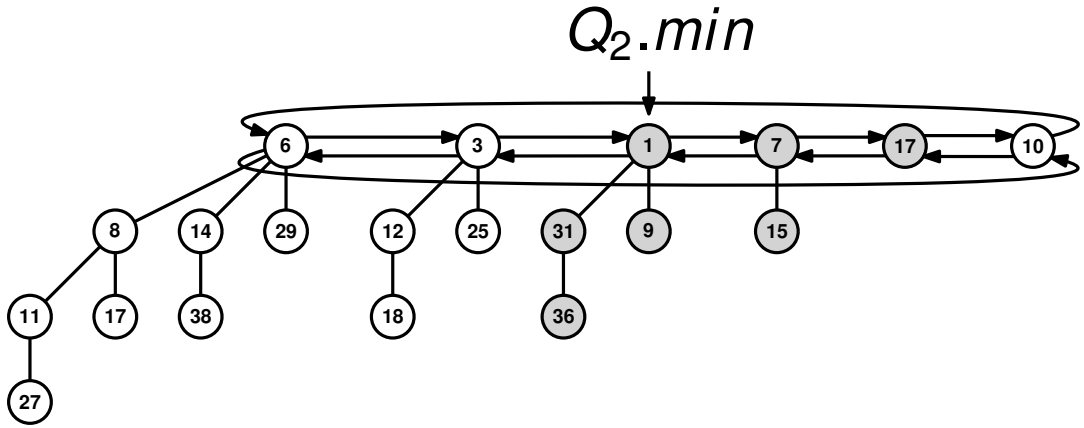

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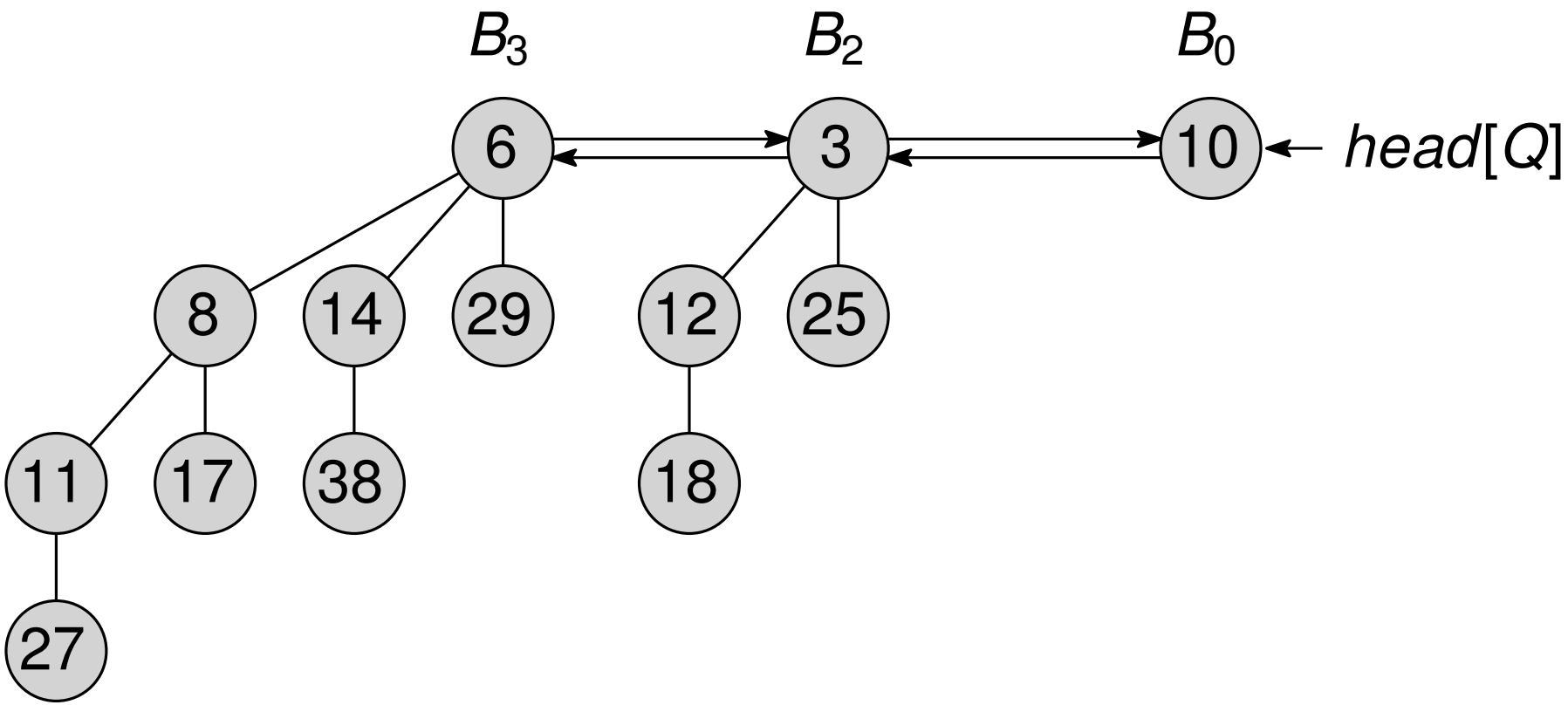
DECREASE-KEY(Q, v, k)



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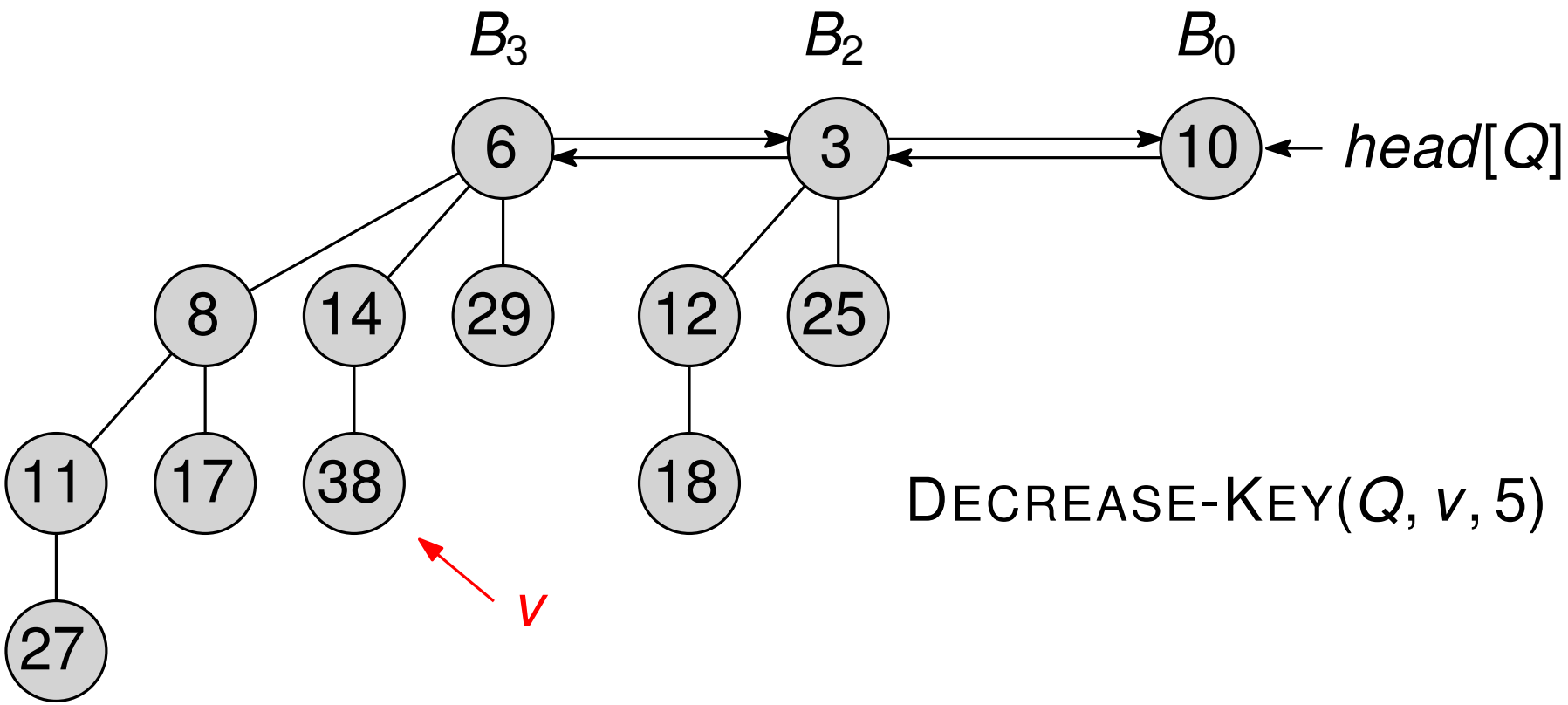

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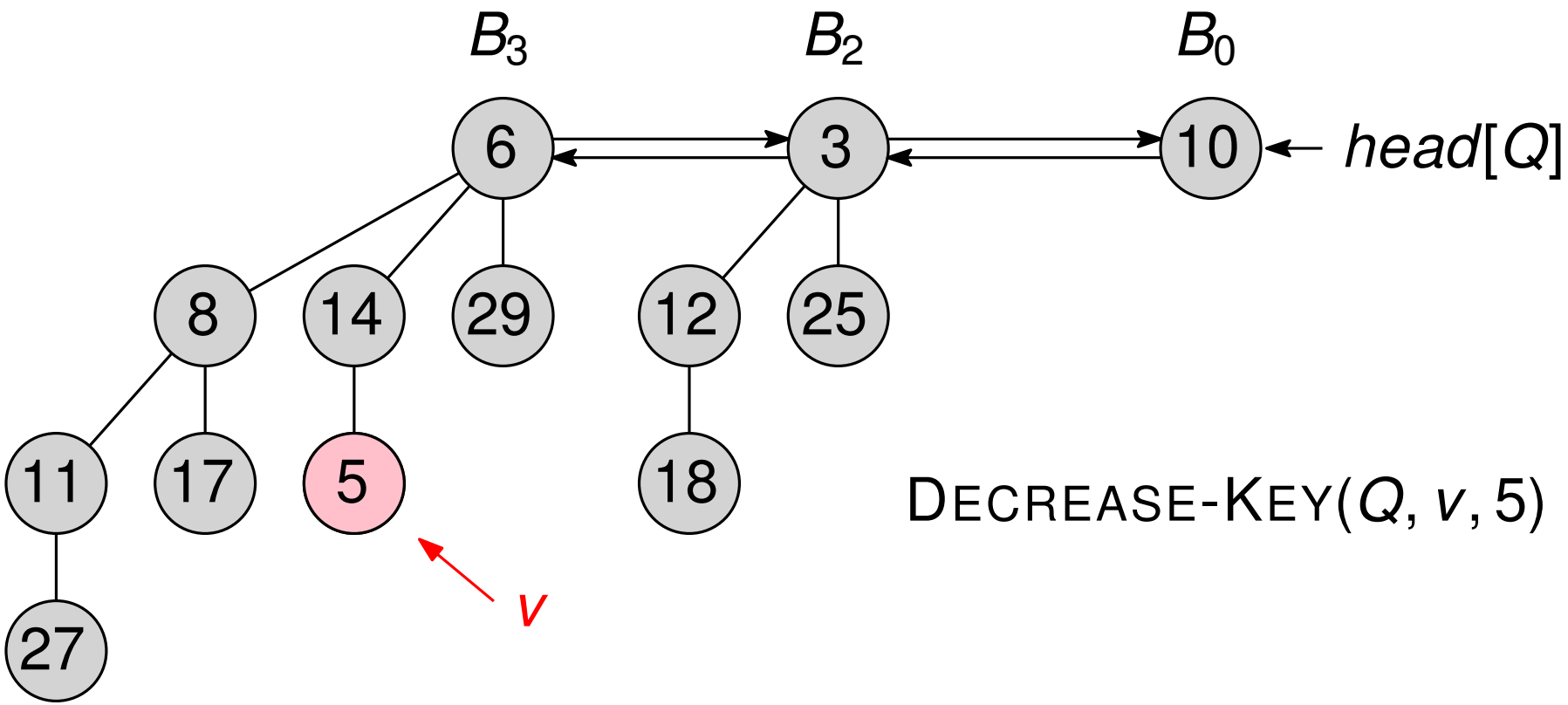

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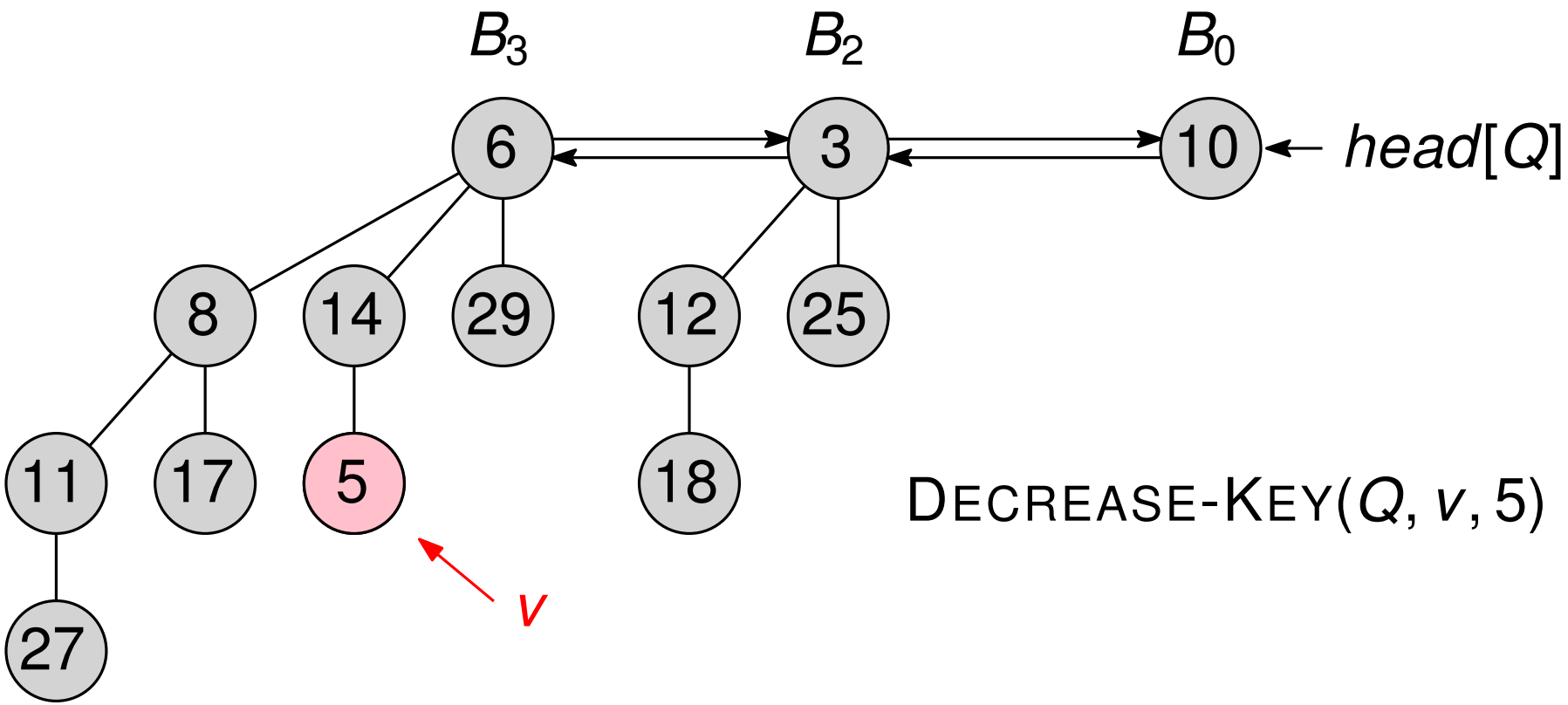

- Fix heap order

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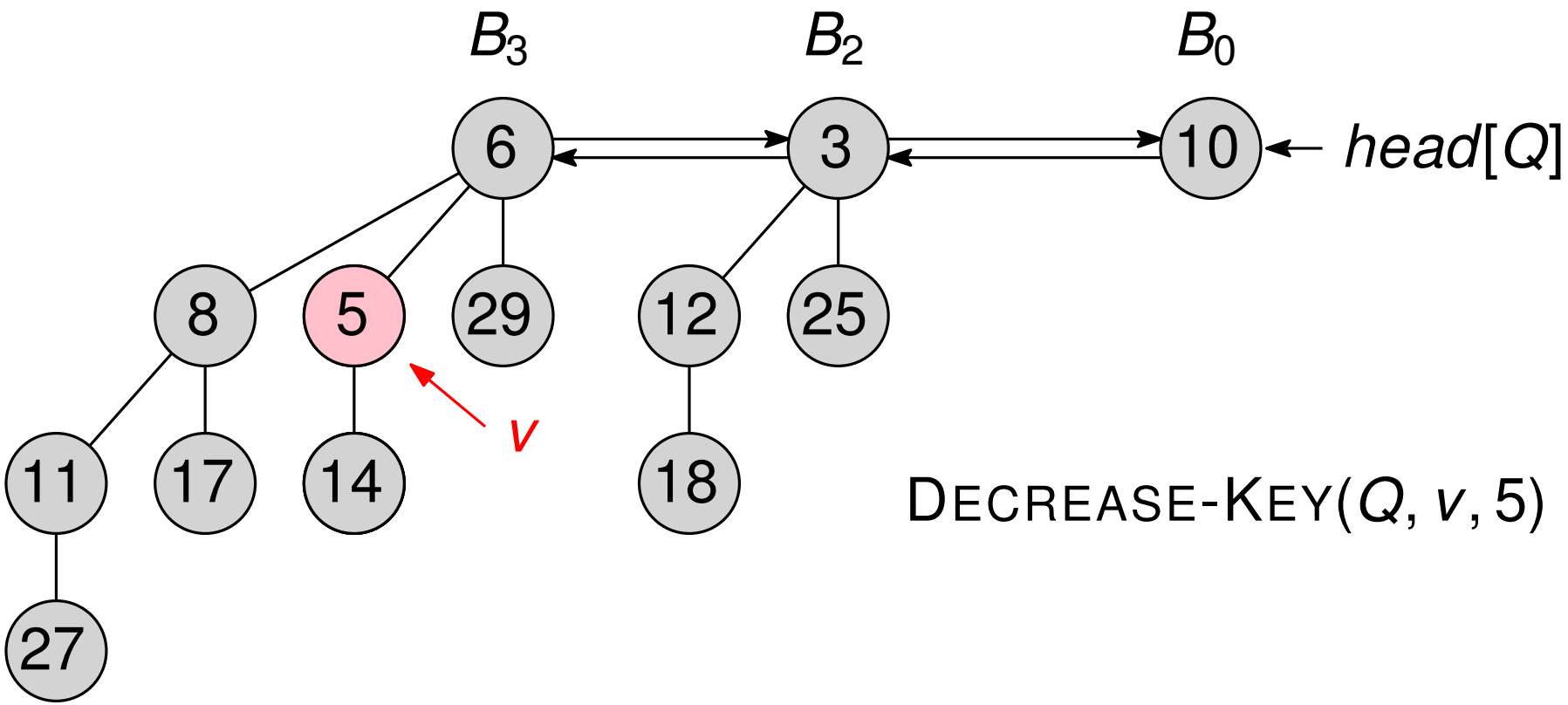

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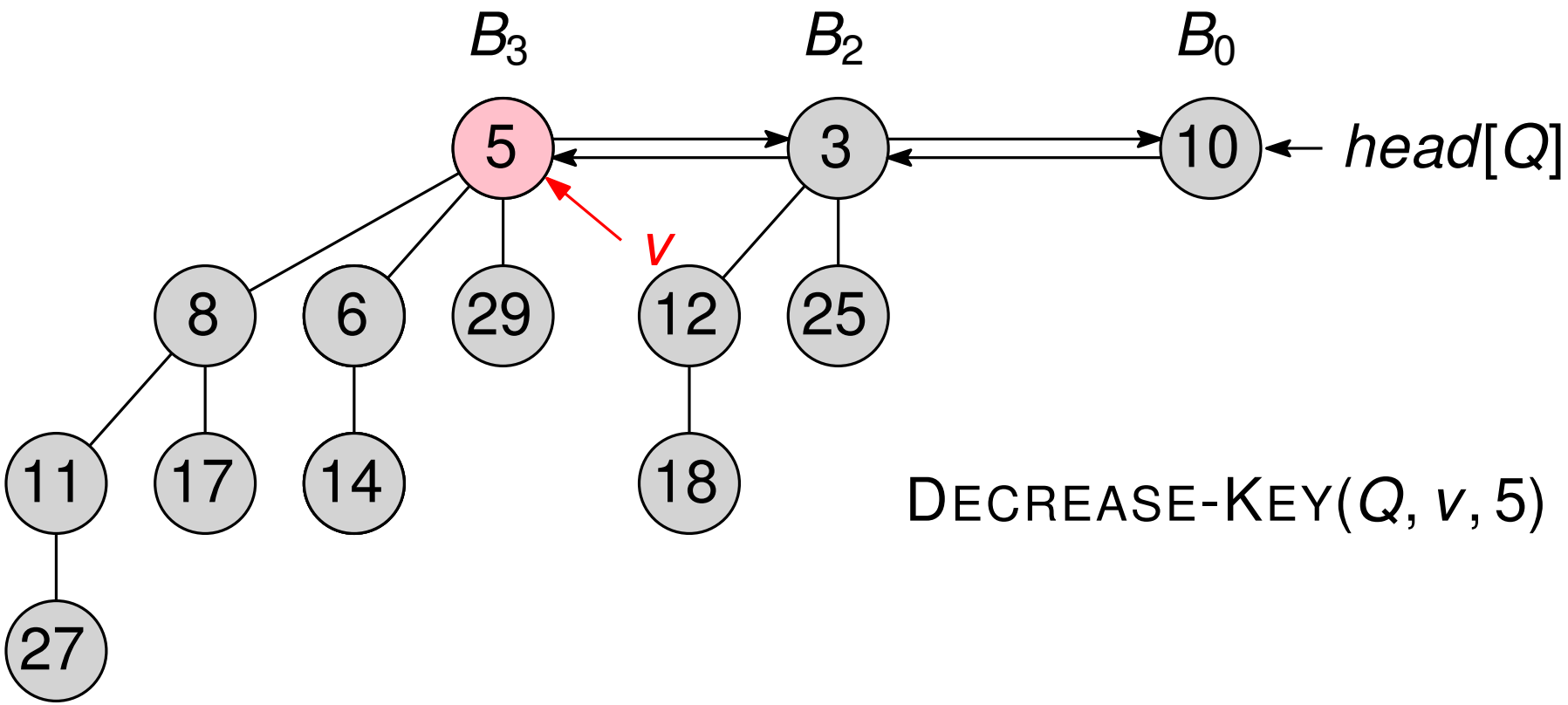
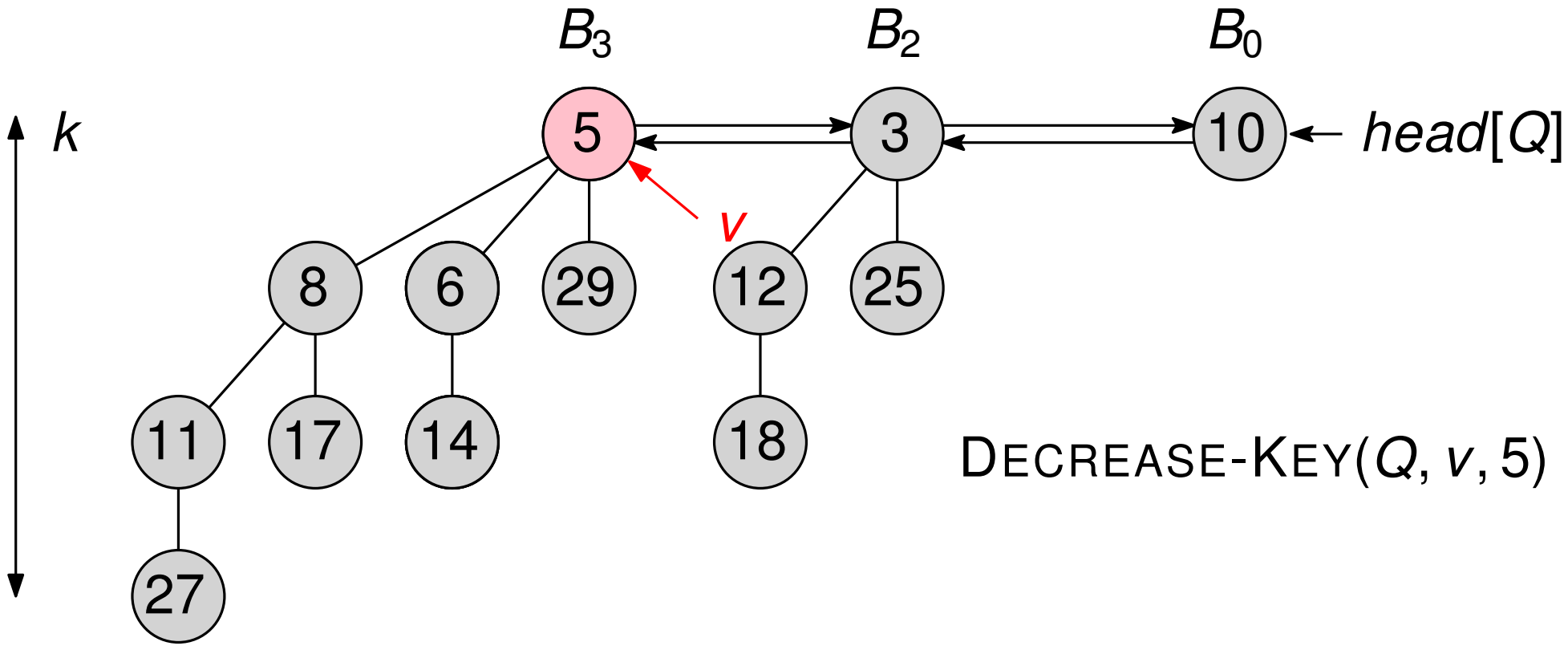


- Fix heap order

DECREASE-KEY(Q, v, k)

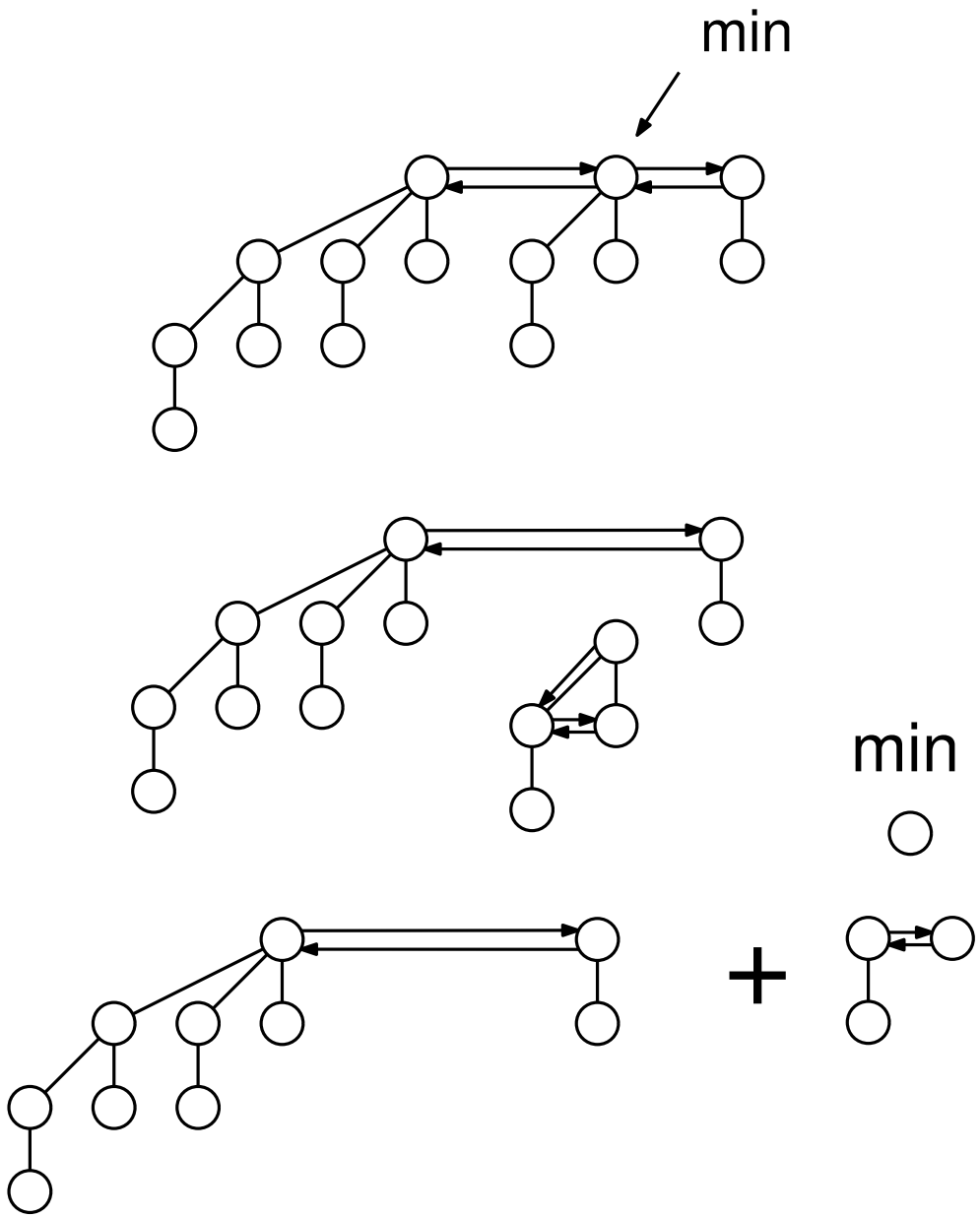
Depth of B_k is $k \leq \log n$



- Fix heap order

Reminder: EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)
  x = MINIMUM(Q)
  Q' = MAKE()
  $Q'.head = x.leftchild$
  LINKEDLIST-EXTRACT(x)
  for each child y of x **do**
    $y.parent$ = NIL
  Q = UNION(Q, Q')
  return x



Analysis: $O(\log n)$

Lazy EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)
    v = MINIMUM(Q)
    $c = v.child$
    $L_v = v.left$
    $R_v = v.right$
    if $c \neq$ NIL **then**
        $L_c = c.left$
        $L_v.right = c$
        $c.left = L_v$
        $R_v.left = L_c$
        $L_c.right = R_v$
    else
        $L_v.right = R_v$
        $R_v.left = L_v$
    $Q.min = R_v$ ▷ arbitrary head
    CONSOLIDATE(Q)
    return v

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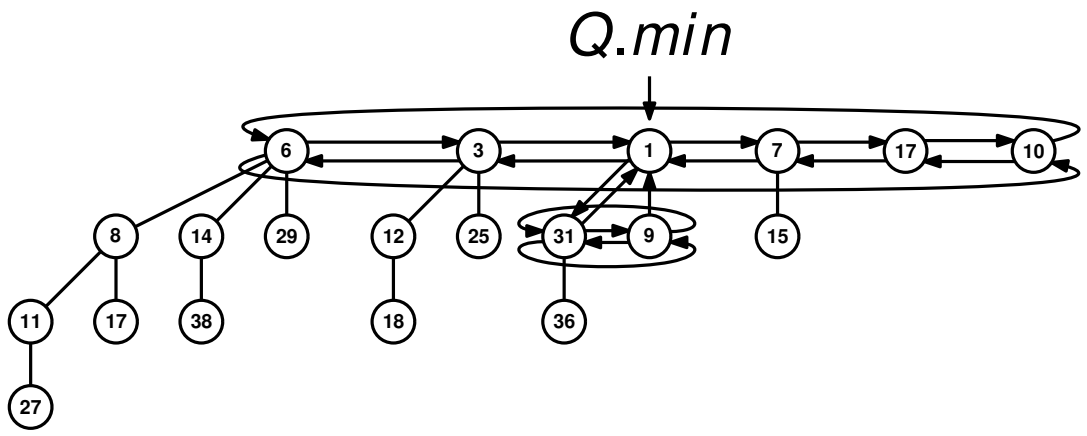

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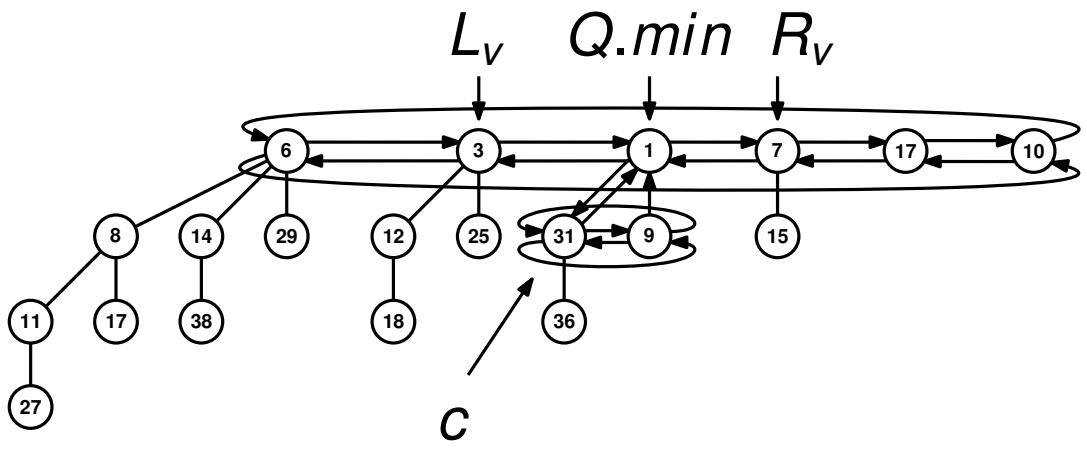

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  $L_v = v.left$
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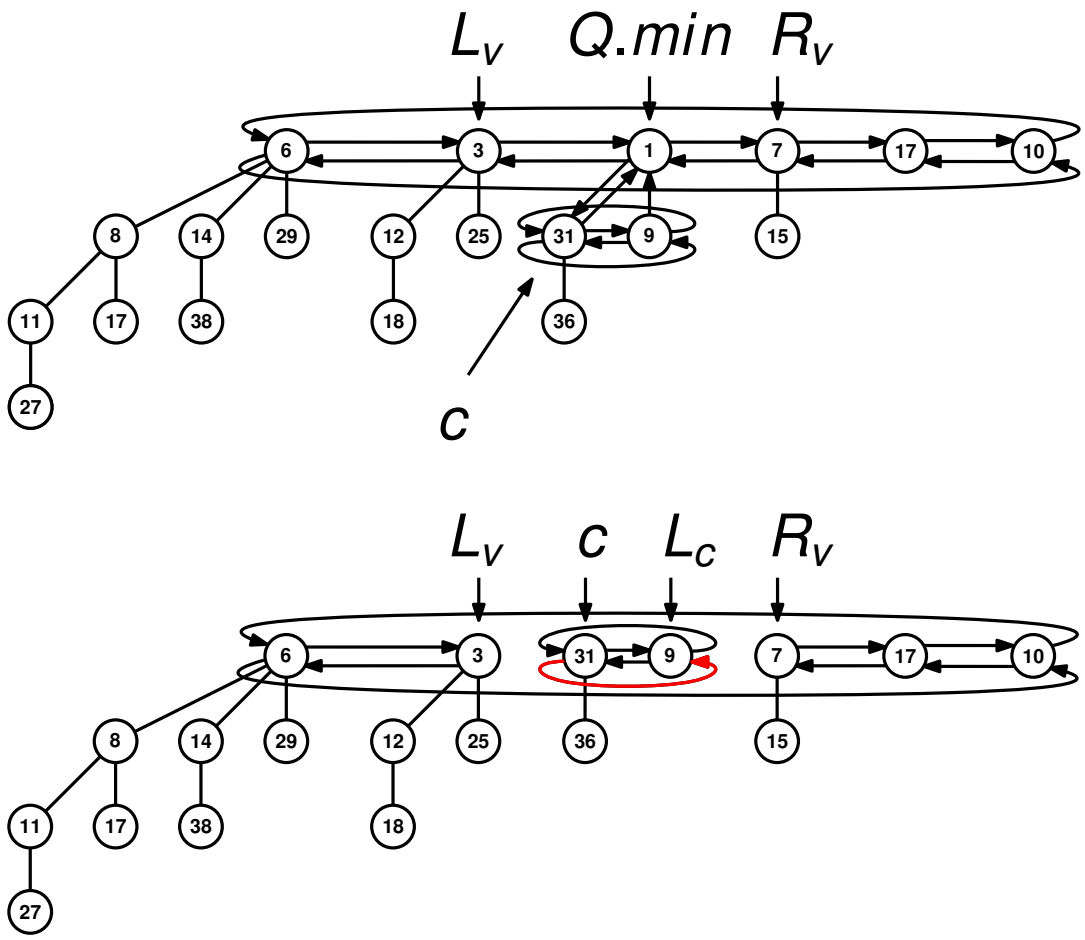
  if $c \neq$ NIL **then**
    $L_c = c.left$
    $L_v.right = c$
    $c.left = L_v$
    $R_v.left = L_c$
    $L_c.right = R_v$
  else
    $L_v.right = R_v$
    $R_v.left = L_v$
  $Q.min = R_v$ ▷ arbitrary head
  CONSOLIDATE(Q)
  return v



Lazy EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)
- v = MINIMUM(Q)
- $c = v.child$
- $L_v = v.left$
- $R_v = v.right$
- **if** $c \neq$ NIL **then**
  - $L_c = c.left$
  - $L_v.right = c$
  - $c.left = L_v$
  - $R_v.left = L_c$
  - $L_c.right = R_v$
- **else**
  - $L_v.right = R_v$
  - $R_v.left = L_v$
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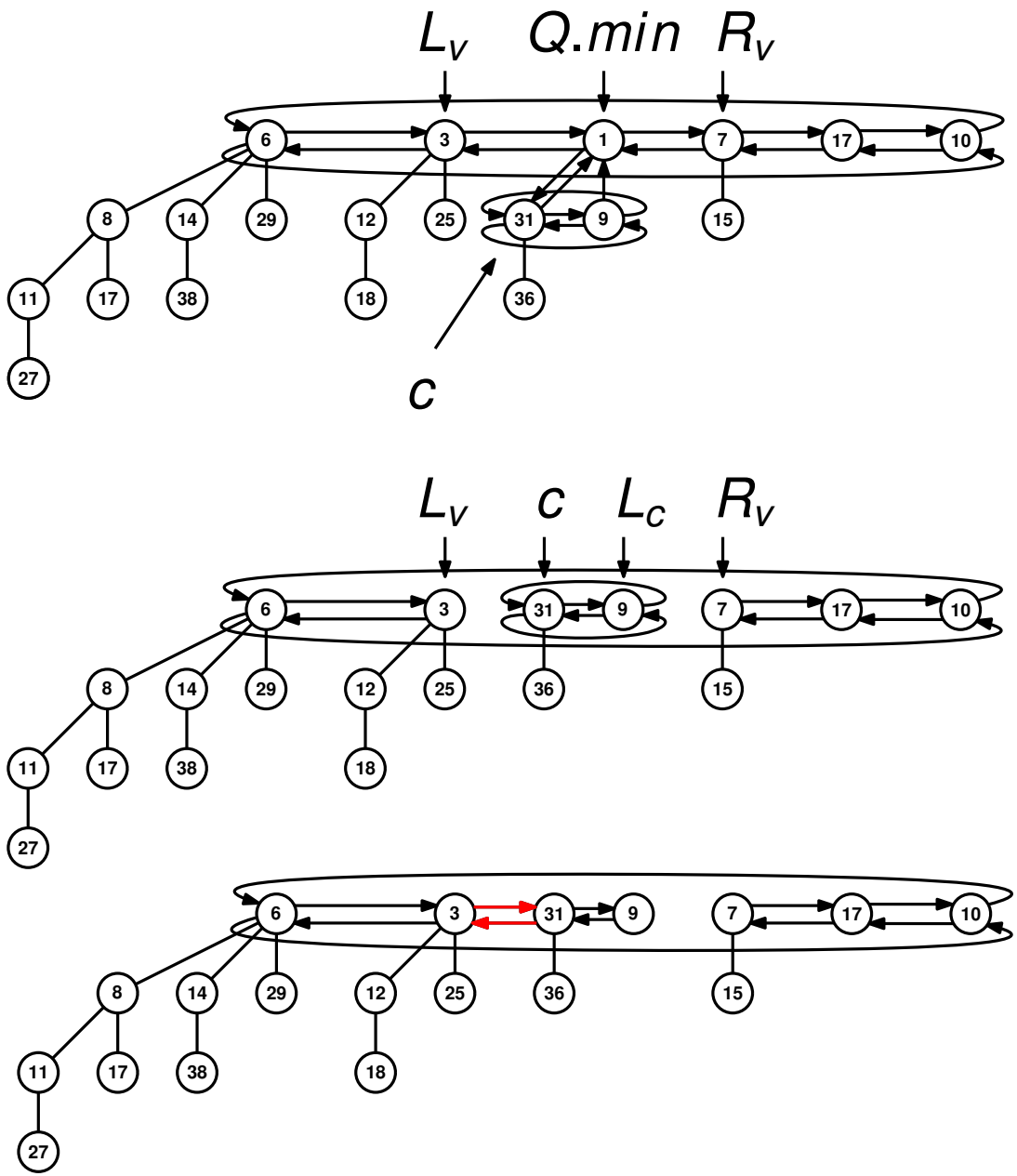
- CONSOLIDATE(Q)
- **return** v



Lazy EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)

v = MINIMUM(Q)

$c = v.child$

$L_v = v.left$

$R_v = v.right$

if $c \neq$ NIL **then**

$L_c = c.left$

$L_v.right = c$

$c.left = L_v$

→ $R_v.left = L_c$

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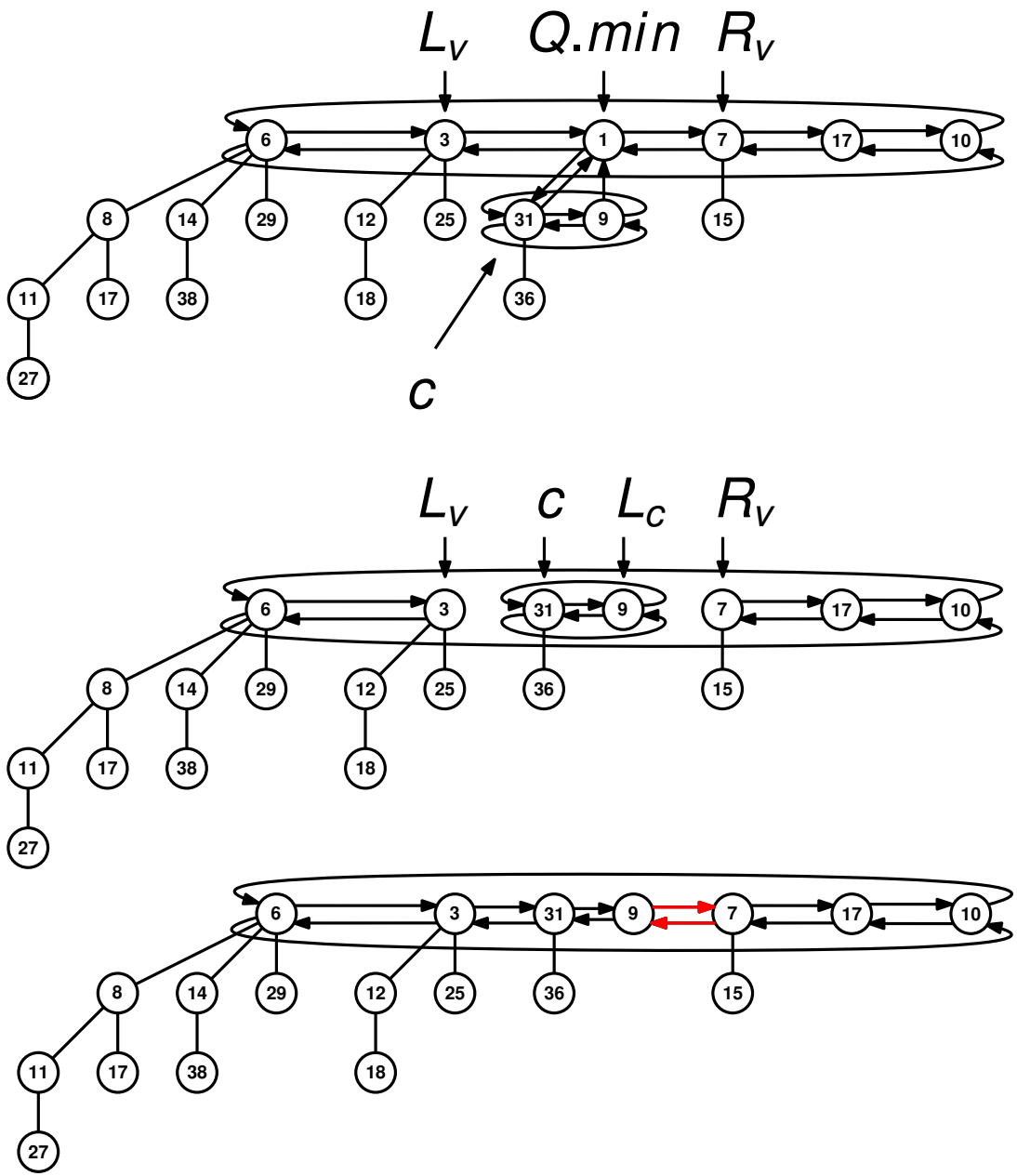
CONSOLIDATE(Q)

return v



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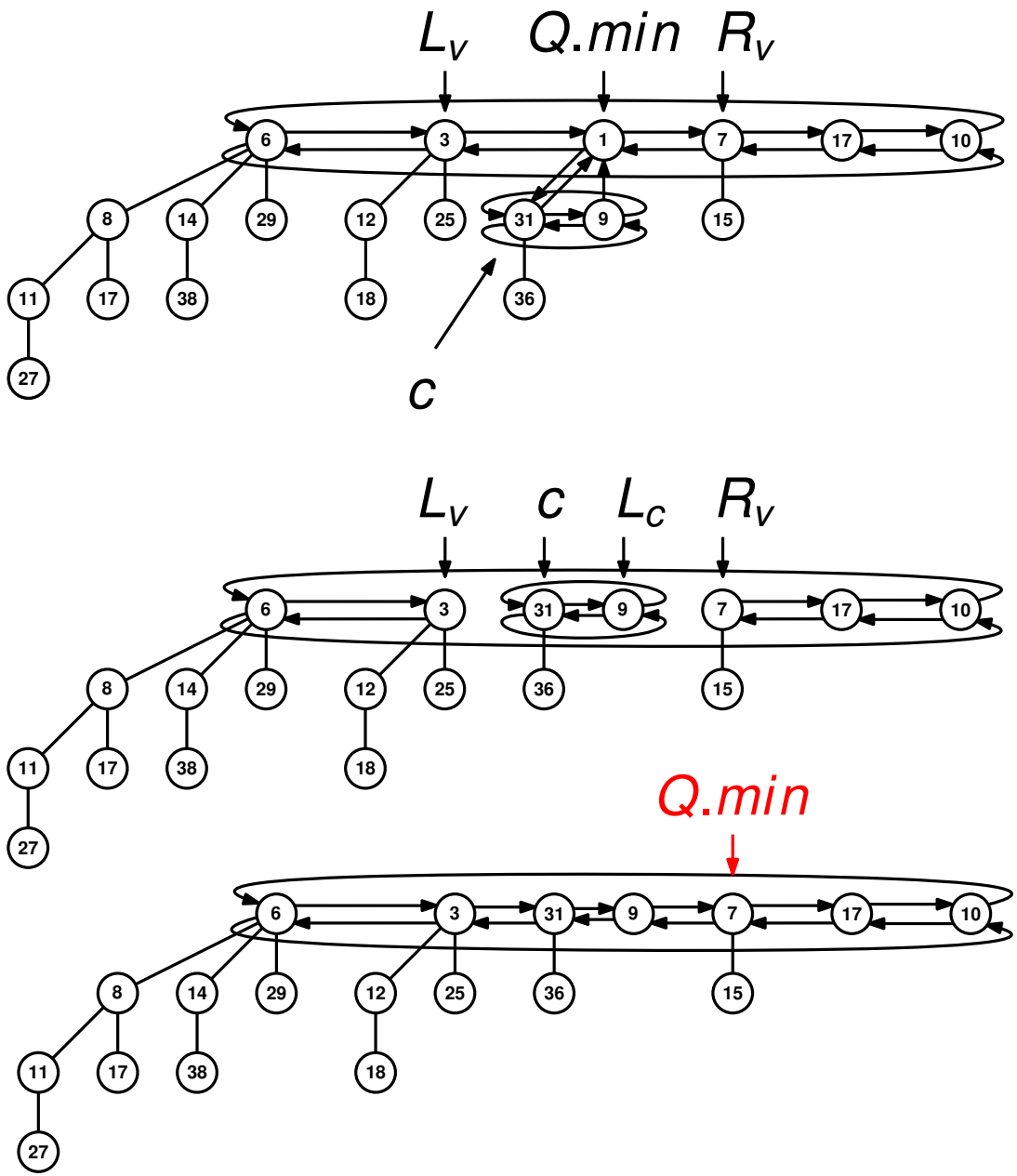
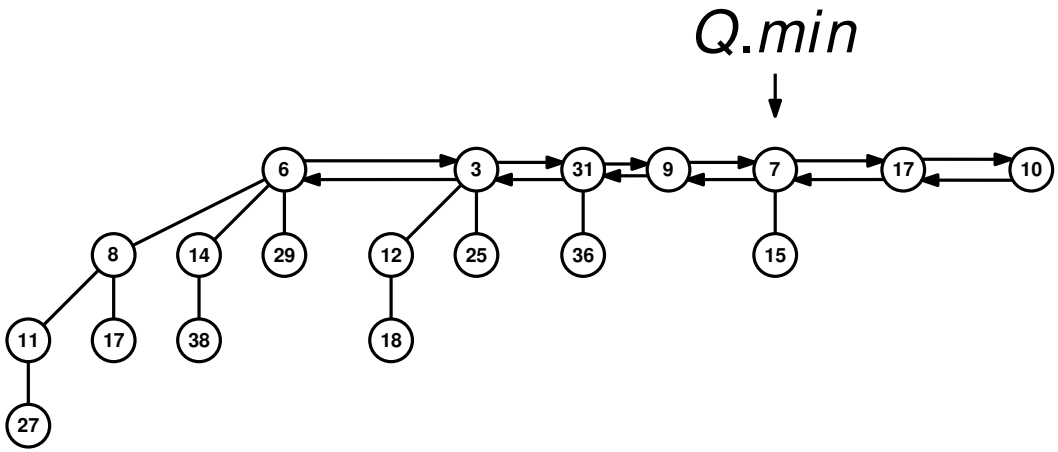


CONSOLIDATE(Q)

Root list:

- At most log n distinct tree orders
- Use log n-sized array with pointers to each tree order



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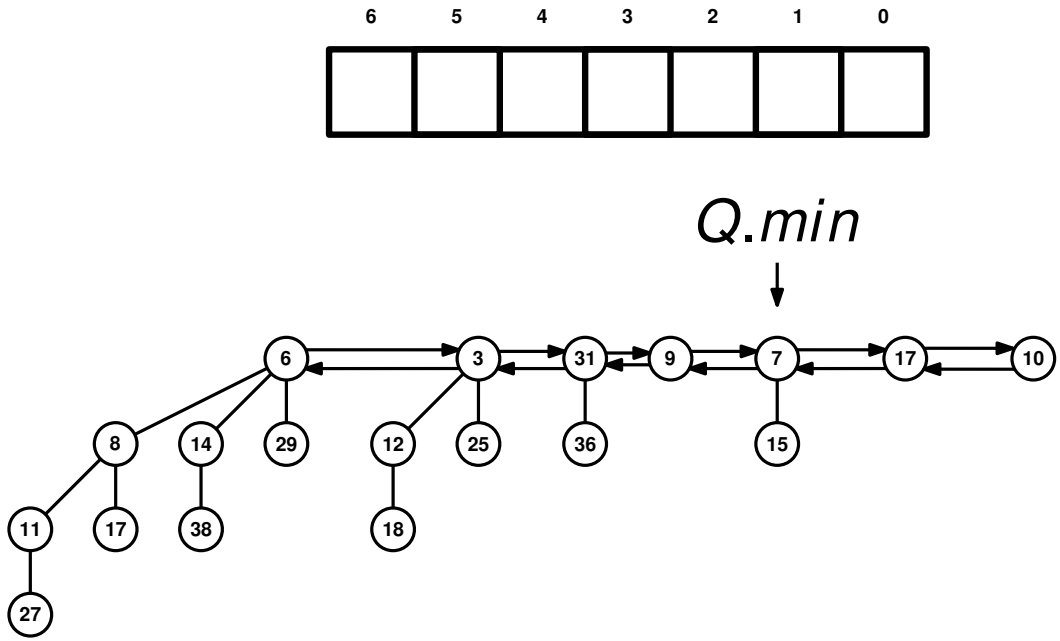

CONSOLIDATE(Q)

```
function CONSOLIDATE(Q)
    Initialize log n-sized array A to NIL
    for each v in root list do
        d = v.degree
        while A[d] ≠ NIL do
            v = LINK(v, A[d])
            A[d] = NIL
            d = d + 1
        A[d] = v; v.parent = NIL
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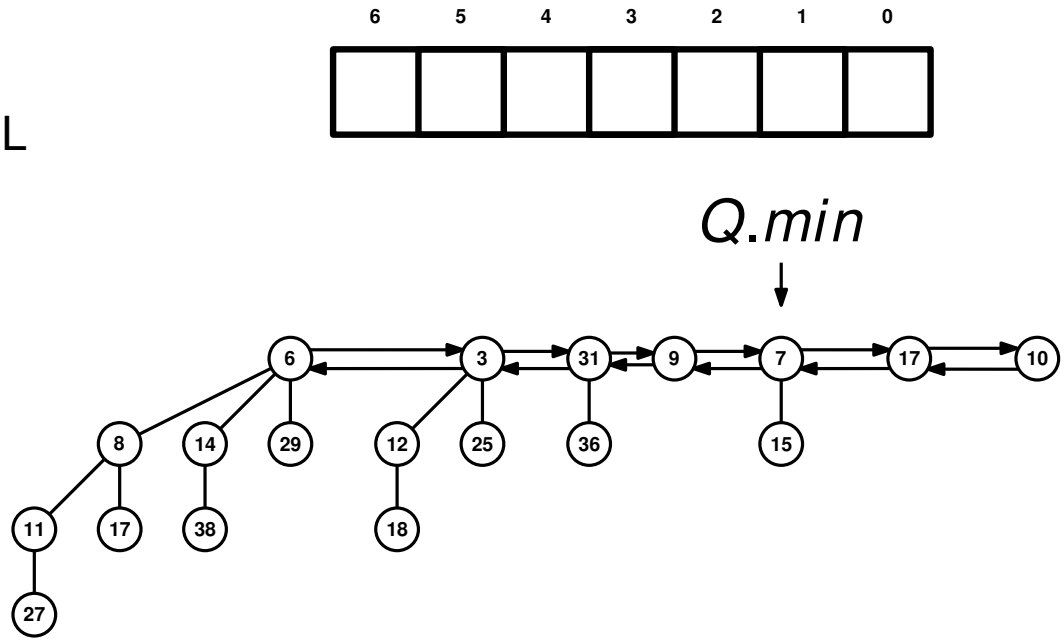

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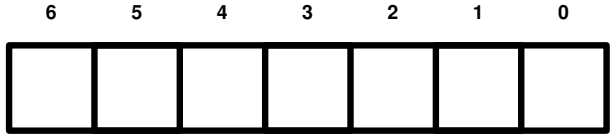

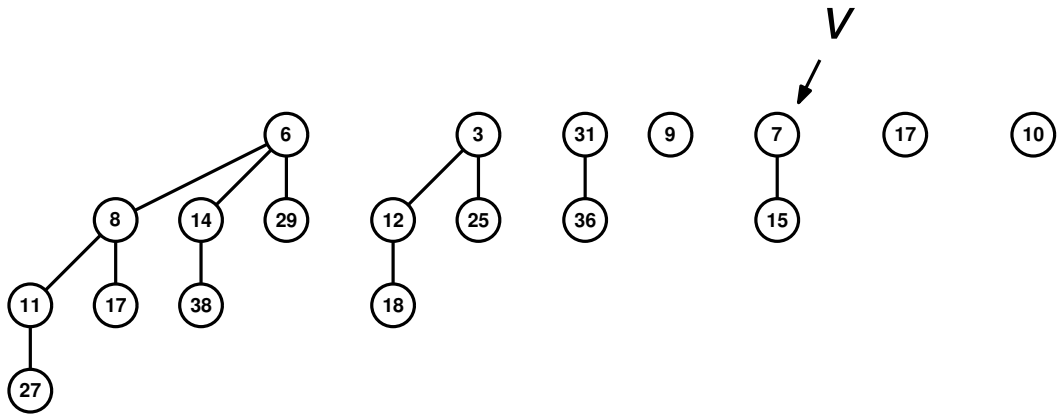

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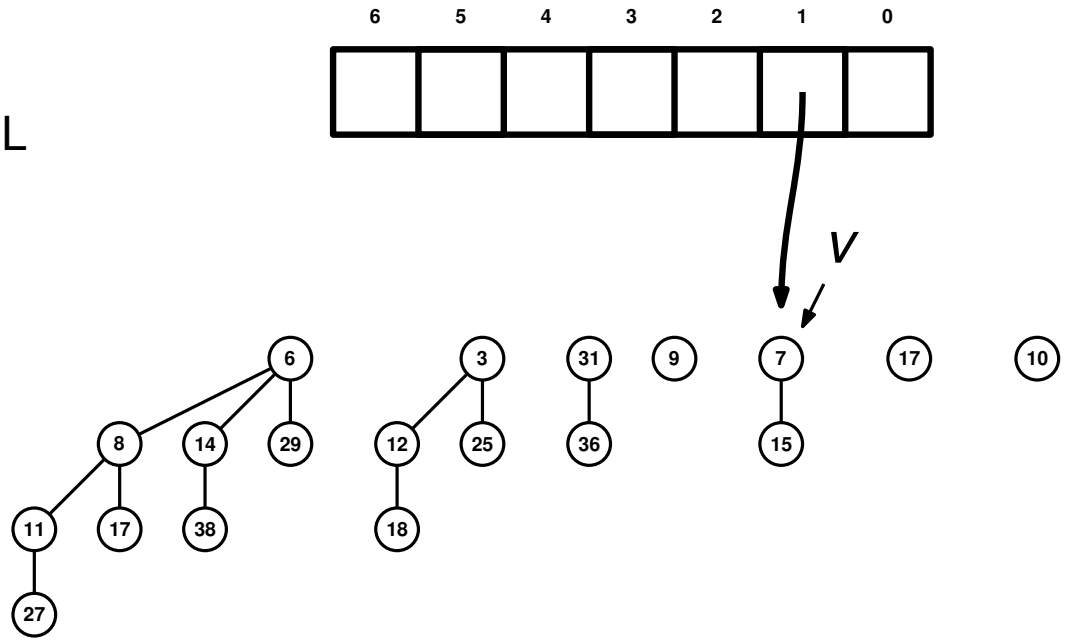

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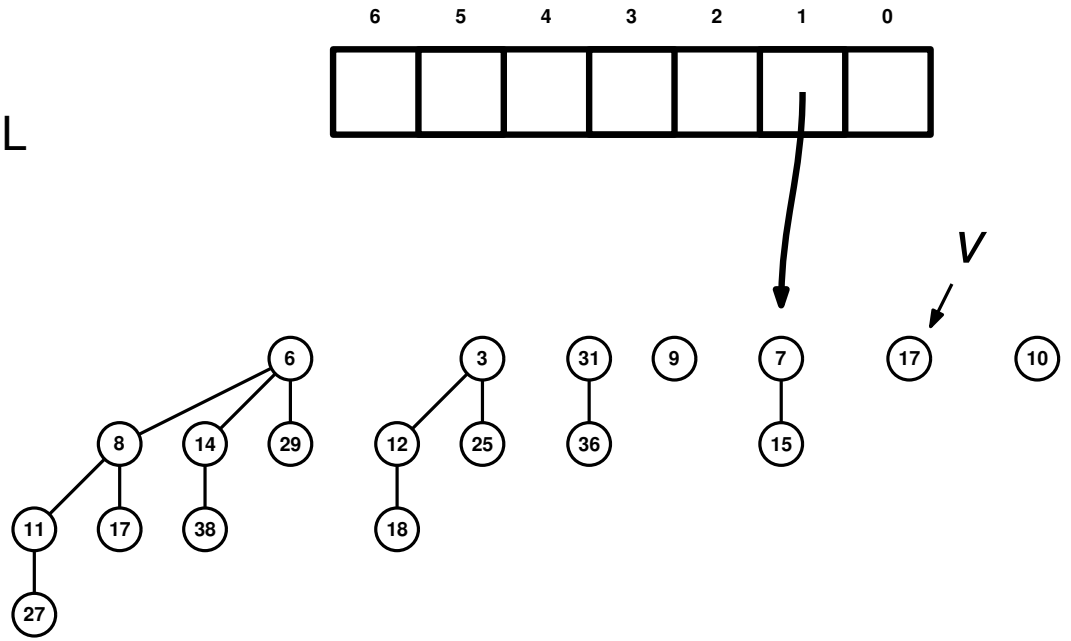

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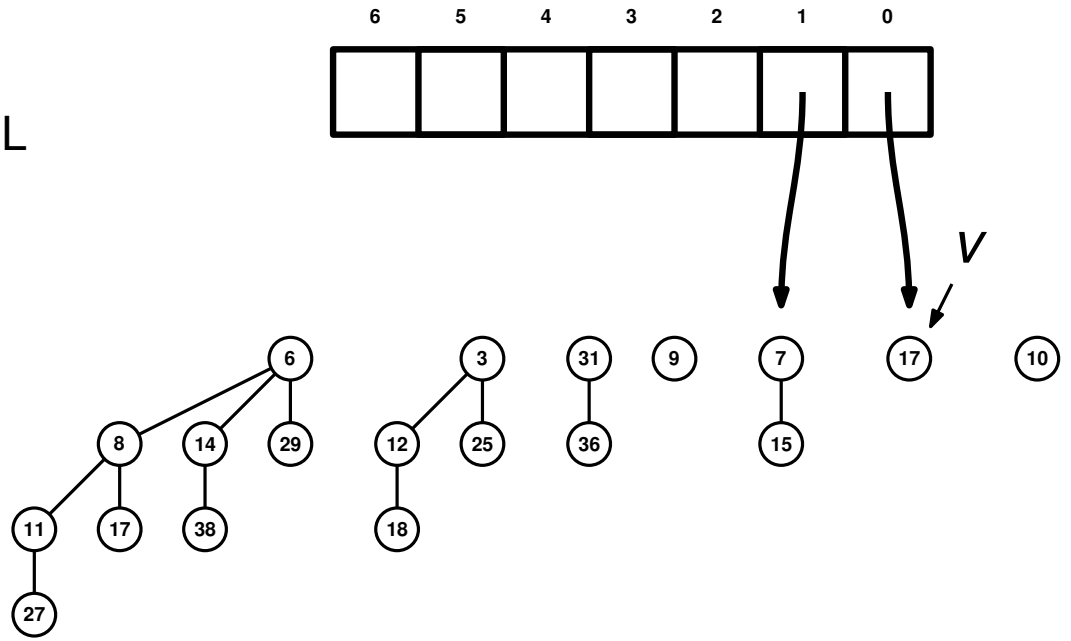

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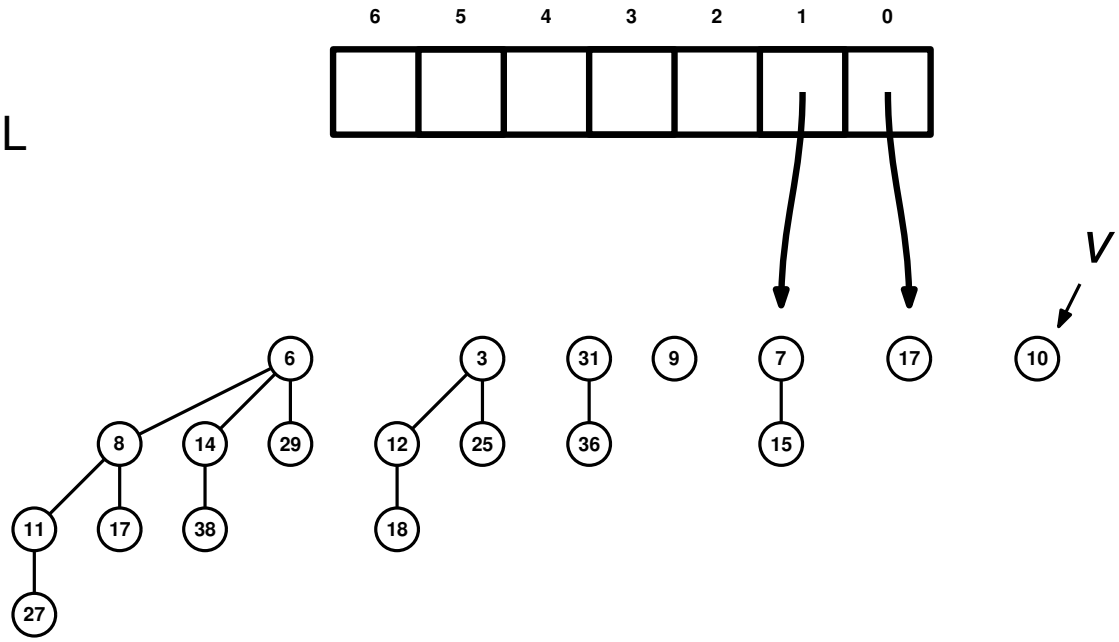

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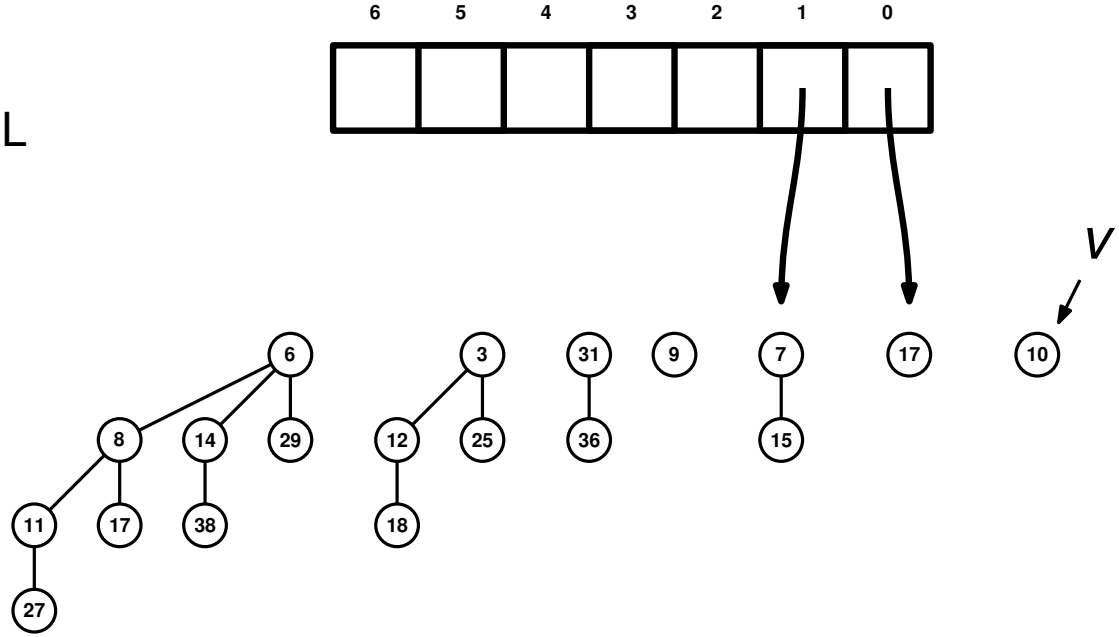

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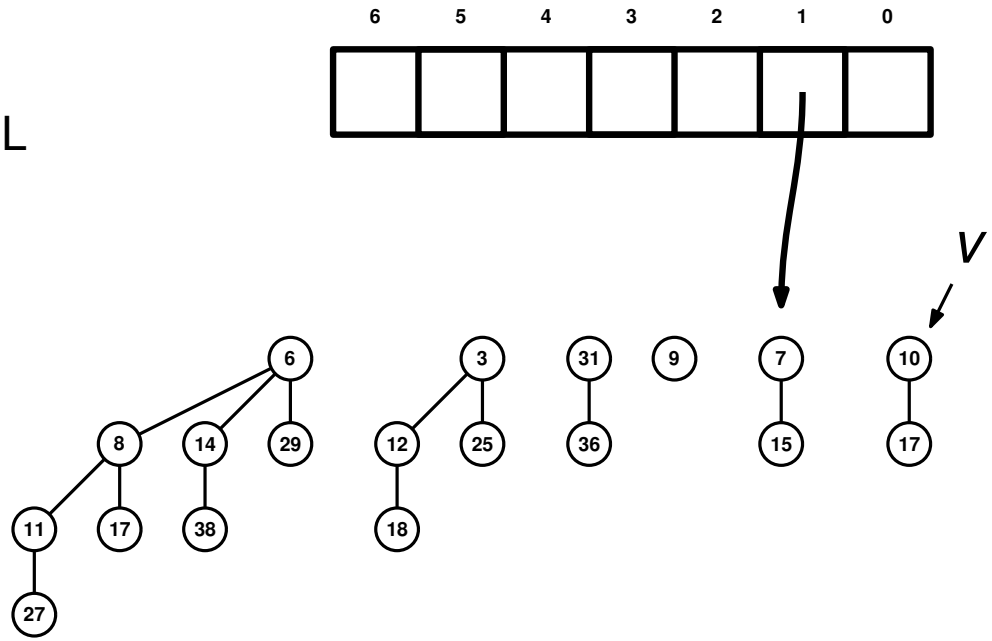

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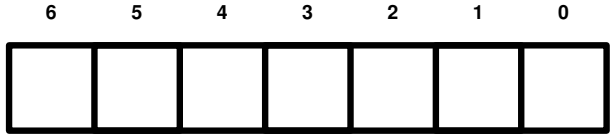

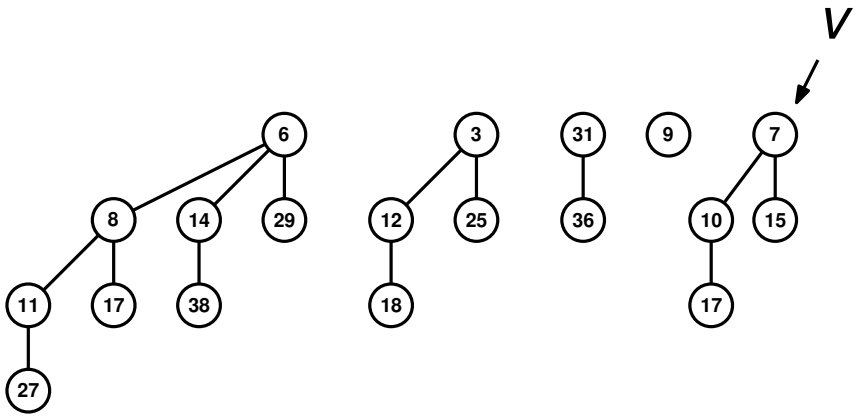

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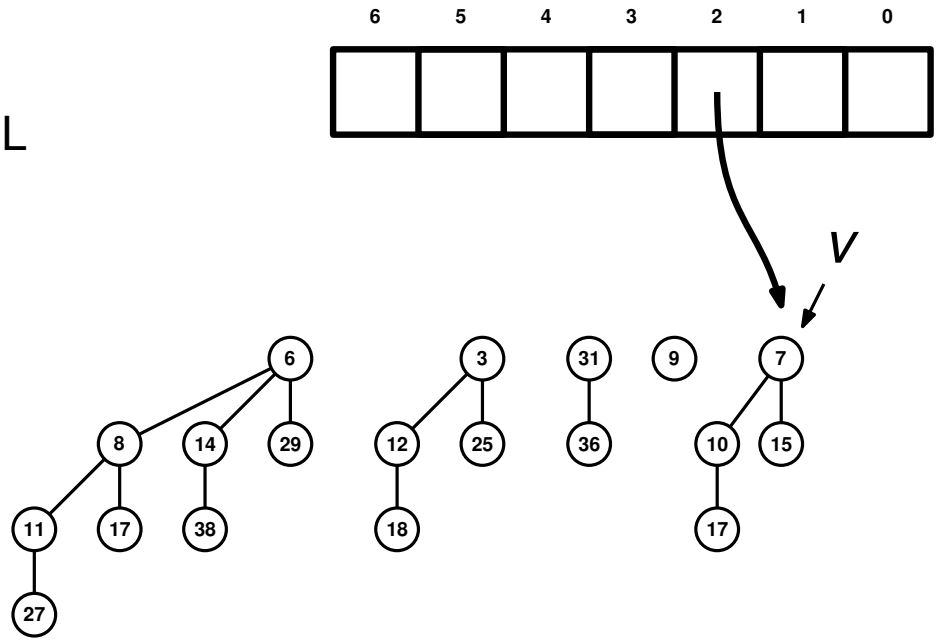

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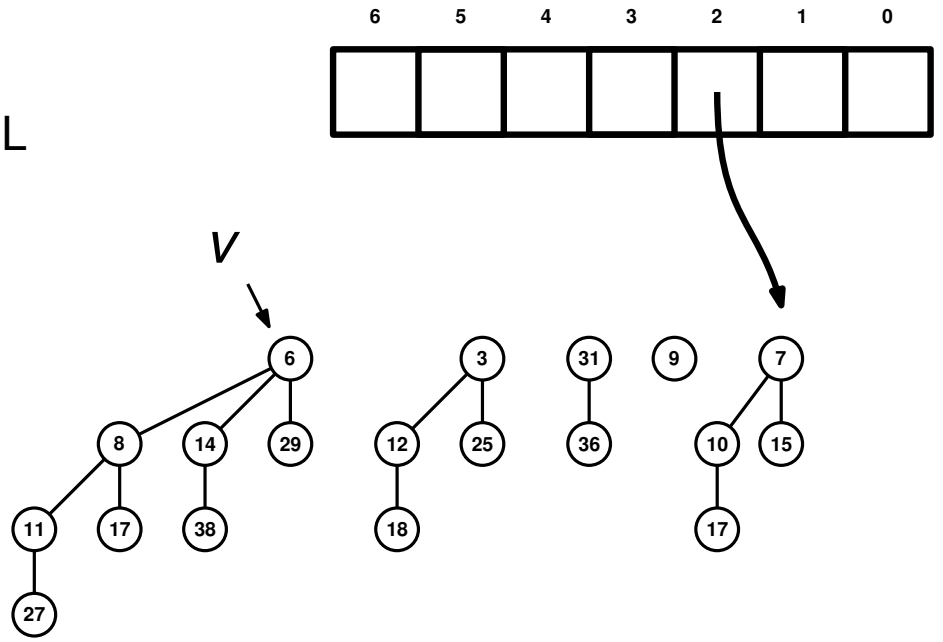

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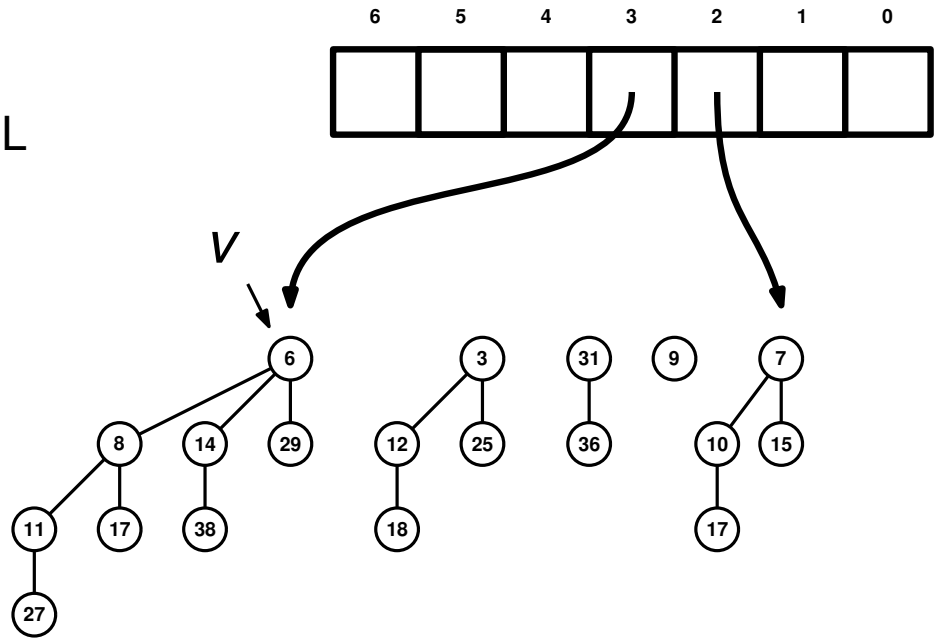

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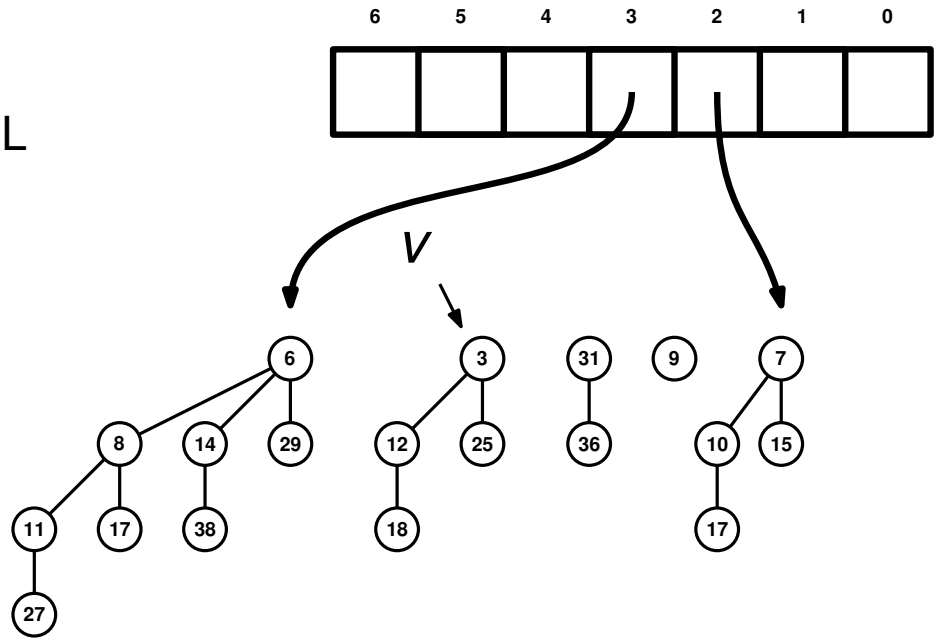

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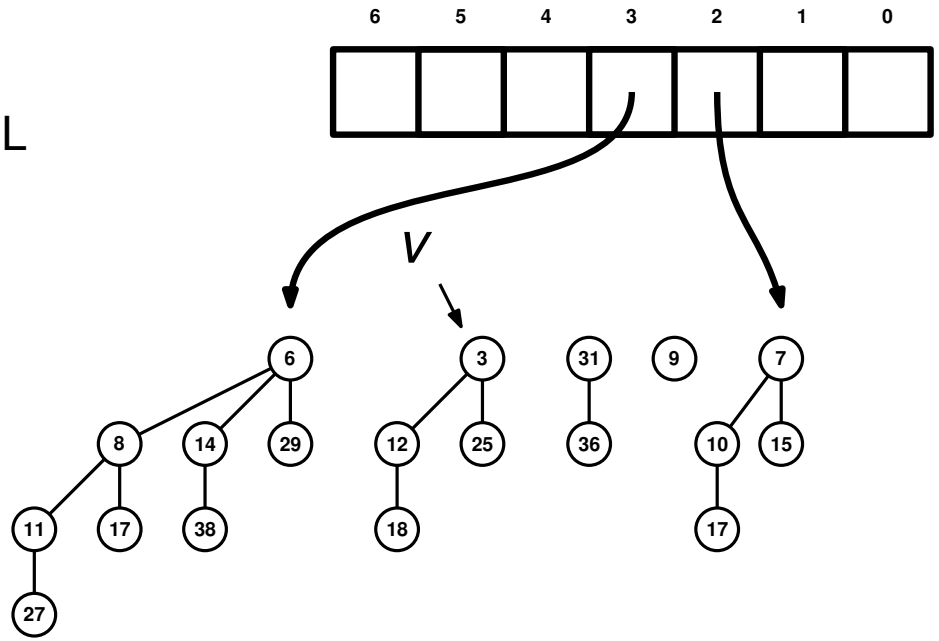

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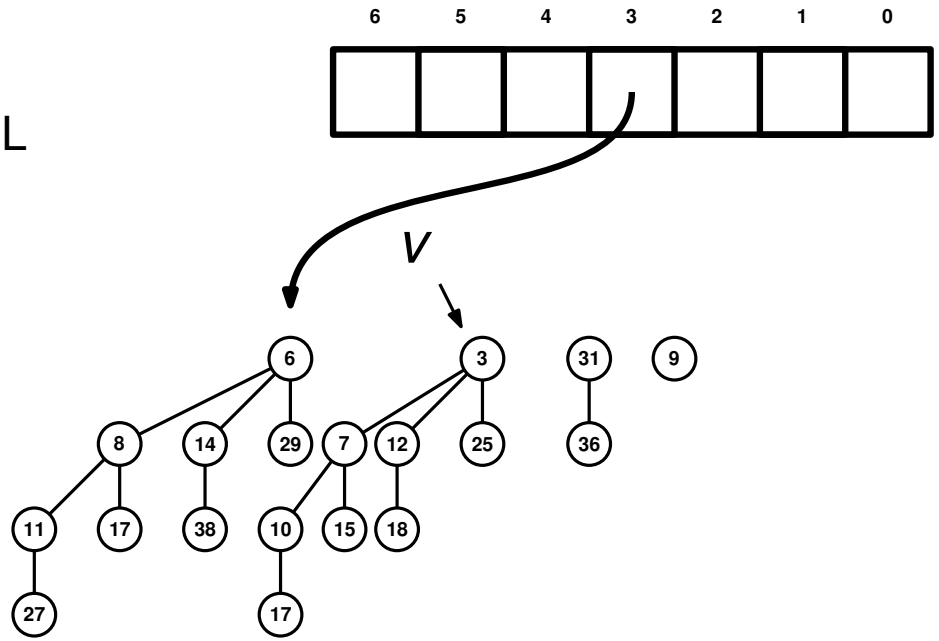

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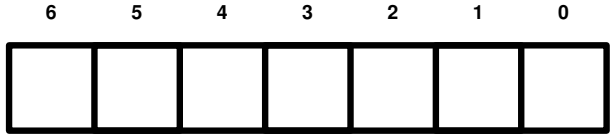

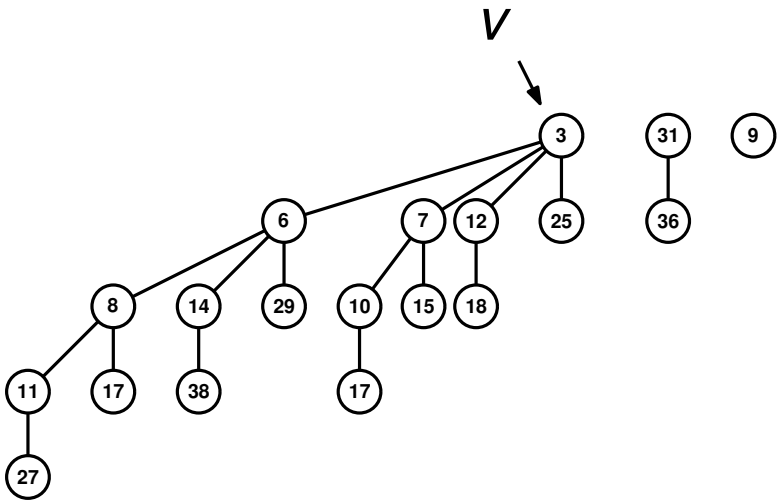

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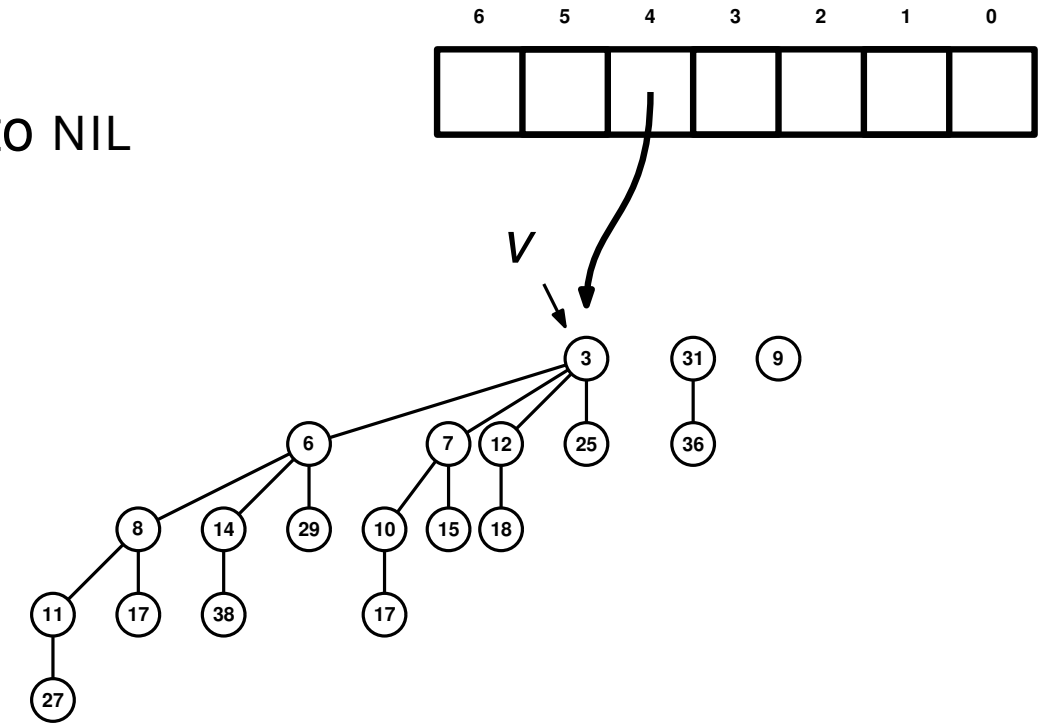

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function CONSOLIDATE(Q)
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    for each v in root list **do**
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        while $A[d] \neq$ NIL **do**
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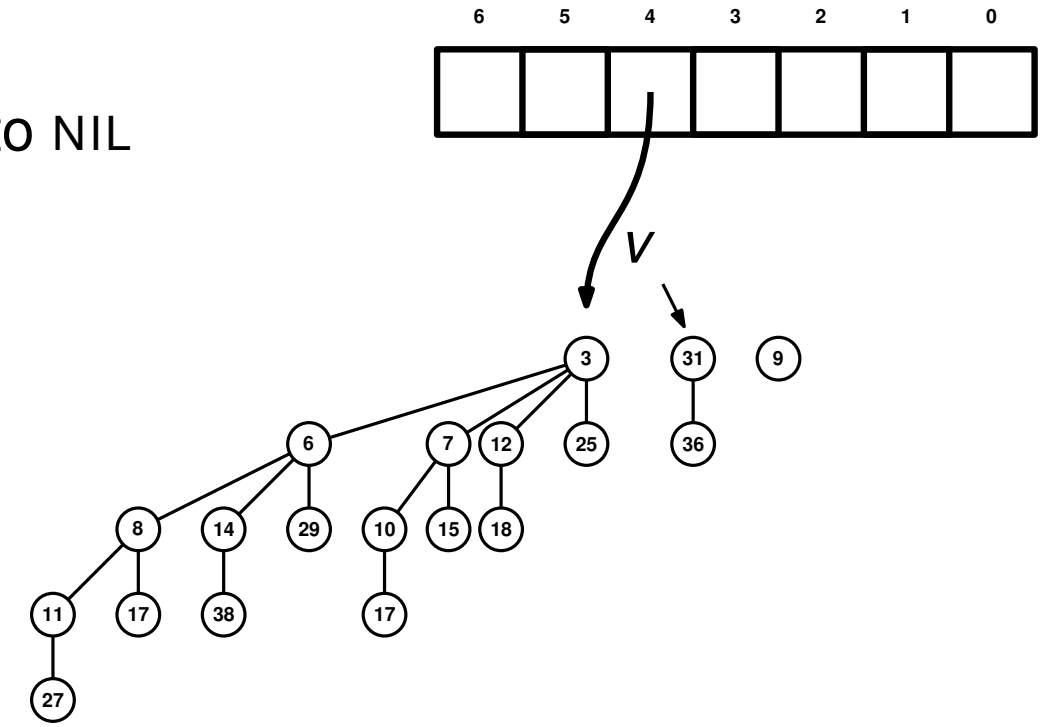

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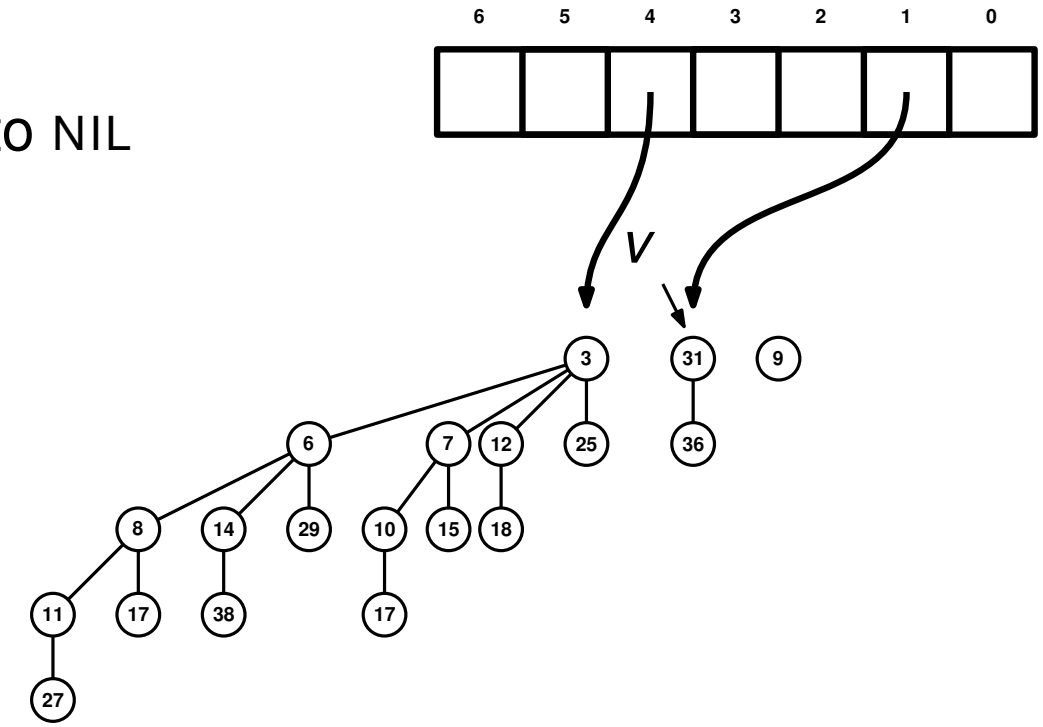

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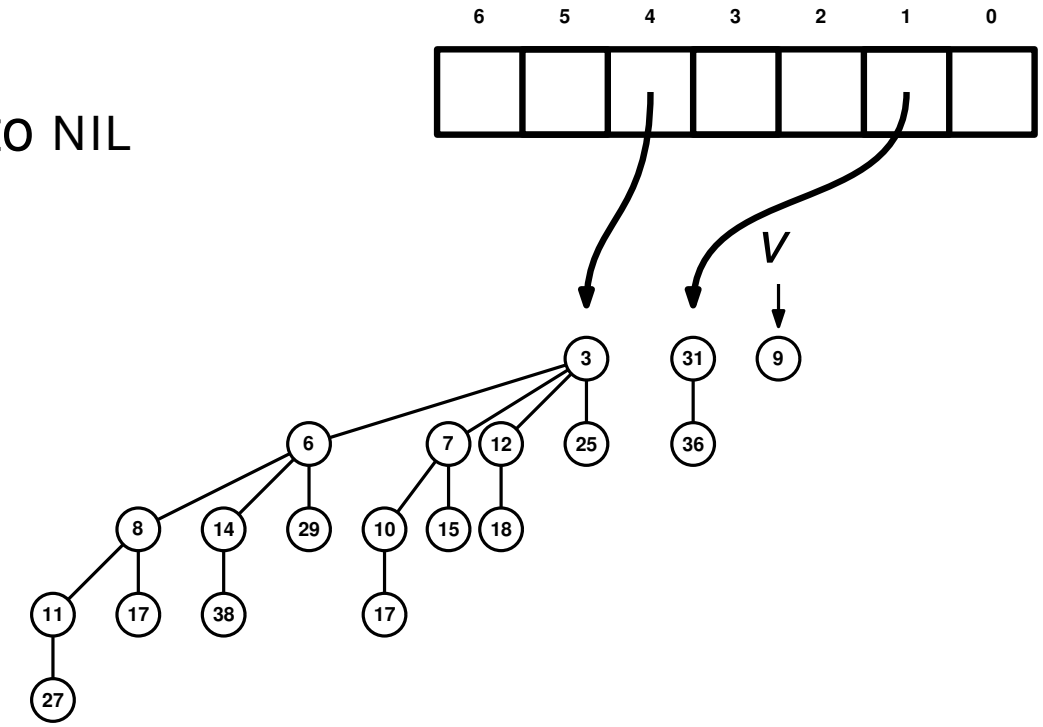

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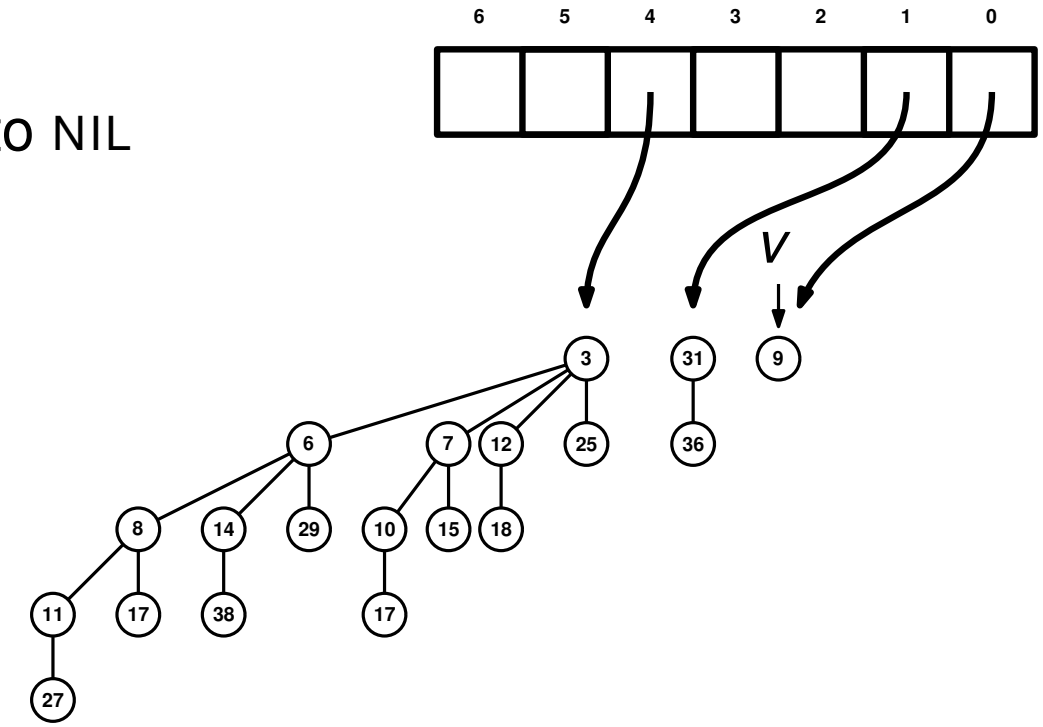

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            d = d + 1
        A[d] = v; v.parent = NIL
    min = +∞
    for i = 0 to log n − 1 do
        if A[i] ≠ NIL then
            Add A[i] to the root list
            if A[i].key < min then
                Q.min ← A[i]; min = A[i].key
```

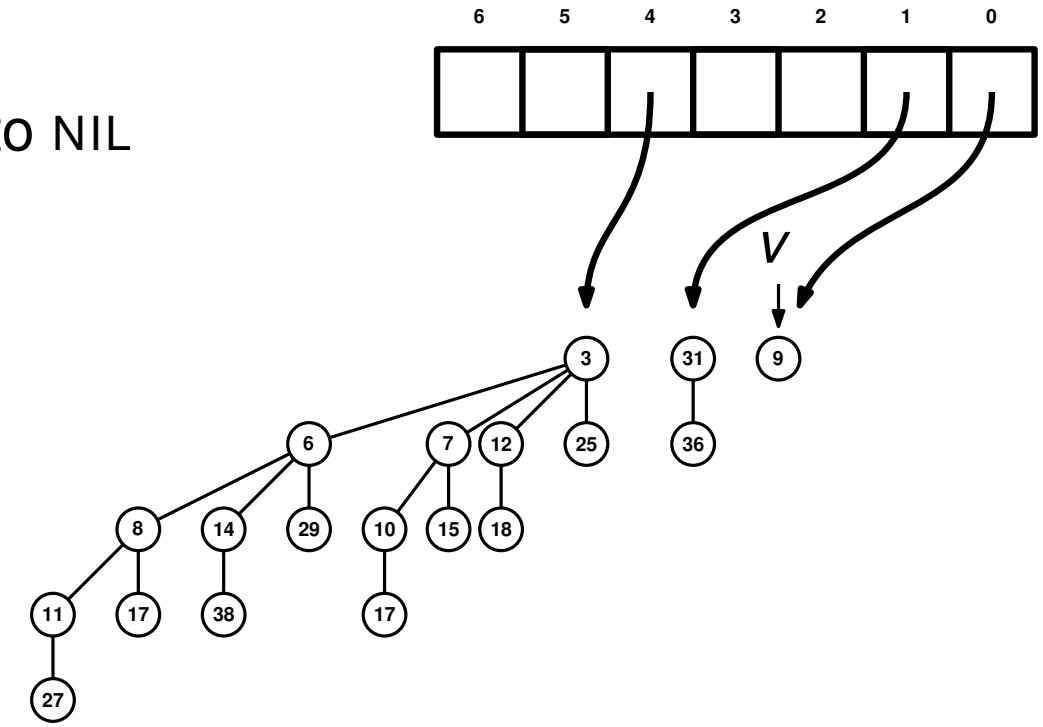

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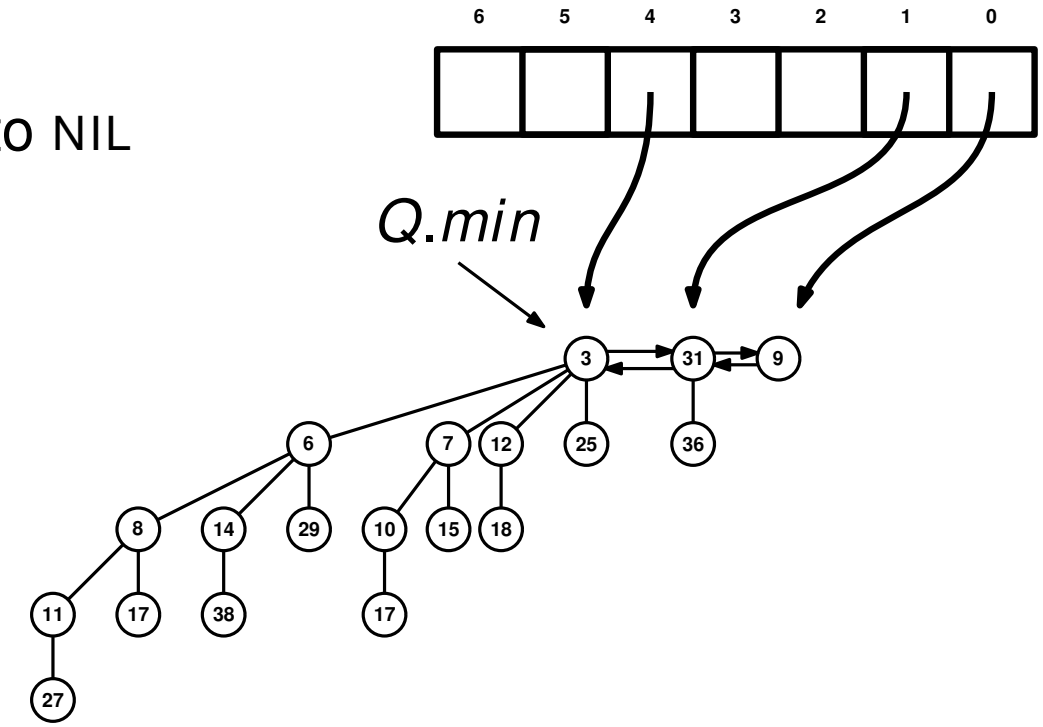
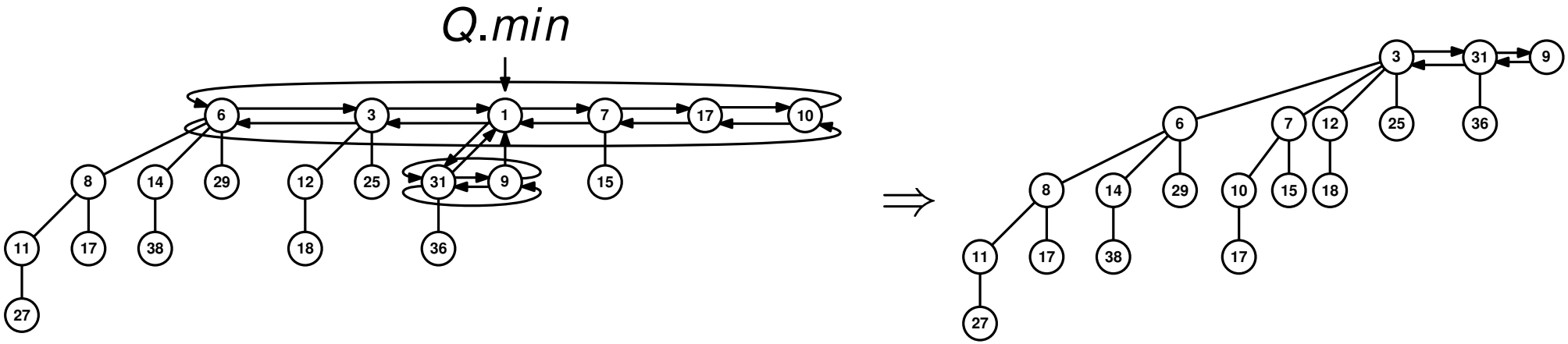
            $d = d + 1$
        $A[d] = v$; $v.parent =$ NIL
    $min = +\infty$
    for $i = 0$ **to** $\log n - 1$ **do**
        if $A[i] \neq$ NIL **then**
            Add $A[i]$ to the root list
            if $A[i].key < min$ **then**
                $Q.min \leftarrow A[i]$; $min = A[i].key$



Lazy EXTRACT-MIN(Q) Analysis

We will use the Potential Method with $\Phi_i = t_i$ = number of trees in the root list after the i-th operation.

Let d be the number of children of the $Q.min$ ($d \leq \log n$)

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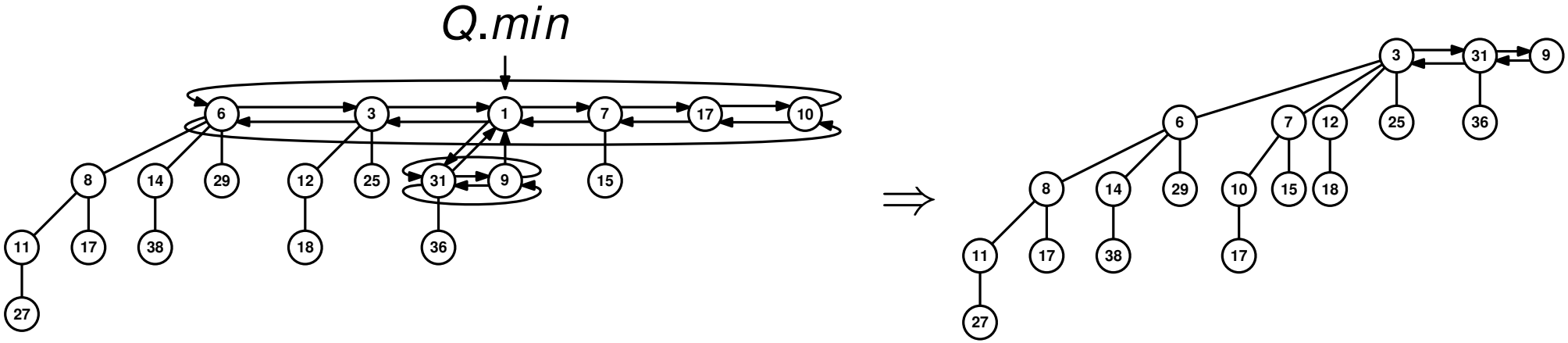
Lazy EXTRACT-MIN(Q) Analysis

We will use the Potential Method with $\Phi_i = t_i$ = number of trees in the root list after the i-th operation.

Let d be the number of children of the $Q.min$ ($d \leq \log n$)



- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log n \leq O(1) + t_{i-1} + 2 \log n$
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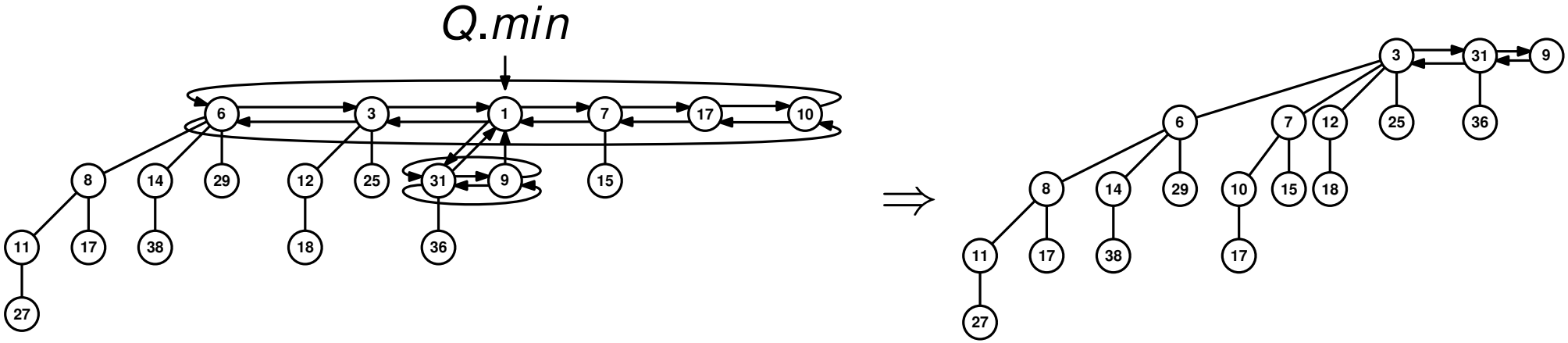

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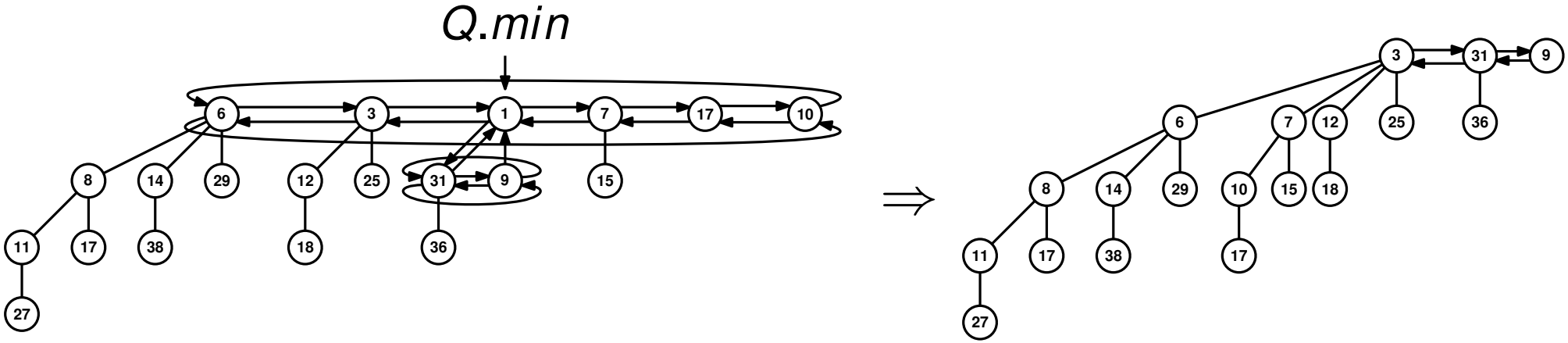

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there are $t_i \leq \log n$ trees in the root list after consolidation

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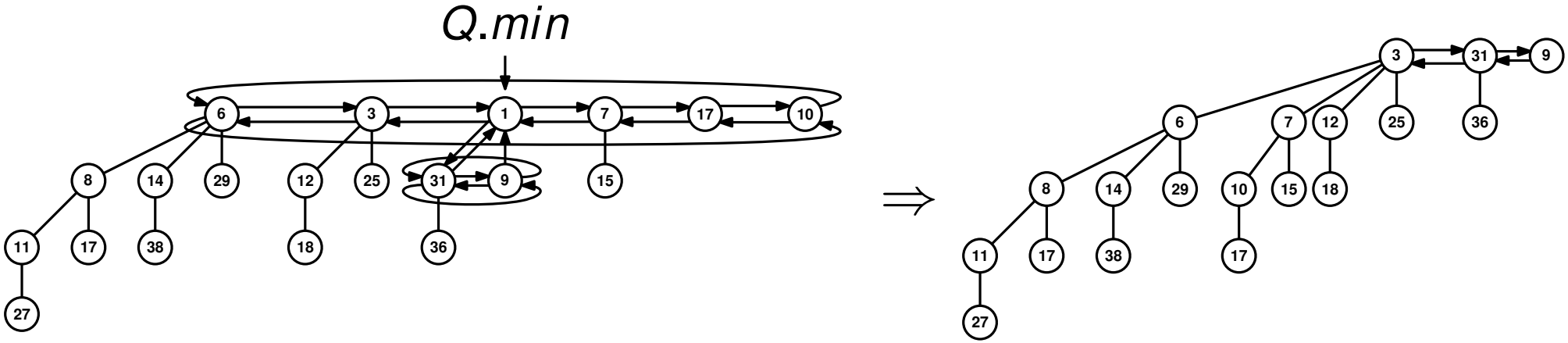

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$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2 \log n + t_i \leq O(1) + 3 \log n = O(\log n)$$

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Heaps

	Binomial	Lazy Binomial	Fibonacci
MAKE()	$O(1)$	$O(1)$	$O(1)$
INSERT(Q, x)	$O(1)^*$	$O(1)$	$O(1)$
MINIMUM(Q)	$O(1)$	$O(1)$	$O(1)$
EXTRACT-MIN(Q)	$O(\log n)$	$O(\log n)^*$	$O(\log n)^*$
DECREASE-KEY(Q, x, k)	$O(\log n)$	$O(\log n)$	$O(1)^*$
DELETE(Q, x)	$O(\log n)$	$O(\log n)^*$	$O(\log n)^*$
UNION(Q_1, Q_2)	$O(\log n)$	$O(1)$	$O(1)$

* Amortized cost

Fibonacci Heaps

Goal: DECREASE-KEY(Q, v, k) in $O(1)$ (amortized) time

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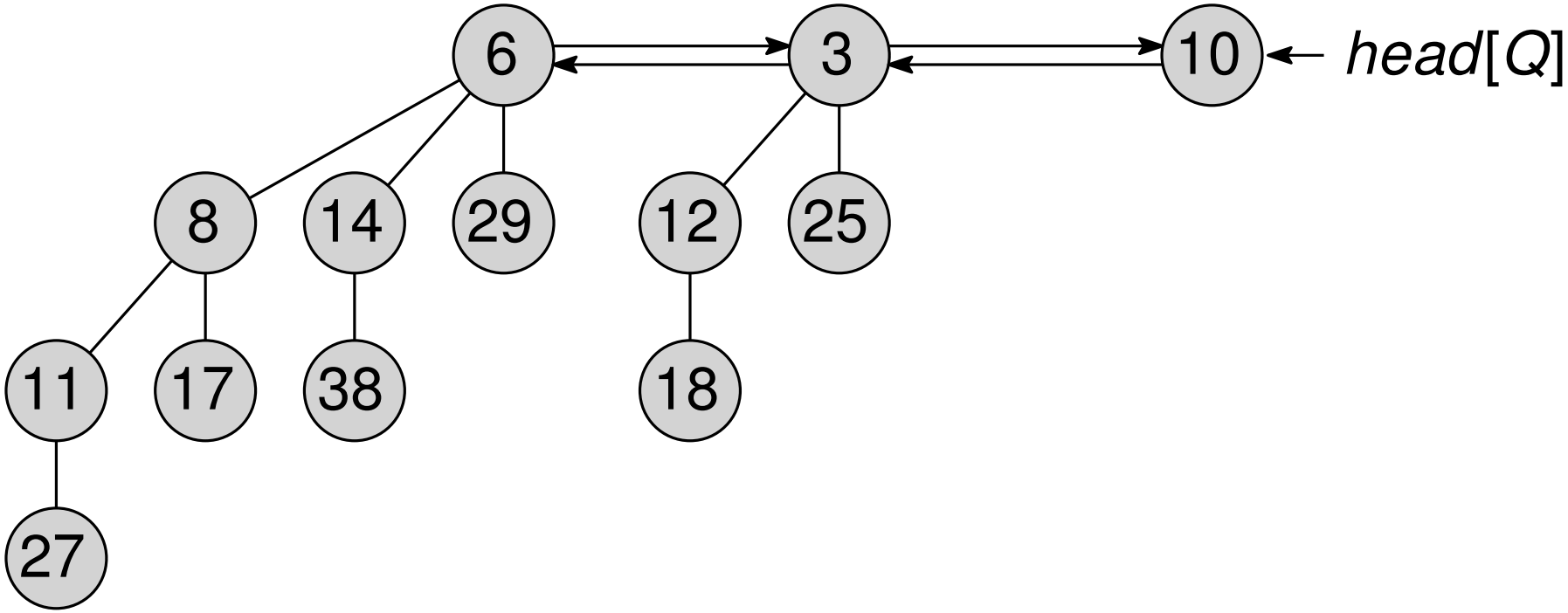

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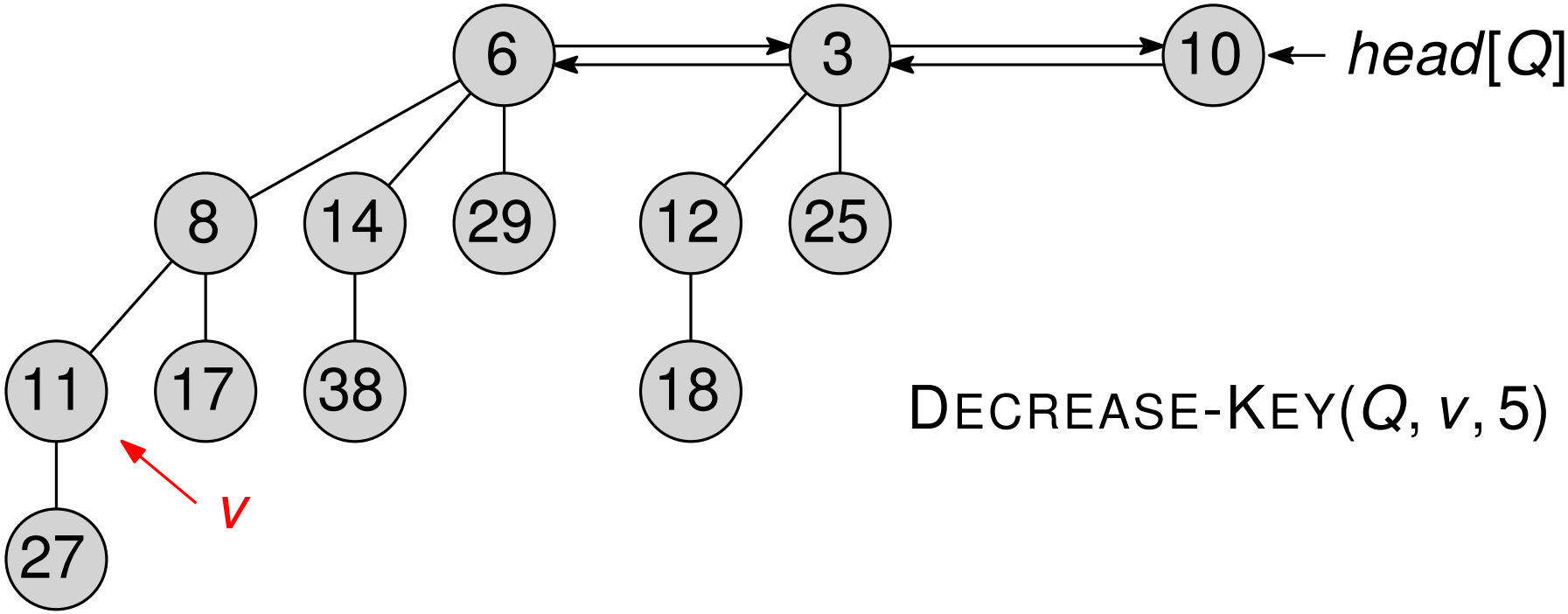

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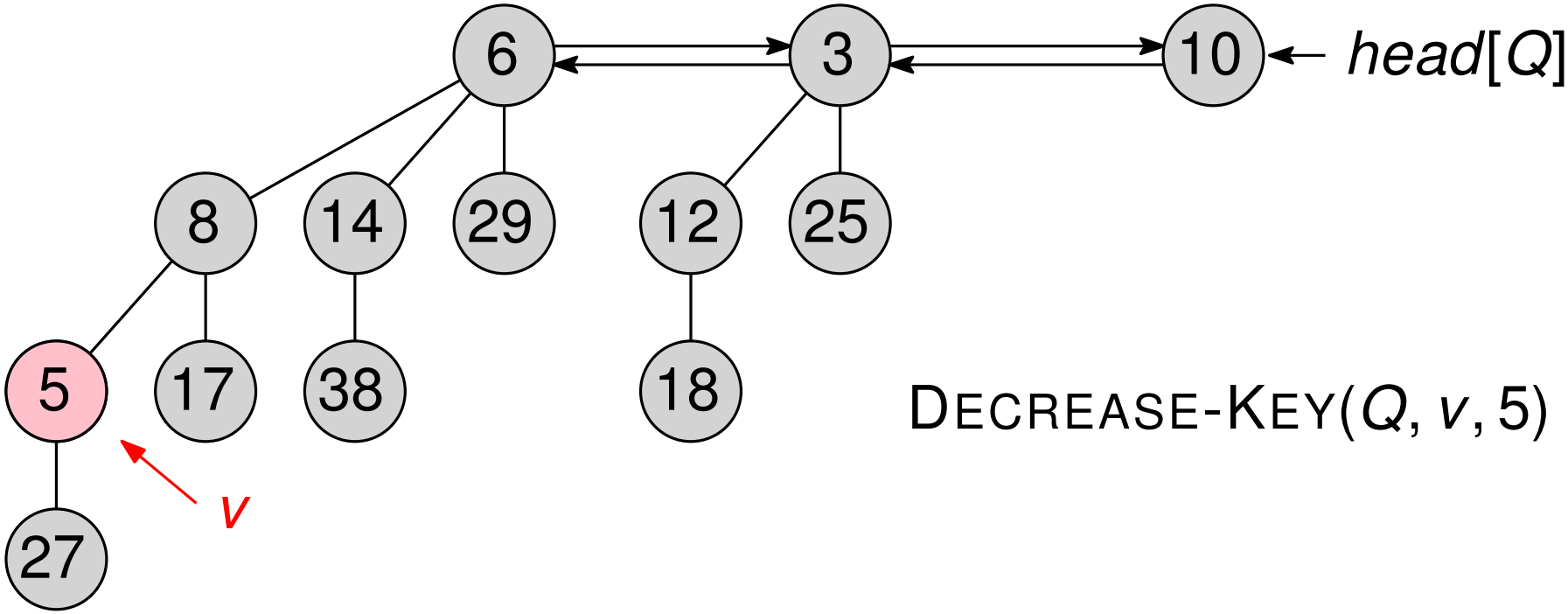

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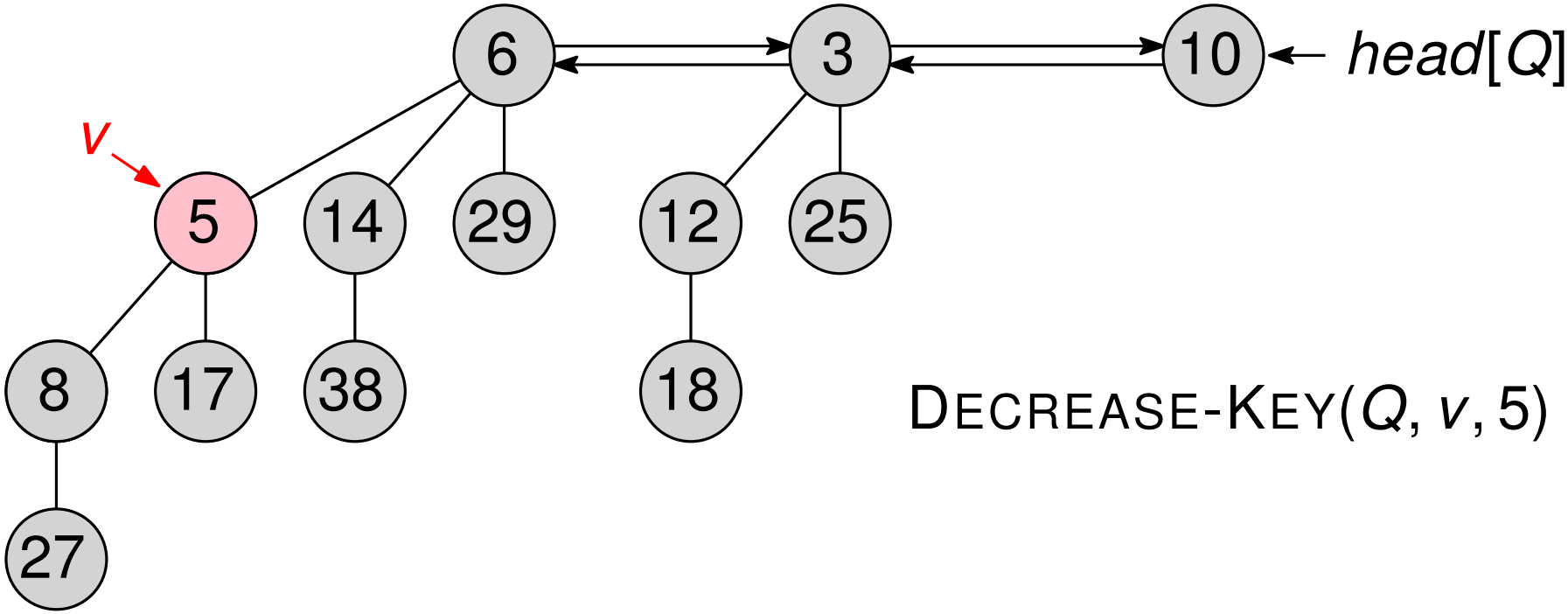

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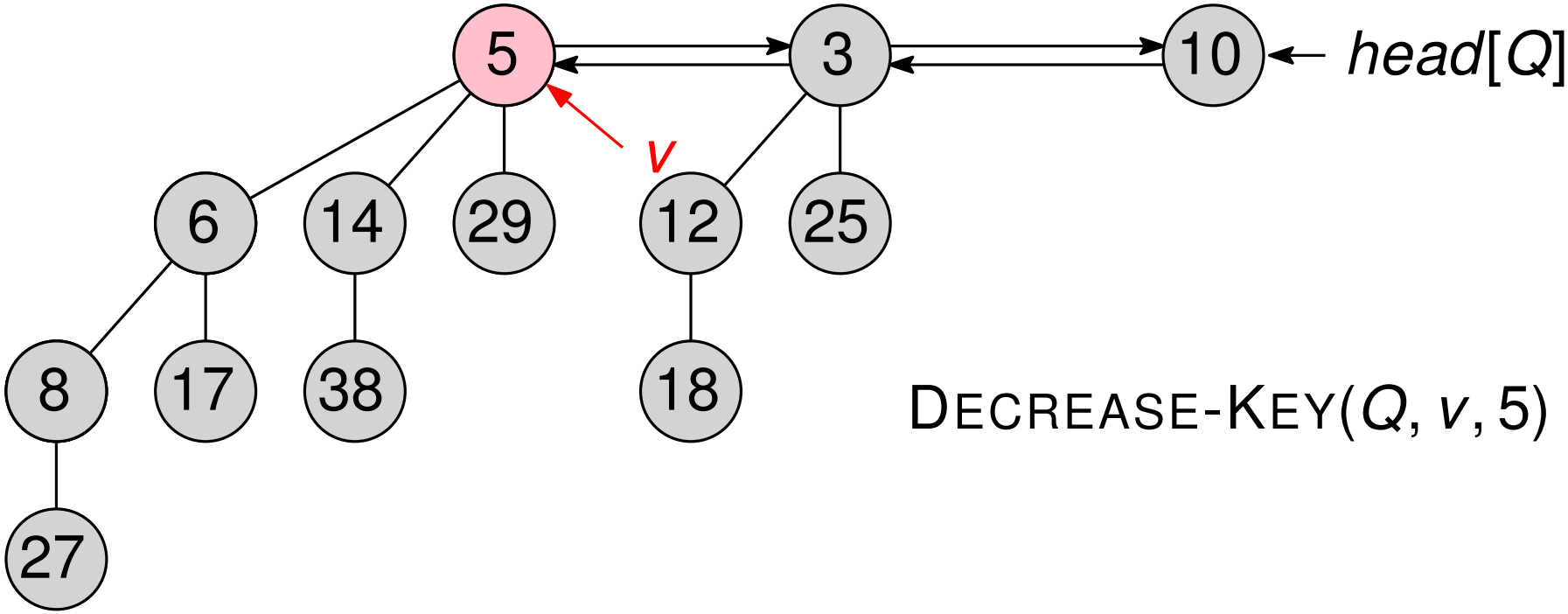

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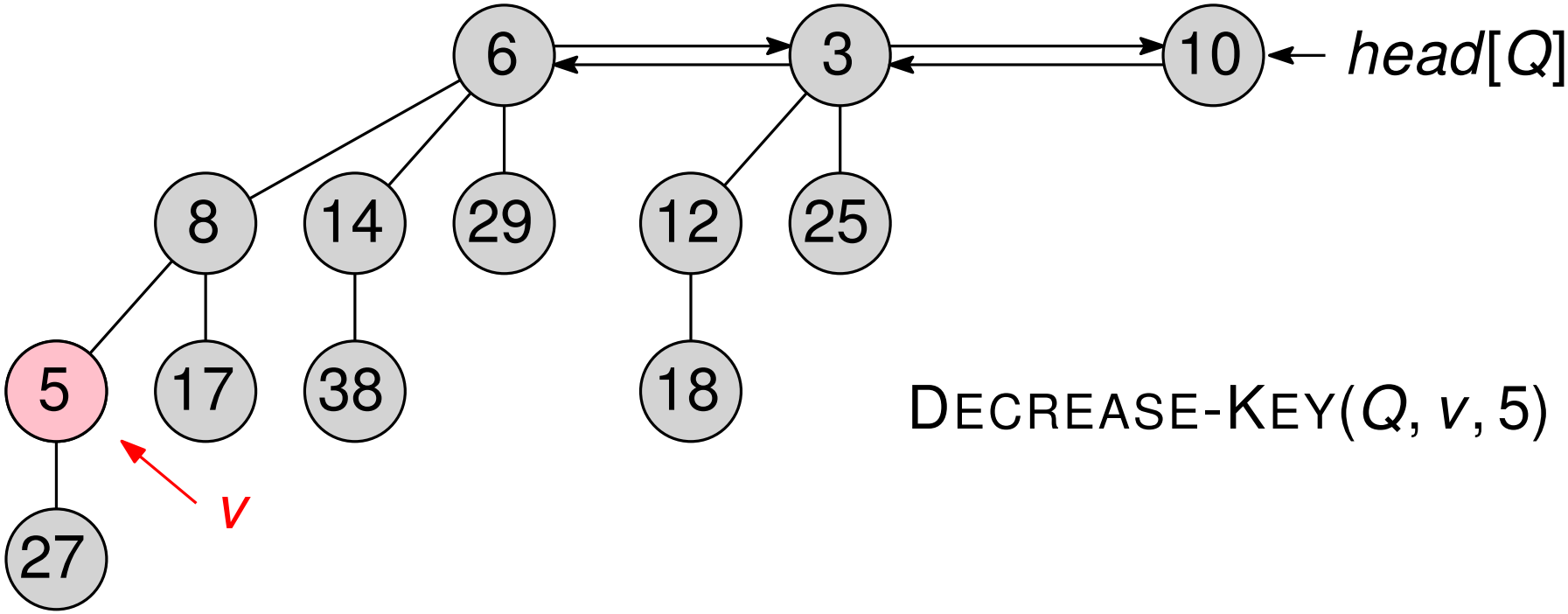

Idea: If the new key is smaller than parent's, DECREASE-KEY(Q, v, k) will splice-out v and add it to the root list

Fibonacci Heaps

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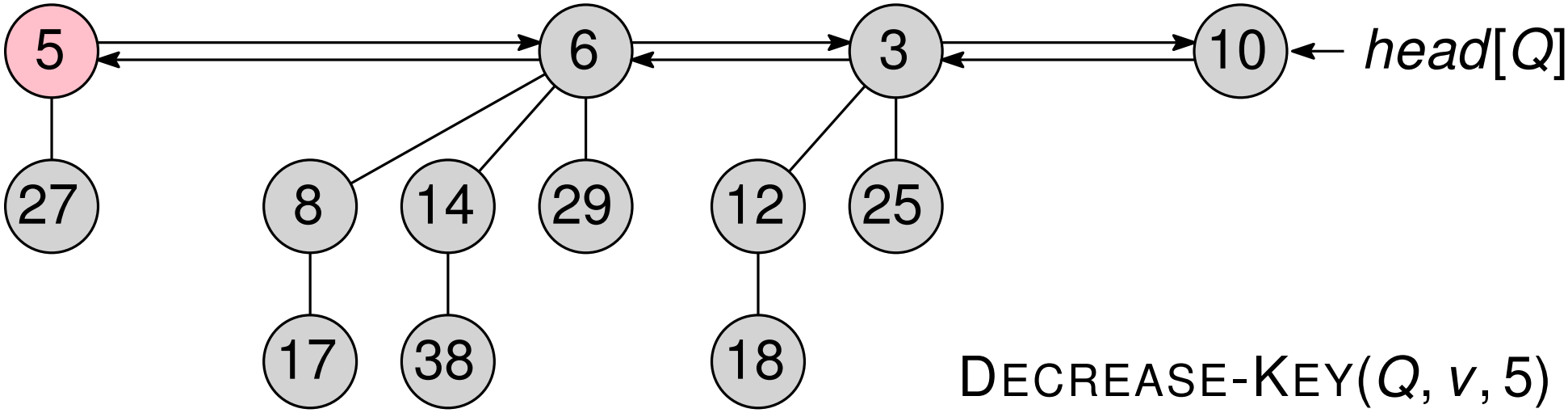

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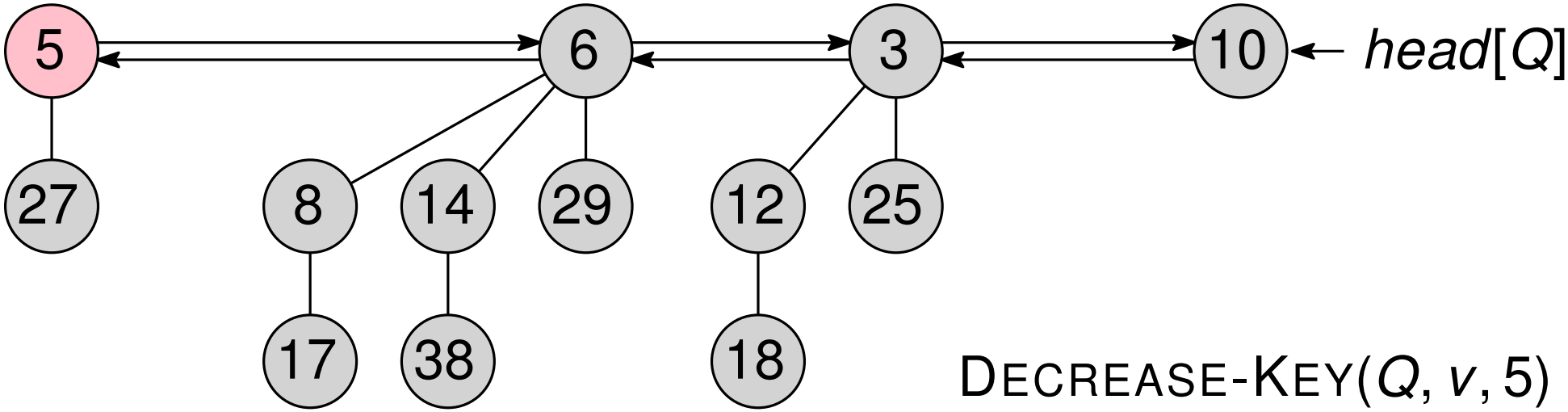

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Almost!

Fibonacci Heaps

Goal: DECREASE-KEY(Q, v, k) in $O(1)$ (amortized) time

Fibonacci Heaps

Goal: DECREASE-KEY(Q, v, k) in $O(1)$ (amortized) time

Collection of heap-ordered (binomial) trees:

- Each tree is heap-ordered
- Arbitrary number of trees in the root list
- Sibling lists are doubly-linked circular lists
- Each node v stores
  - A pointer to parent ($v.parent$)
  - A pointer to one (arbitrary) child ($v.child$)

Fibonacci Heaps

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- Each node v stores
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Each node v:

- stores number of children ($v.degree$)
- can be "marked" (`boolean` $v.marked$)
  - If v lost a child since becoming a child of another node
  - If so, next time it loses another child, it will be spliced out and added to root list (and will become unmarked)

FIB-DECREASE-KEY(Q, x, k)

function FIB-DECREASE-KEY(Q, v, k) ▷ Assert $k < v.key$
    $v.key = k$
    if $v.parent \neq$ NIL **and** $v.key < v.parent.key$ **then**
        RECURSIVE-CUT(Q, v, $v.parent$)

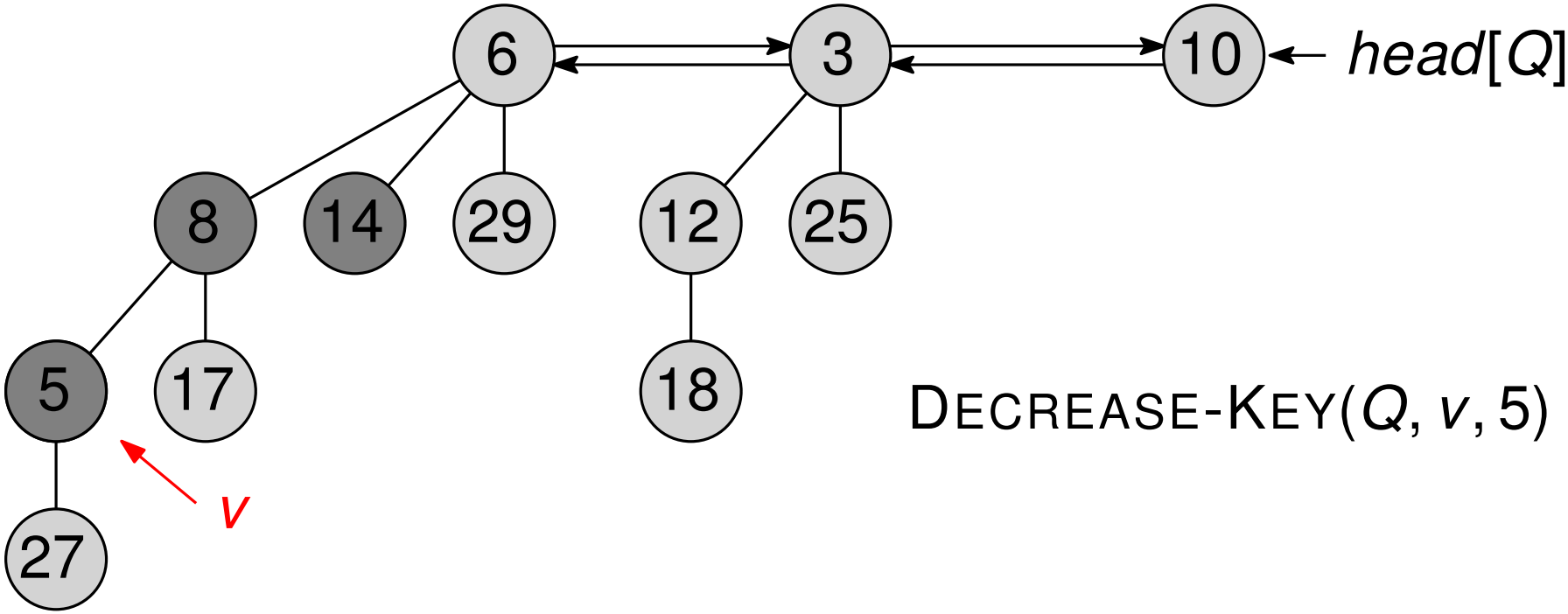
FIB-DECREASE-KEY(Q, x, k)

```
function FIB-DECREASE-KEY(Q, v, k)                    ▷ Assert k < v.key
    v.key = k
    if v.parent ≠ NIL and v.key < v.parent.key then
        RECURSIVE-CUT(Q, v, v.parent)

function RECURSIVE-CUT(Q, v, p)                       ▷ p is the parent of v
    Remove v from child list of p
    Add v to the root list of Q
    v.mark = FALSE; v.parent = NIL                    ▷ Unmark v

    if p.parent ≠ NIL then
        if p.mark == FALSE then
            p.mark = TRUE                             ▷ p just lost a child, so mark it
        else                                          ▷ p just lost the second child
            RECURSIVE-CUT(Q, p, p.parent)             ▷ so add it to the root list too
```

Example: FIB-DECREASE-KEY(Q, v, k)



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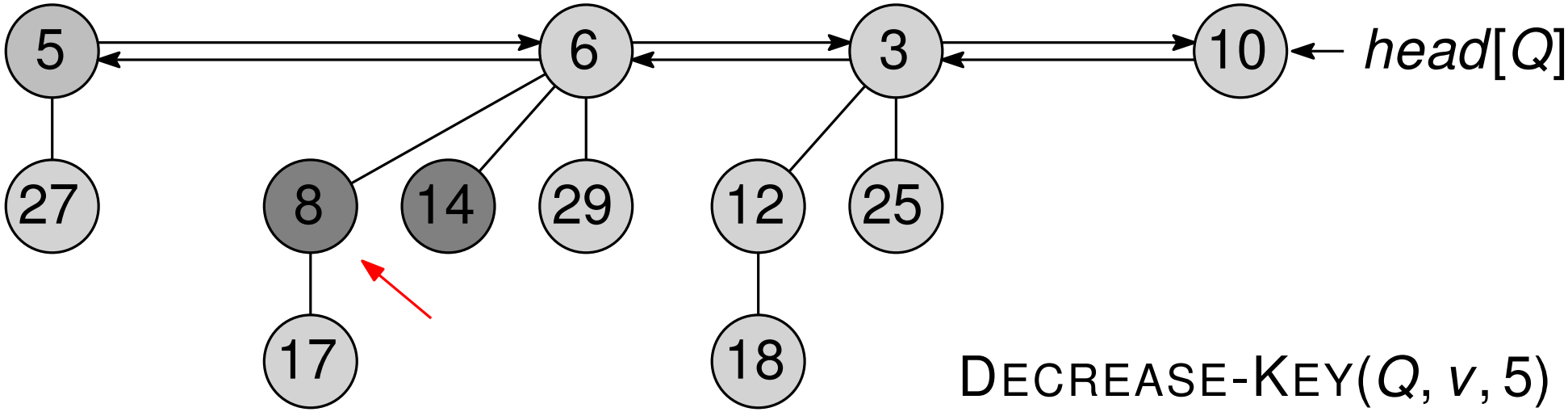

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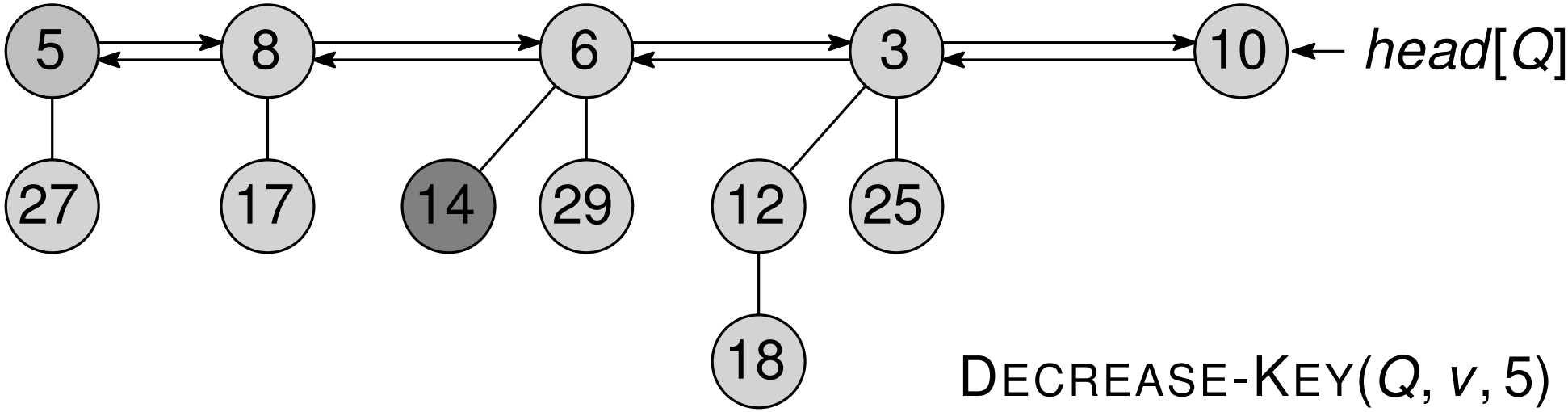

Unmarking Vertices

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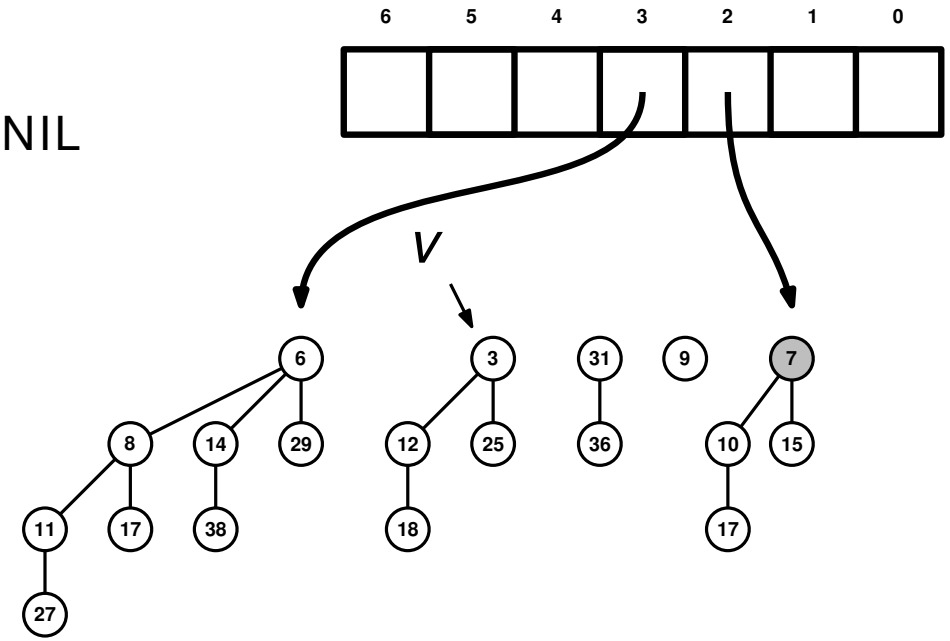

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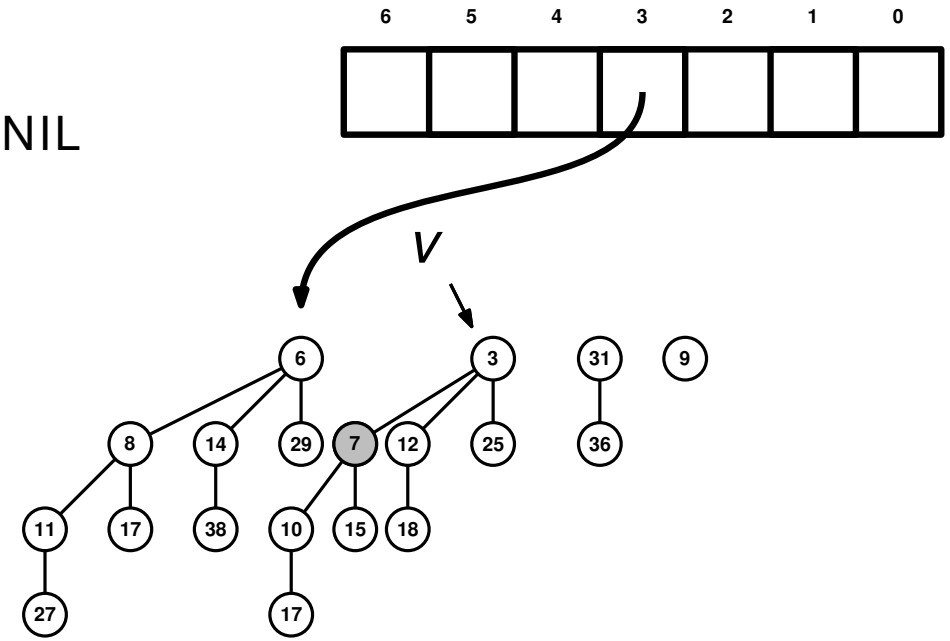

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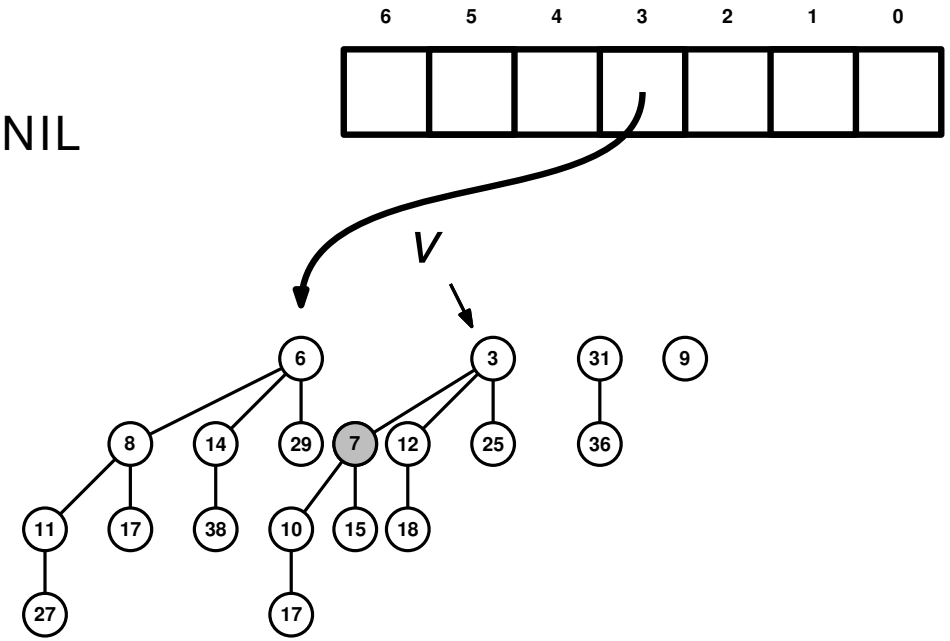

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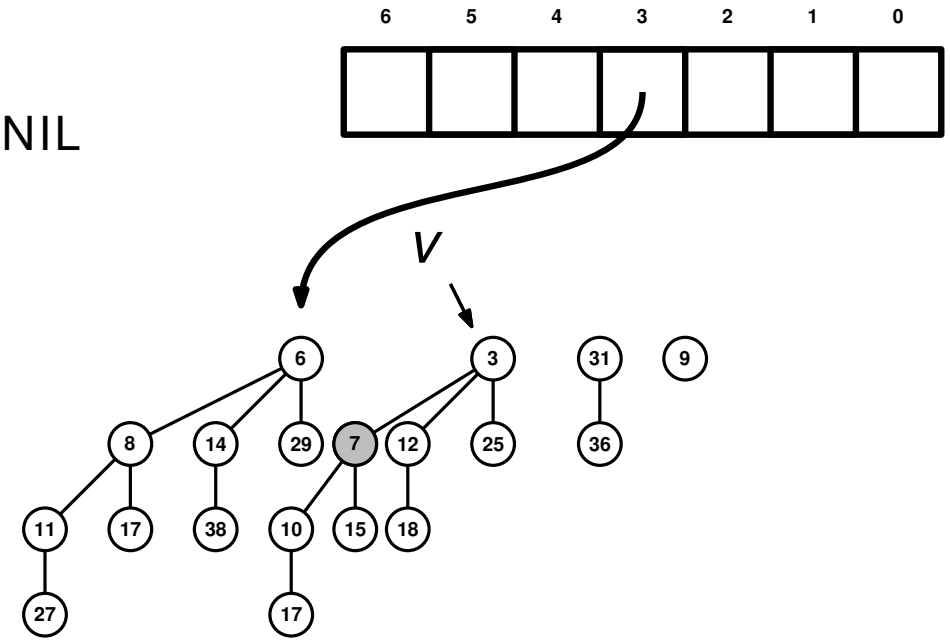


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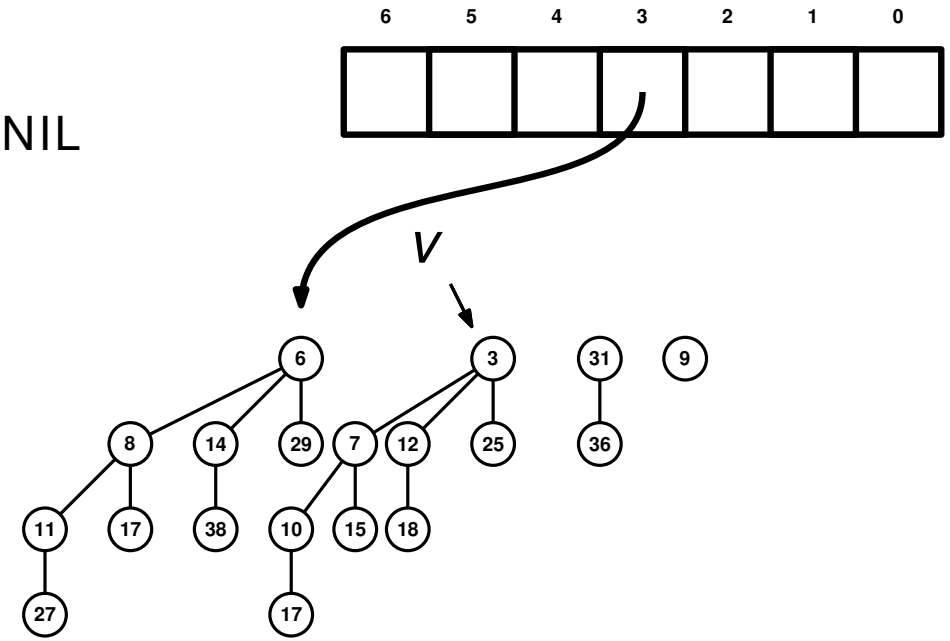


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FIB-DECREASE-KEY(Q, v, k) Analysis

Potential: $\Phi_i = k(t_i + 2m_i)$

- t_i = # of root list trees
- m_i = # of marked nodes
- k = constant TBD later

```
function RECURSIVE-CUT(Q, v, p)
    Remove v from child list of p
    Add v to the root list of Q
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    if p.parent ≠ NIL then
        if p.mark == FALSE then
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            RECURSIVE-CUT(Q, p, p.parent)
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$$\Delta\Phi = k(t_i - t_{i-1}) + 2k(m_i - m_{i-1}) \leq k(t' + 2 \cdot (2 - t')) = k(4 - t')$$

Actual cost: $c_i = O(t')$, i.e., $c_i \leq \bar{k} \cdot t'$ for some constant $\bar{k}$

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```

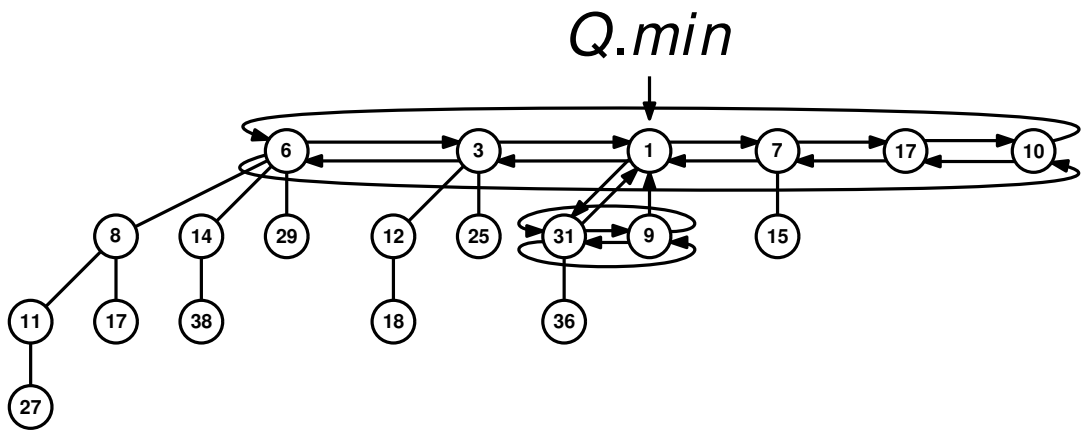
$$\Delta\Phi = k(t_i - t_{i-1}) + 2k(m_i - m_{i-1}) \leq k(t' + 2 \cdot (2 - t')) = k(4 - t')$$

Actual cost: $c_i = O(t')$, i.e., $c_i \leq \bar{k} \cdot t'$ for some constant $\bar{k}$

By setting $k = \bar{k}$, we get:

$\hat{c}_i = c_i + \Delta\Phi \leq \bar{k} \cdot t' + \bar{k}(4 - t') = 4\bar{k} = O(1)$

EXTRACT-MIN(Q)



EXTRACT-MIN(Q)

function EXTRACT-MIN(Q)

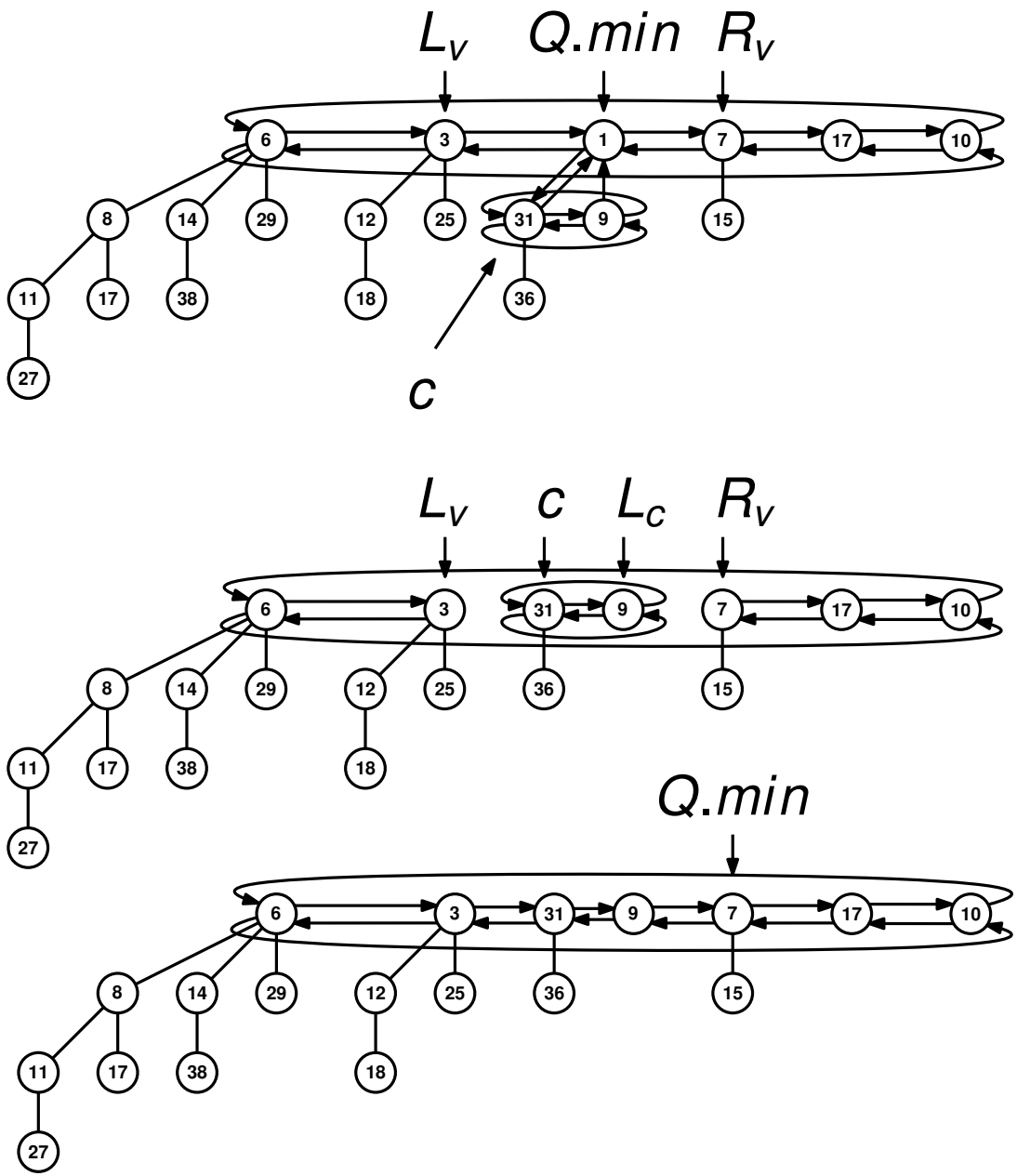
v = MINIMUM(Q)

Extract v from root list

Add v's children into root list

CONSOLIDATE(Q)

return v



CONSOLIDATE(Q)

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function CONSOLIDATE(Q)
    Initialize O(log n)-sized array A to NIL
    for each v in root list do
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        while A[d] ≠ NIL do
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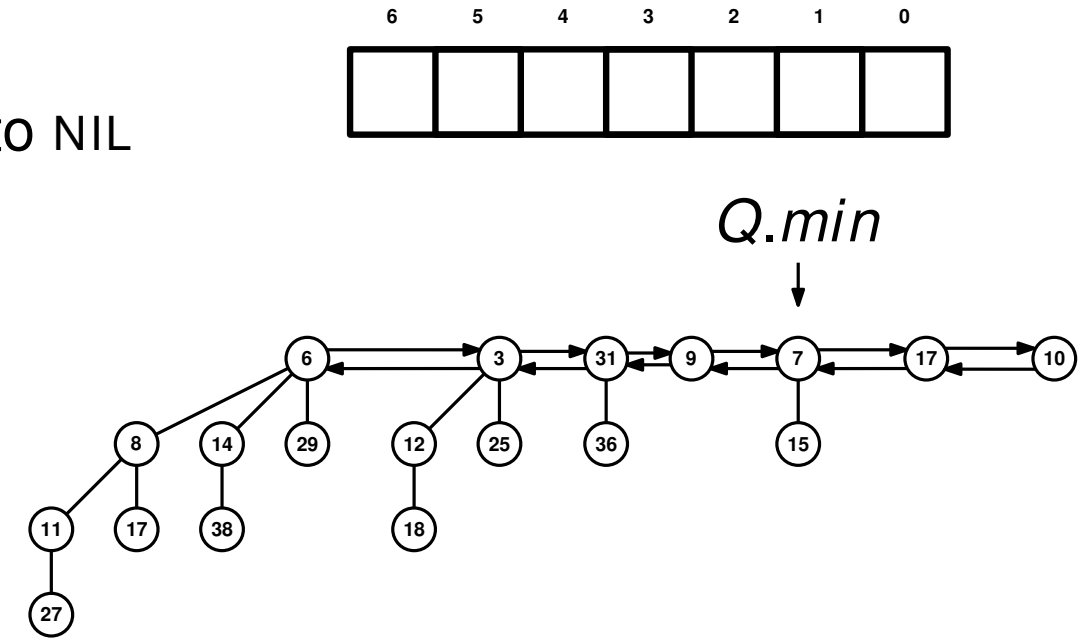

CONSOLIDATE(Q)

function CONSOLIDATE(Q)
    Initialize $O(\log n)$-sized array A to NIL
    for each v in root list **do**
        $d = v.degree$
        while $A[d] \neq$ NIL **do**
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            $A[d] =$ NIL
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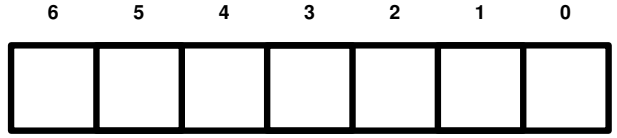

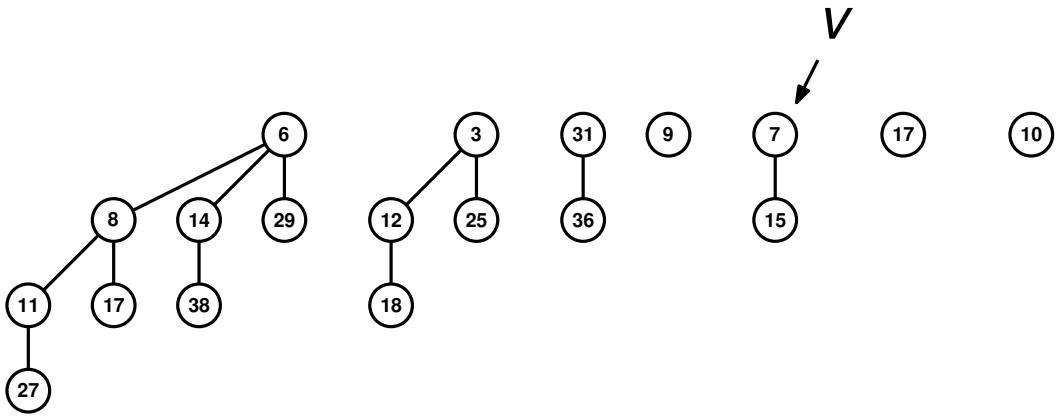

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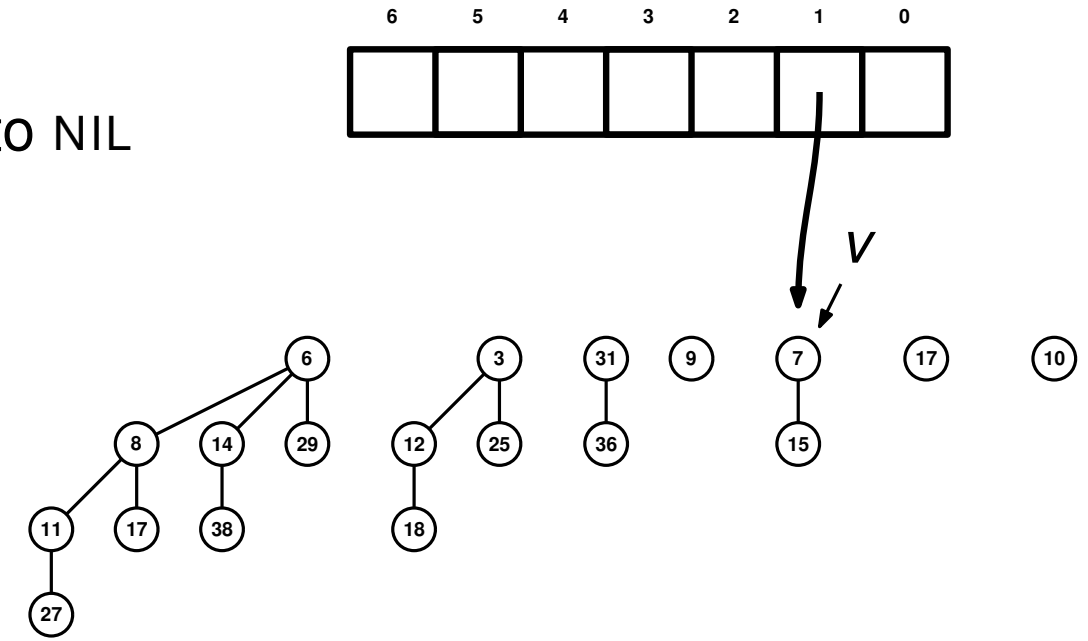

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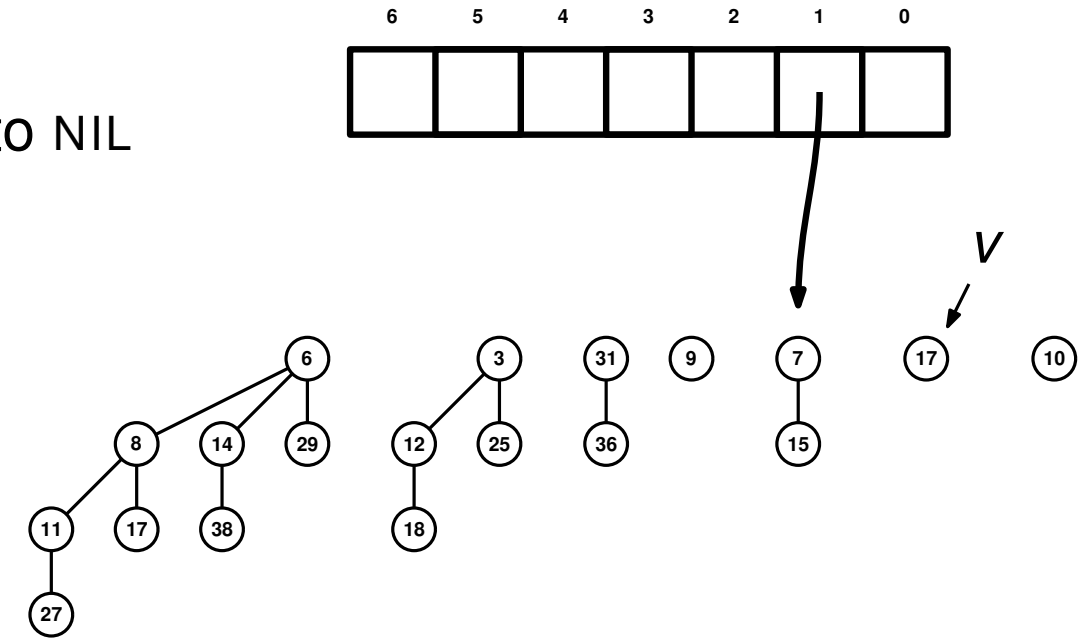

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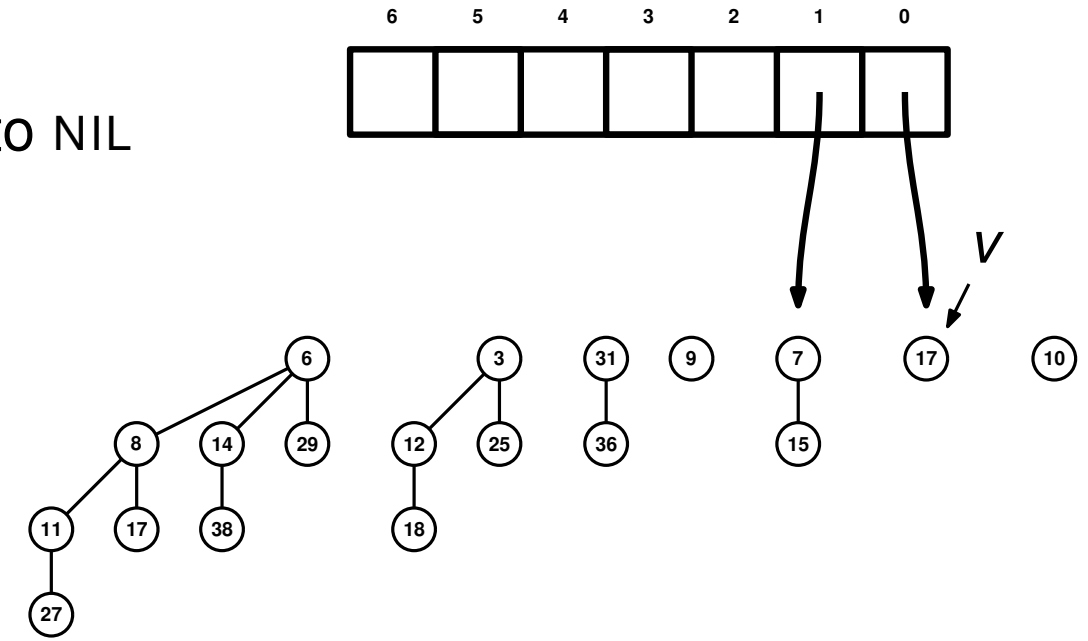

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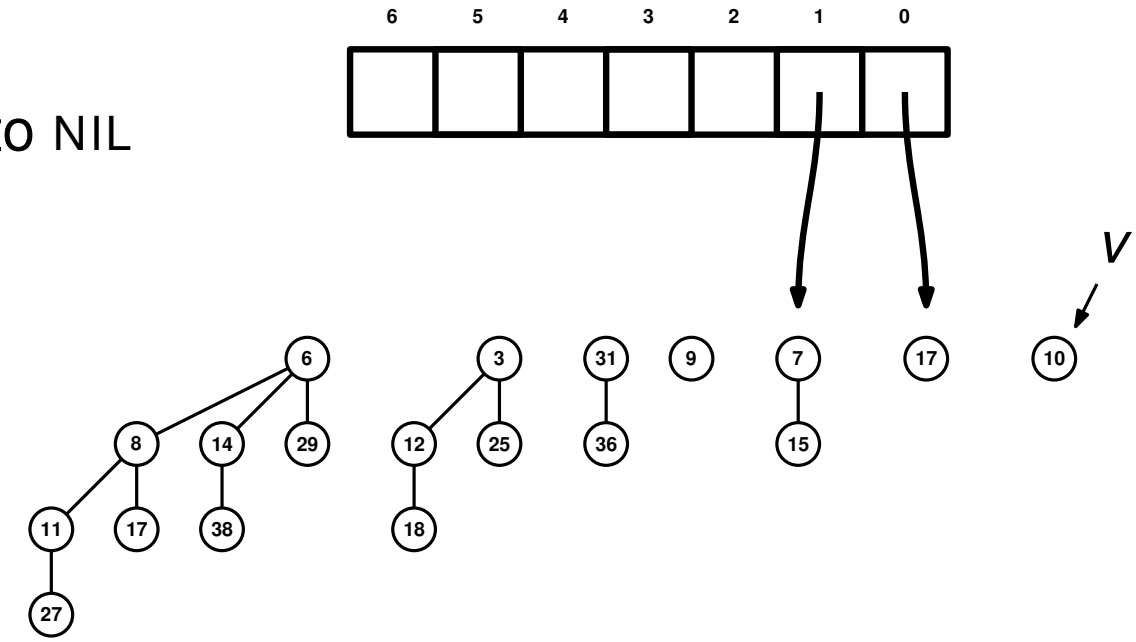

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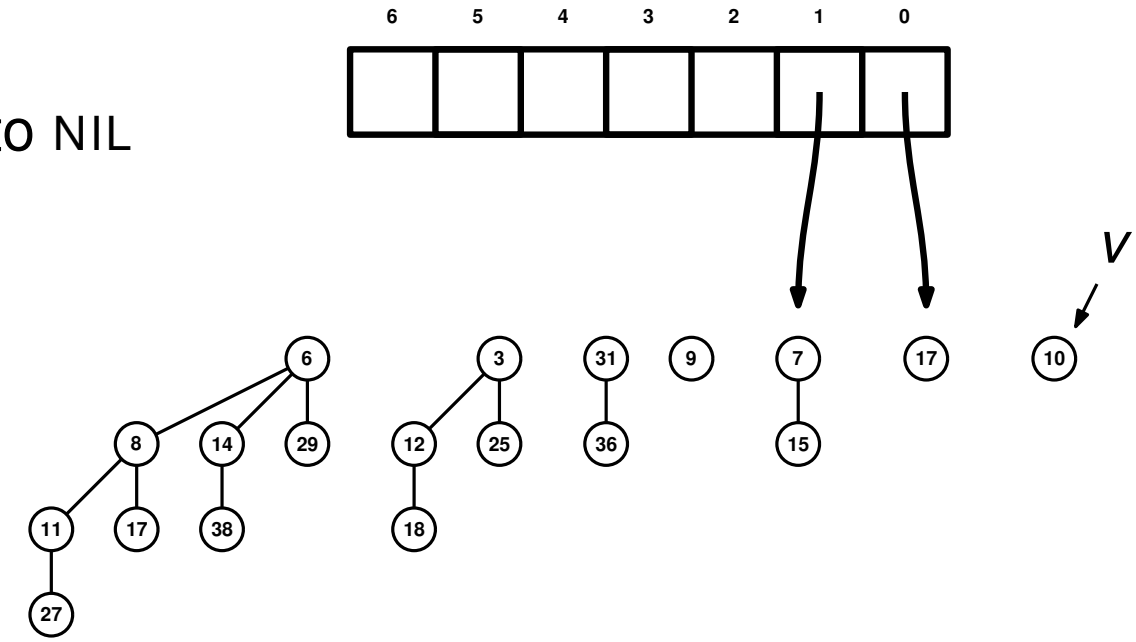

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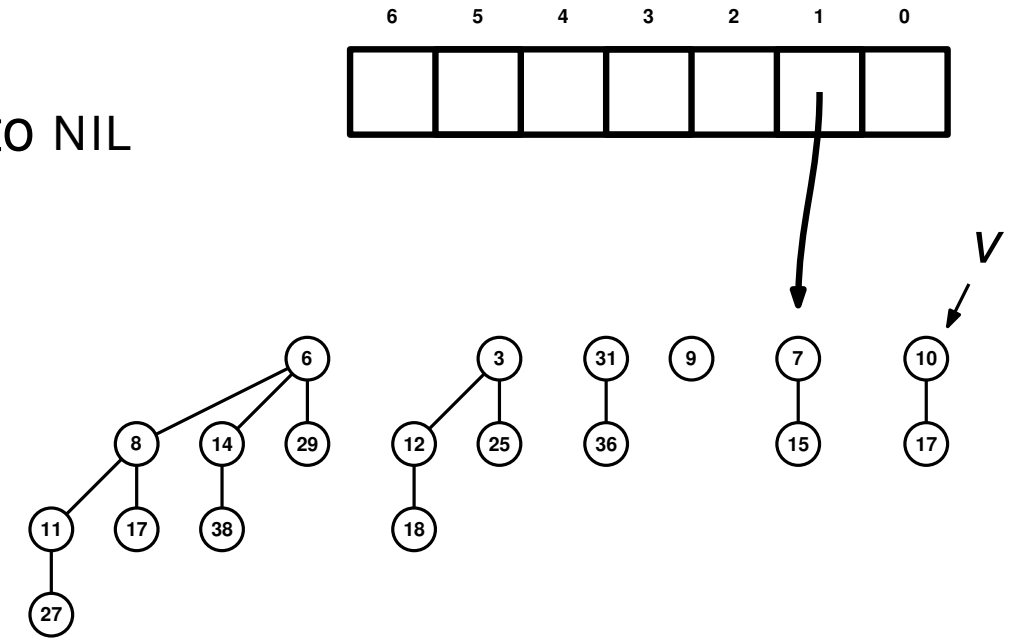

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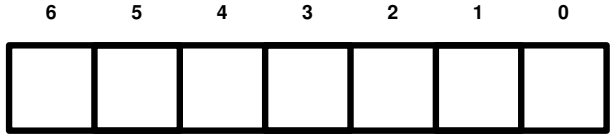

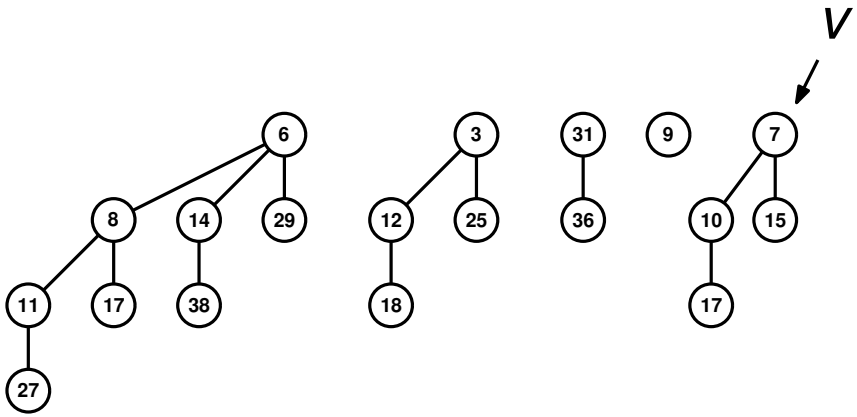

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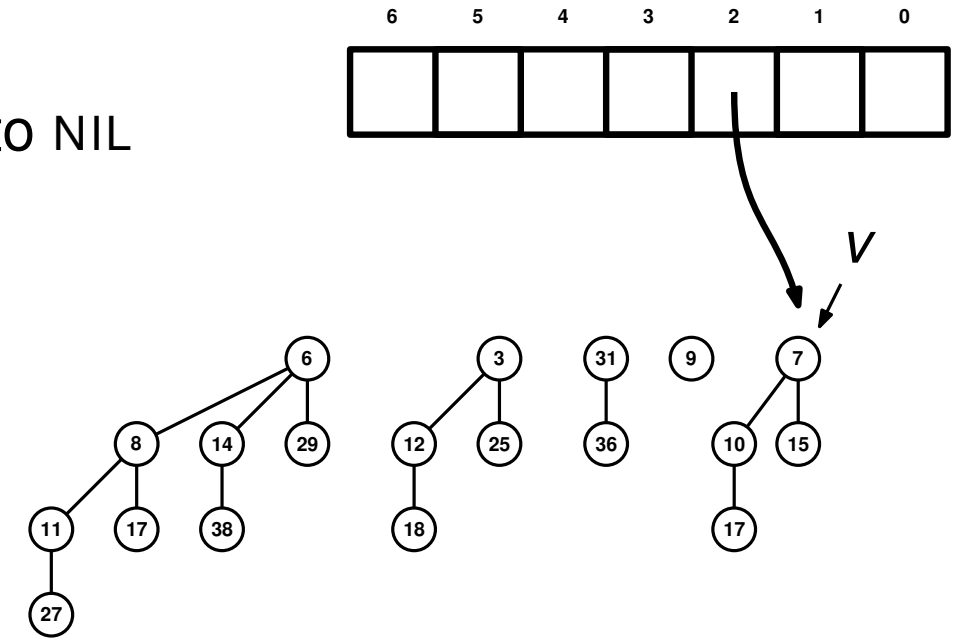

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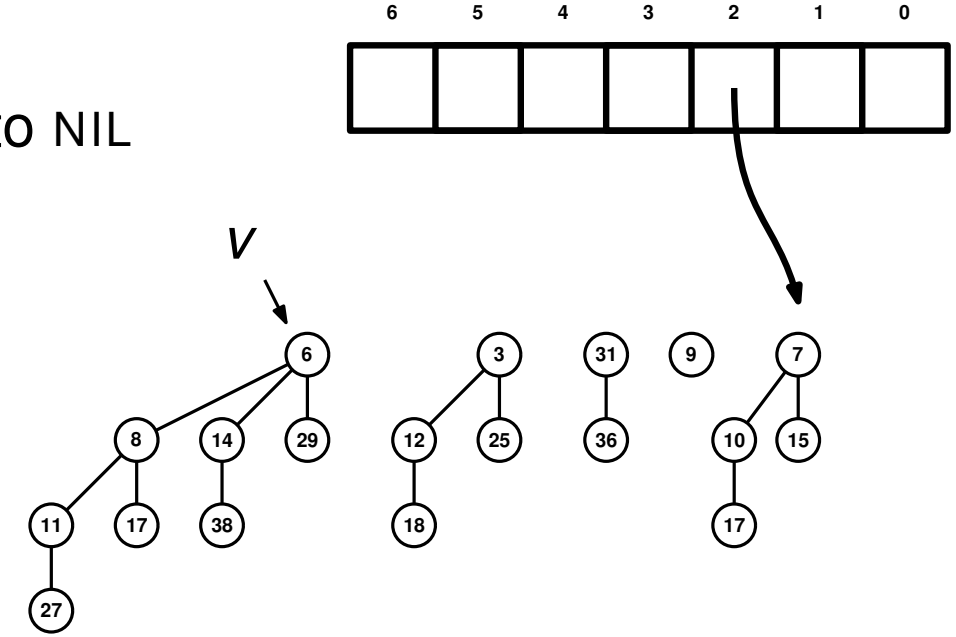

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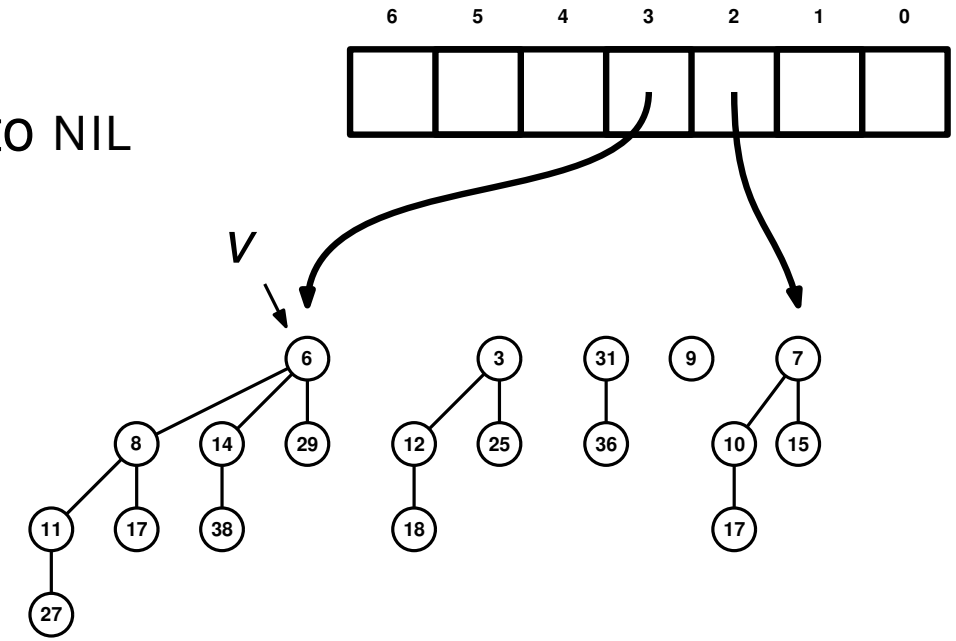

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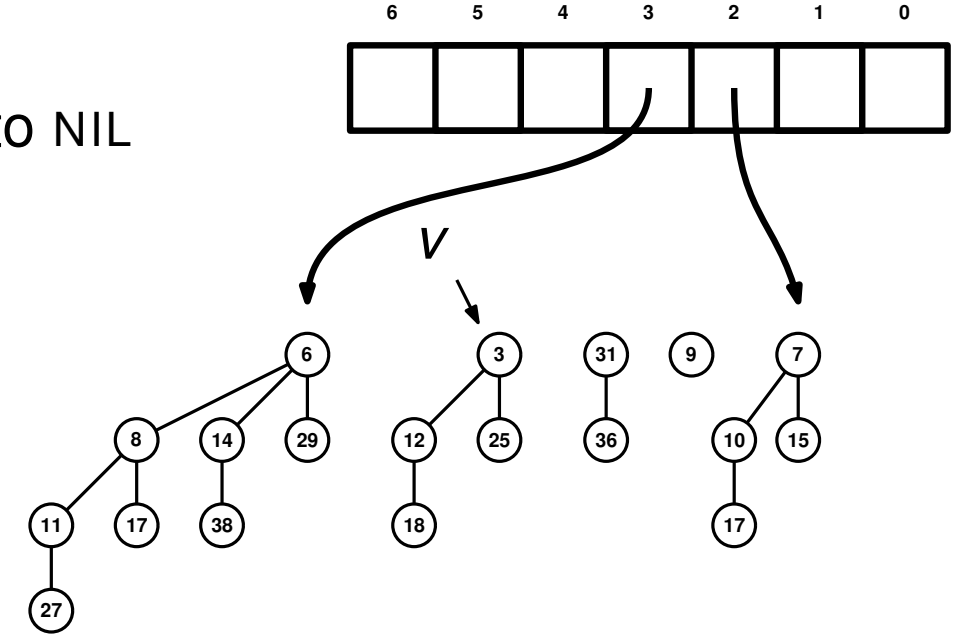

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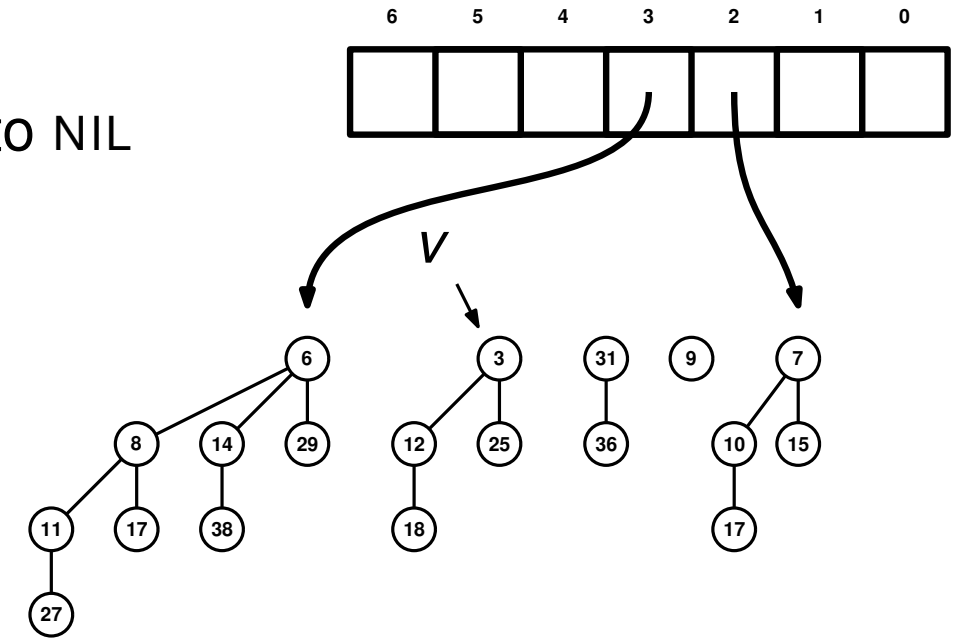

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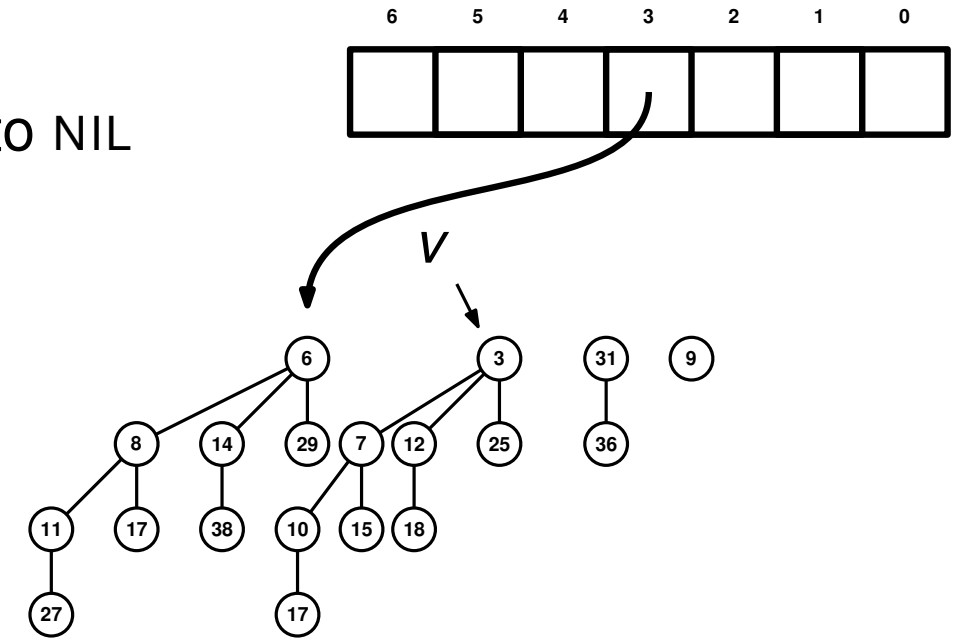

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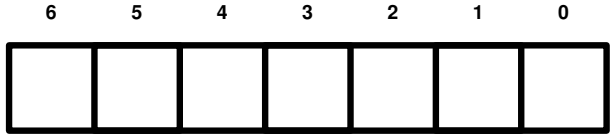

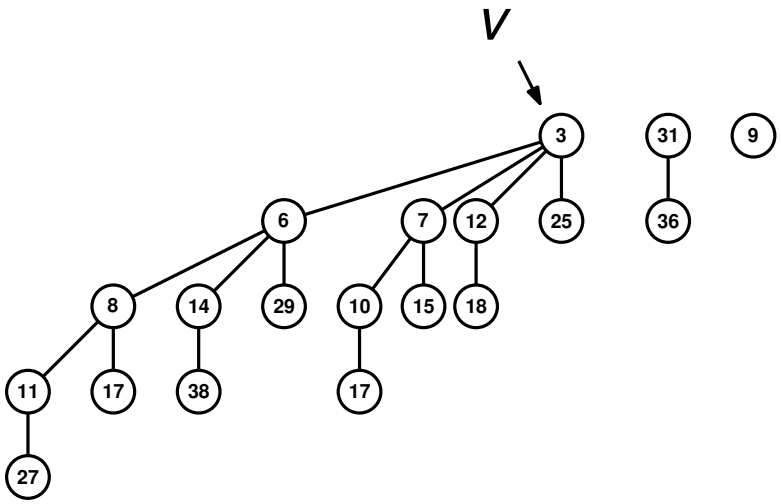

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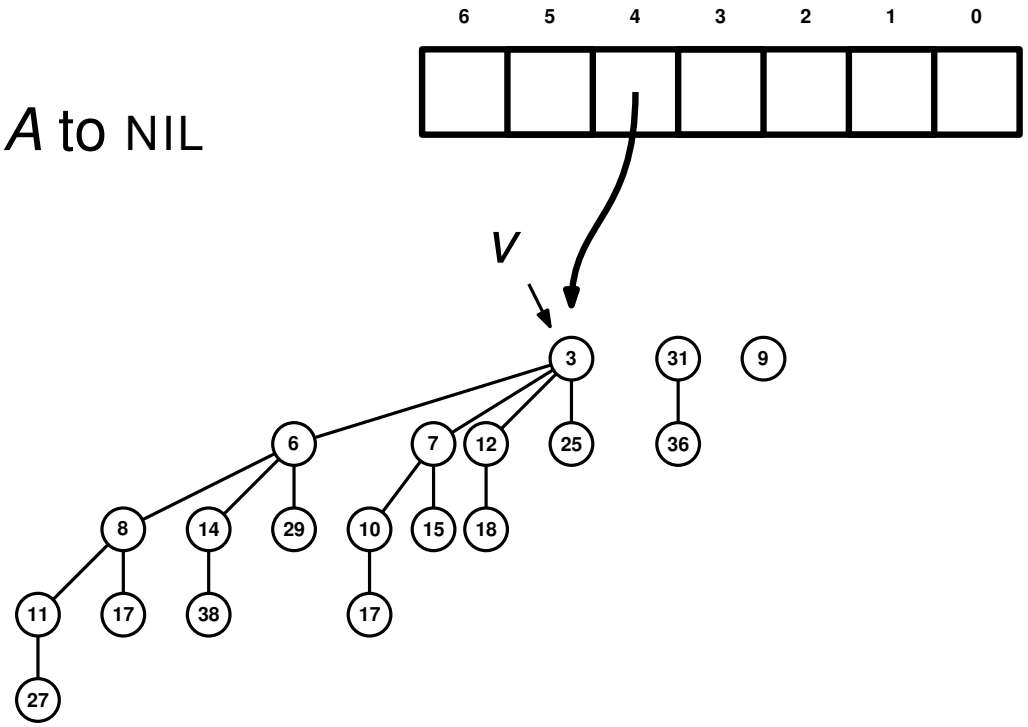

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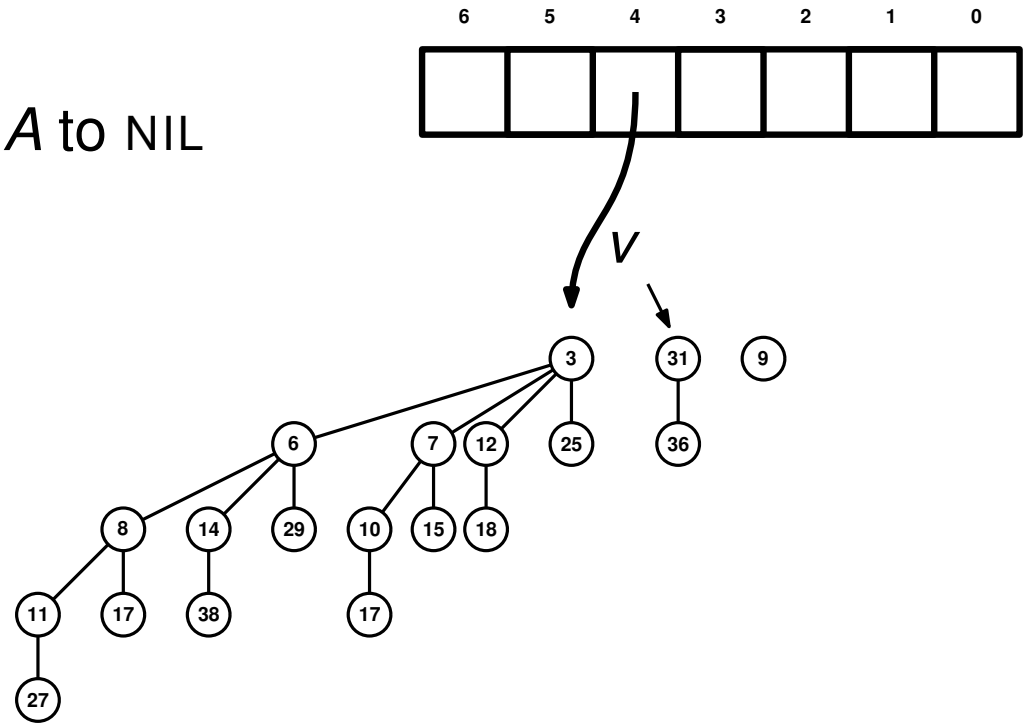

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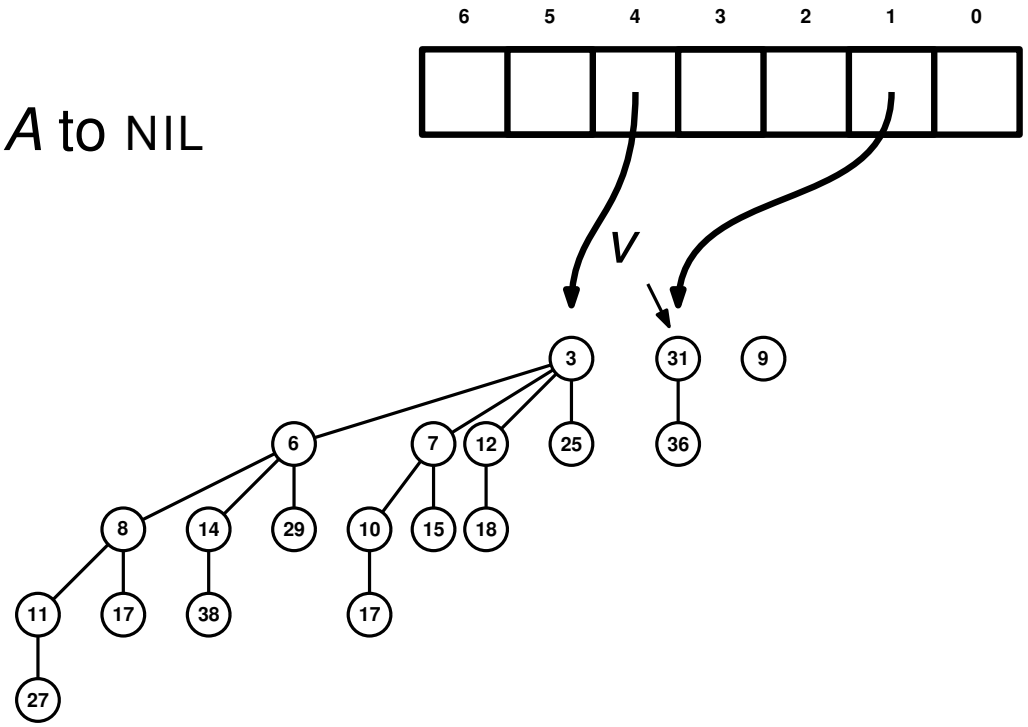

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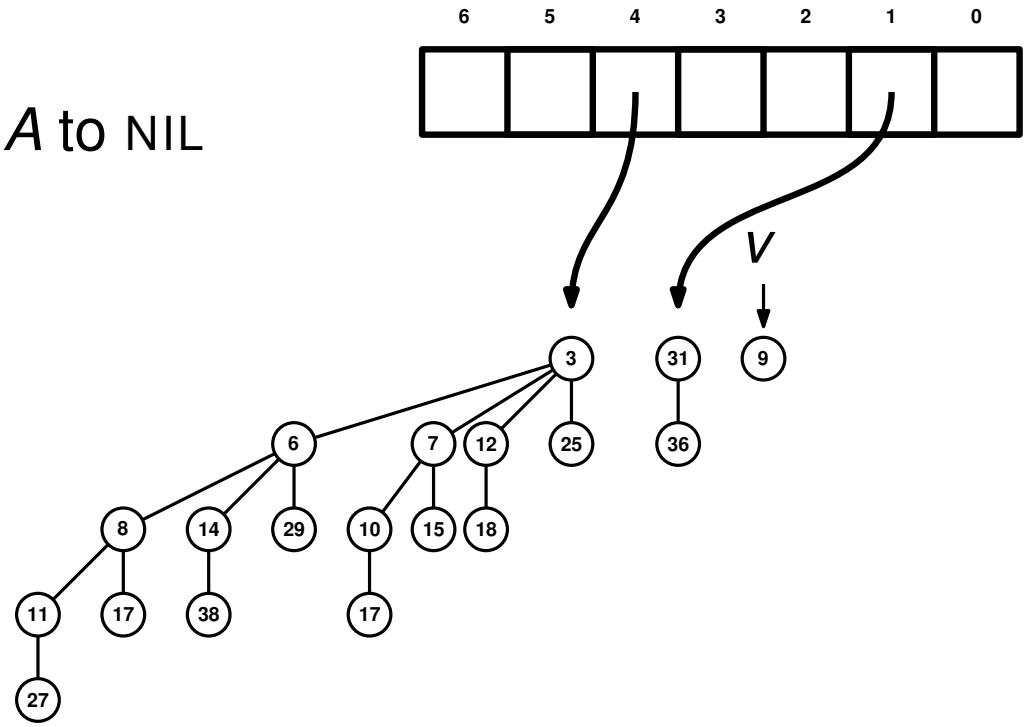

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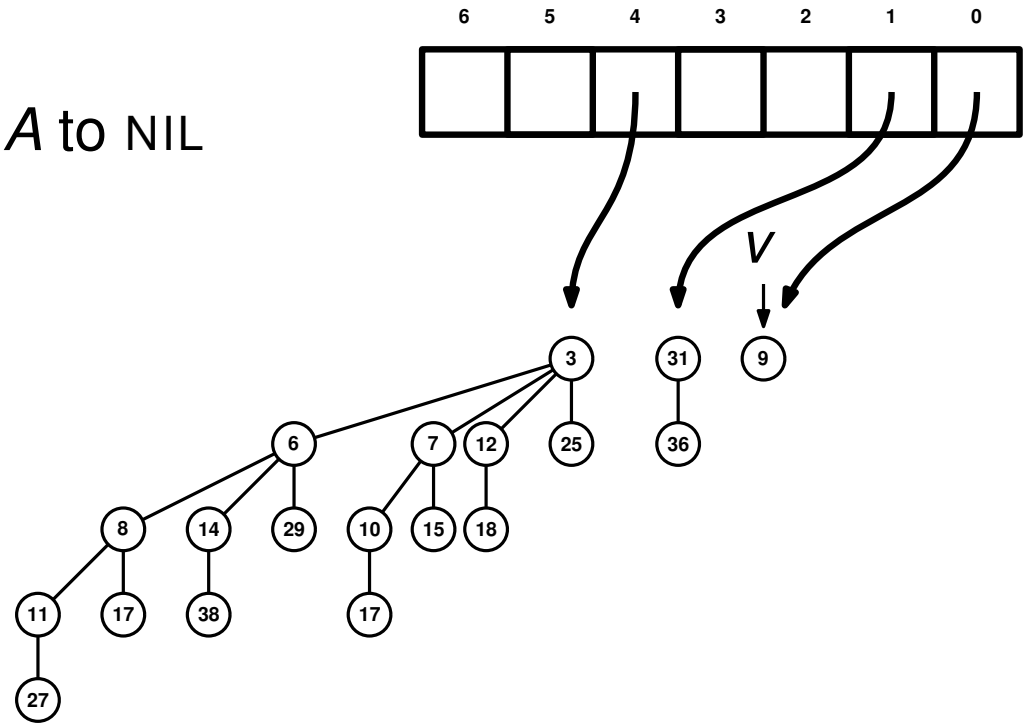

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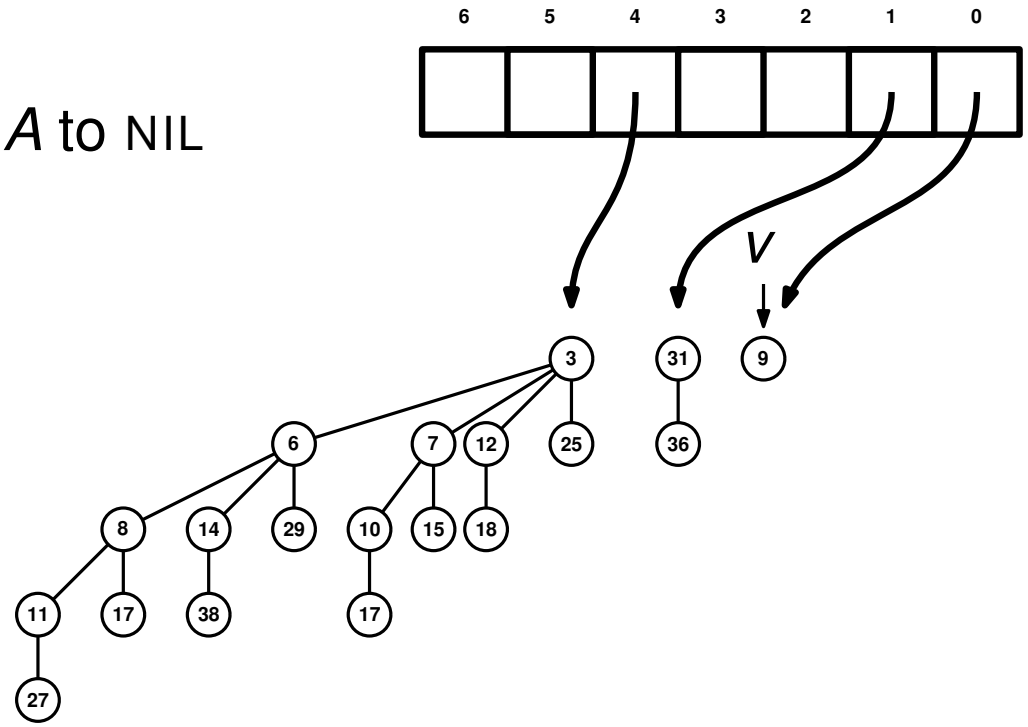

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        $A[d] = v$; $v.parent =$ NIL
    $min = +\infty$
    for $i = 0$ **to** $|A| - 1$ **do**
        if $A[i] \neq$ NIL **then**
            Add $A[i]$ to the root list
            if $A[i].key < min$ **then**
                $Q.min \leftarrow A[i]$; $min = A[i].key$

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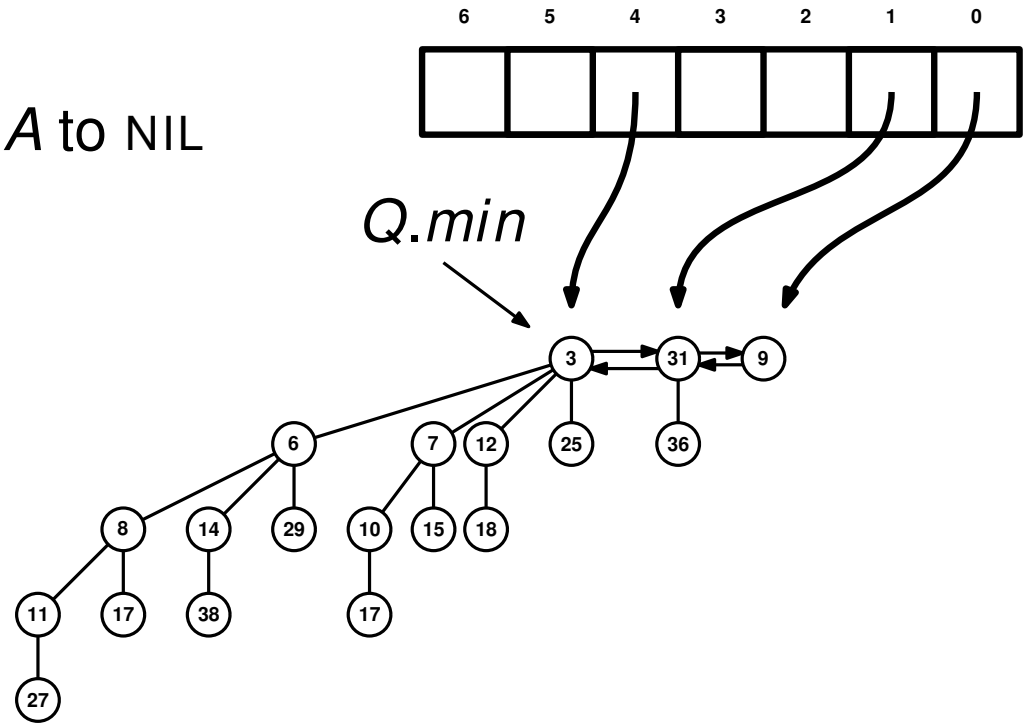

Bounding maximum degree

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Lemma 1. *Let $k = v.degree$. Then for every node v in Fibonacci heap,* $\text{SIZE}(v) \geq F_{k+2} \geq \phi^k$, *where* $\phi = \frac{1+\sqrt{5}}{2}$

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Corollary 1. $k \leq \log_\phi n$

Analysis of EXTRACT-MIN(Q)

Potential: $\Phi_i = \bar{k}(t_i + 2m_i)$

- t_i = # of root list trees
- m_i = # of marked nodes
- $\bar{k}$ is a constant

Let d be the number of children of the $Q.min$ ($d \leq \log_\Phi n$)

because there are $t_i \leq \log_\Phi n$ trees in the root list after consolidation

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- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log_\phi n \leq O(1) + t_{i-1} + 2\log_\phi n$

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$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2\log_\phi n + \bar{k} \cdot t_i - (\bar{k} - 1) \cdot t_{i-1}$$
$$\leq O(1) + 2\log_\phi n + \bar{k} \cdot t_i$$

for any $\bar{k} \geq 1$

because there are $t_i \leq \log_\phi n$ trees in the root list after consolidation

Analysis of EXTRACT-MIN(Q)

Potential: $\Phi_i = \bar{k}(t_i + 2m_i)$

- t_i = # of root list trees
- m_i = # of marked nodes
- $\bar{k}$ is a constant

Let d be the number of children of the $Q.min$ ($d \leq \log_\phi n$)

$$\Delta\Phi_i = \bar{k}(t_i - t_{i-1}) + 2\bar{k}(m_i - m_{i-1}) \leq \bar{k}(t_i - t_{i-1})$$

because $m_i \leq m_{i-1}$

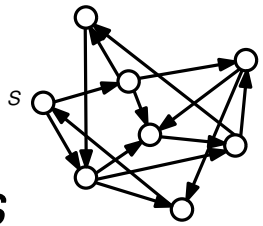
- Actual cost $c_i \leq O(1) + (t_{i-1} + d) + \log_\phi n \leq O(1) + t_{i-1} + 2\log_\phi n$

$$\hat{c}_i = c_i + \Delta\Phi_i \leq O(1) + 2\log_\phi n + \bar{k} \cdot t_i - (\bar{k} - 1) \cdot t_{i-1}$$
$$\leq O(1) + 2\log_\phi n + \bar{k} \cdot t_i \leq O(1) + 2\log_\phi n + \bar{k}\log_\phi n = O(\log n)$$

for any $\bar{k} \geq 1$

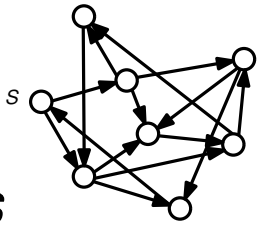
because there are $t_i \leq \log_\phi n$ trees in the root list after consolidation

Application: Single Source Shortest Paths



- Input: Graph G with n vertices, m edges with weights, vertex s
- Output: minimum-weight distance from s to every other vertex of G

Application: Single Source Shortest Paths



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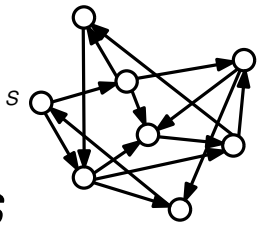
Dijkstra's Algorithm

Use prioirity queue using

- n calls to INSERT
- n calls to EXTRACT-MIN
- m calls to DECREASE-KEY

Application: Single Source Shortest Paths

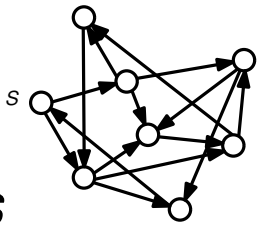


- Input: Graph G with n vertices, m edges with weights, vertex s
- Output: minimum-weight distance from s to every other vertex of G

Dijkstra's Algorithm

Use prioirity queue using	Binary	Binomial	Fibonacci
n calls to INSERT	$O(1)$	$O(1)^*$	$O(1)$
n calls to EXTRACT-MIN	$O(\log n)$	$O(\log n)$	$O(\log n)^*$
m calls to DECREASE-KEY	$O(\log n)$	$O(\log n)$	$O(1)^*$

Application: Single Source Shortest Paths



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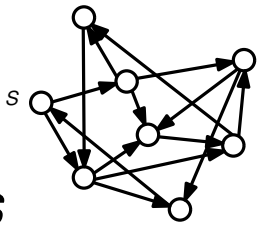
Binary Binomial Fibonacci

Application: Single Source Shortest Paths



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m calls to DECREASE-KEY	$O(\log n)$	$O(\log n)$	$O(1)^*$

Binary	Binomial	Fibonacci
$O(n + n\log n + m\log n)$ $= O(m\log n)$	$O(n + n\log n + m\log n)$ $= O(m\log n)$	$O(n + m + n\log n)$ $= O(m + n\log n)$