INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

AlgoPARC

ICS 621: Analysis of Algorithms

Prof. Nodari Sitchinava

Dynamic Programming

Recursive Backtracking,
Pruned with a Lookup (Memo) Table

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(Faster if there are overlapping subproblems)

Example: Subset Sum

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$$S = \{17, 5, 7, 15, 3, 8\} \qquad x = 16$$

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```
for (i = 0; i < 2^n; i++) {        // i-th subset out of 2^n
  sum = 0;
  for (int j = 0; j < n; j++)
    if (i & (1 << j))              // is j-th bit set in i?
      sum += S[j]                  // j is part of the subset
  if (sum == x) return true;       // or could return i
}
return false;
```

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0 1 0 0 1 1 ← `i = 19`

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Subset Sum: Recursive Solution

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Top-down Solution: Prune search space as you generate partial solutions

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$$\begin{matrix} 0 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

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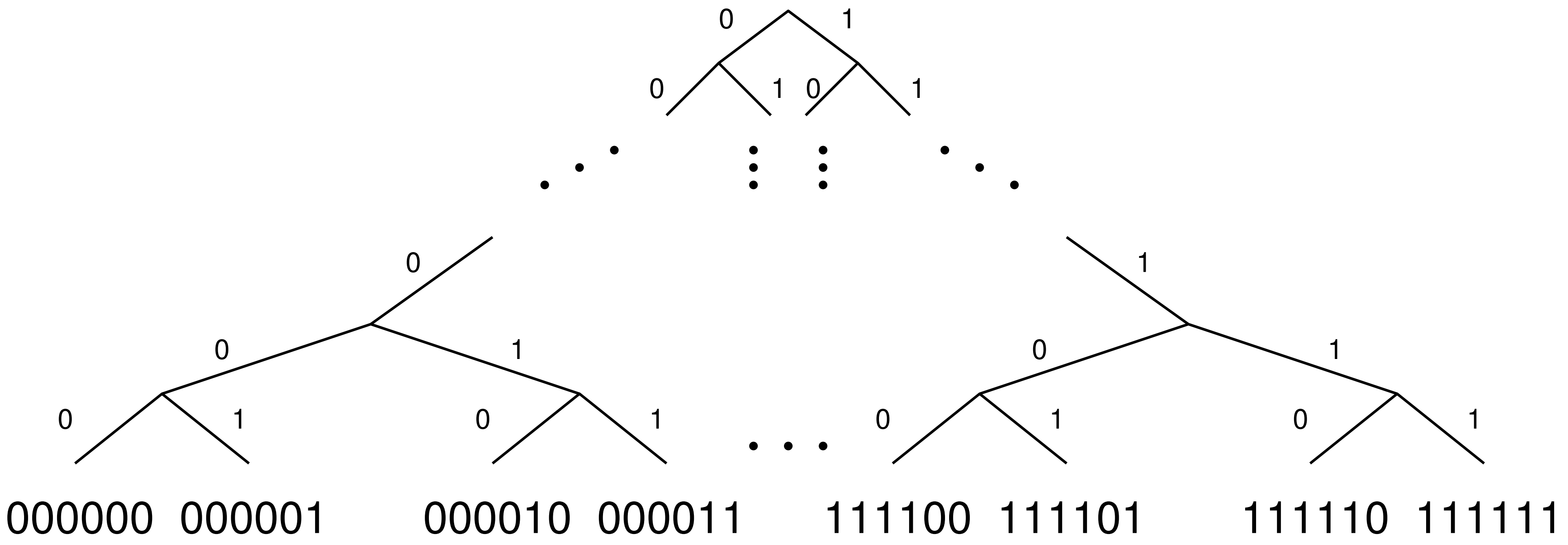

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int S[20] = {...}; int x = ...;     // initialize S & x

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SubsetSum(S[i:n], x):



  else
    bool Si_notSelected = SubsetSum(S[i+1:n], x);
    bool Si_selected = SubsetSum(S[i+1:n], x-S[i]);
    return (Si_notSelected or Si_selected);
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Subset Sum: Dynamic Programming

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int SS[20][MAX_X]; memset(SS, UNDEFINED, sizeof(SS));
// returns true iff exists subset within S[i:n]
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0-1 Knapsack

Problem (0-1 Knapsack)**.** *Given a set S of n items, each with its own value V_i and weight W_i for all $1 \leq i \leq n$ and a maximum knapsack capacity C, compute the maximum value of the items that you can carry. You cannot take fractions of items.*

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- Optimization version of Subset Sum

Example:

$\{(V_i, W_i)\} = \{(10, 17), (5, 7), (3, 8), (9, 15)\} \quad C = 16$

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$$maxV(i, C) = \begin{cases} 0 & \text{if } i > n \\ 0 & \text{if } C \leq 0 \\ maxV(i+1, C) & \text{if } W_i > C \\ \max \left\{ \begin{array}{l} maxV(i+1, C) \\ V_i + maxV(i+1, C - W_i) \end{array} \right\} & \text{if } W_i \leq C \end{cases}$$

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maxV(i,C) {
  if (i > n || C <= 0) return 0;

  if (W[i] > C)                          //can't take i-th item
    return maxV(i+1, C);
  return max(maxV(i+1, C),               //don't take i-th item
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```
maxV(i,C) {
  if (i > n || C <= 0) return 0;
  if (M[i][C] != UNDEFINED) return M[i, C];
  if (W[i] > C)                        //can't take i-th item
    return M[i][C]=maxV(i+1, C);
  return M[i][C]=max(maxV(i+1, C),    //don't take i-th item
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int M[maxN+1][maxC];
maxV(i,C) {
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}
main() { memset(M, UNDEFINED, sizeof(M));
  return maxV(1, C);
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Hint 1: LIS(i) returns the size of the longest increasing subsequence that ***terminates*** on $A[i]$.

Hint 2: When deciding whether to add $A[i]$, find the longest subsequence in $A[1 : i - 1]$ that allows adding $A[i]$.

LIS: Recursive Solution

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$A = \{-7, 10, 9, 2, 3, 8, 8, 1\}$

$$
LIS(i) = \begin{cases} 1 & \text{if } i = 1 \\ \max \begin{Bmatrix} 1 & \\ LIS(1) + 1 & \text{if } A[i] > A[1] \\ LIS(2) + 1 & \text{if } A[i] > A[2] \\ LIS(3) + 1 & \text{if } A[i] > A[3] \\ \ldots & \ldots \\ LIS(i-1) + 1 & \text{if } A[i] > A[i-1] \end{Bmatrix} & \text{if } i > 1 \end{cases}
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```
LIS(i)

  if (i == 1) return 1;
  best = 1;  // LIS = { A[i] }
  for (int j = 1; j < i; j++)
    if (A[i] > A[j]) {
      current = LIS(j) + 1;
      if (best < current)
        best = current;
    }
  return best;
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    }
  return best;
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```
main()
  best = 0;
  for (i = 1; i <= n; i++ ){
    current = LIS(i);
    if (best < current)
      best = current;
  }
  return best;
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LIS(i)
  if (L[i]!=UNDEFINED) return L[i];
  if (i == 1) return L[i] = 1;
  best = 1;  // LIS = { A[i] }
  for (int j = 1; j < i; j++)
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Bottom-up DP

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Iterative DP (no recursion)

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int SS[20][MAX_X]; memset(SS, UNDEFINED, sizeof(SS));
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SubsetSum(i, x):
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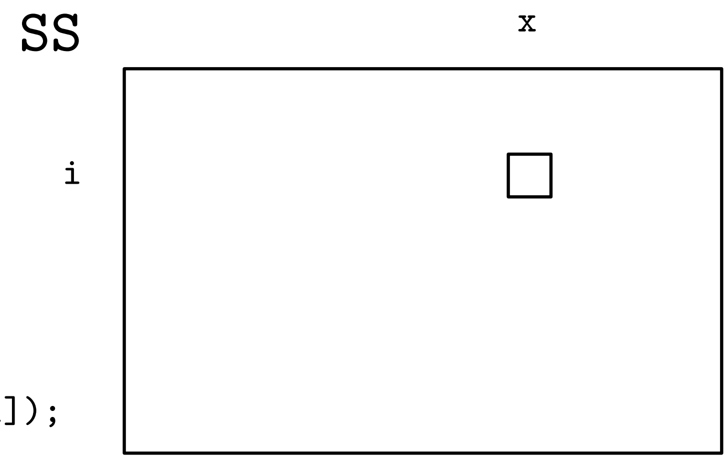

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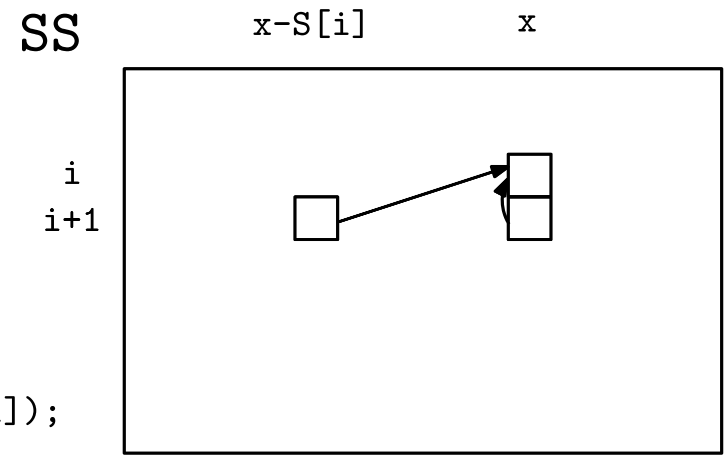

Example: Subset Sum

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int S[20] = {...}; int x = ...;     // initialize S & x
int SS[20][MAX_X]; memset(SS, UNDEFINED, sizeof(SS));
// returns true iff exists subset within S[i:n-1]
// that adds up to x
SubsetSum(i, x):
  if (x < 0 or i > n)  return S[i][x] = false;
  else if (x == 0)     return S[i][x] = true;
  else if (SS[i][x] != UNDEFINED) return SS[i][x];
  else
    return SS[i][x] = SubsetSum(i+1, x) or SubsetSum(i+1, x-S[i]);
main() {
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```

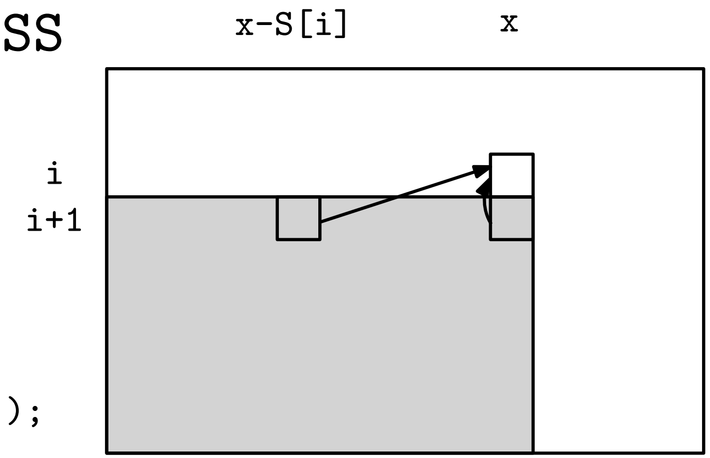

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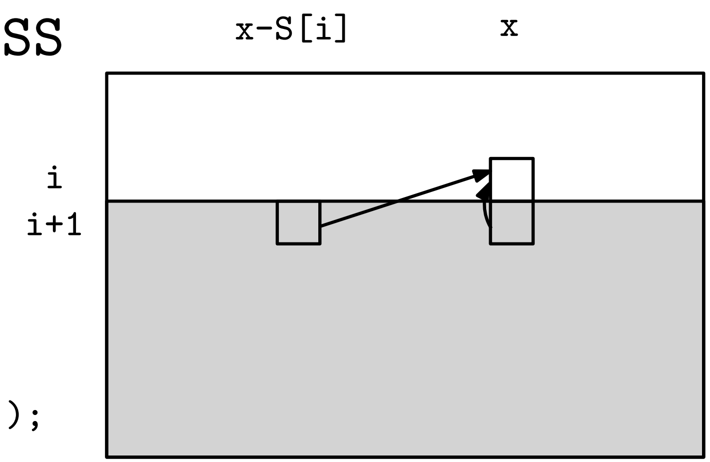

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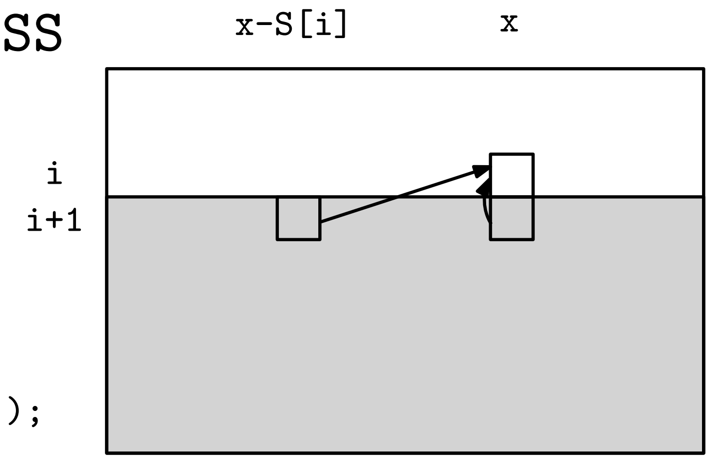


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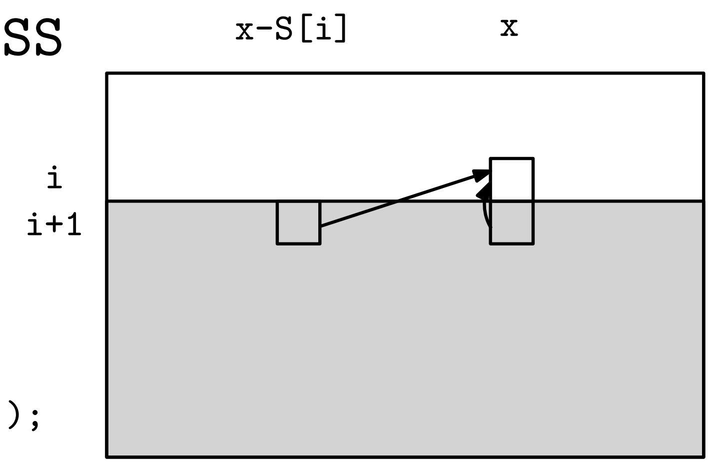


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Can also fill it out in ascending order of `i`, by looking at subproblems `S[0:i]`, instead of `S[i:n-1]`

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Problem (0-1 Knapsack)**.** *Given a set S of n items, each with its own value V_i and weight W_i for all $1 \leq i \leq n$ and a maximum knapsack capacity C, compute the maximum value of the items that you can carry. You cannot take fractions of items.*

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int M[maxN+1][maxC];
maxV(i,C) {
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  if (W[i] > C)
    return  M[i][C]=maxV(i+1, C);
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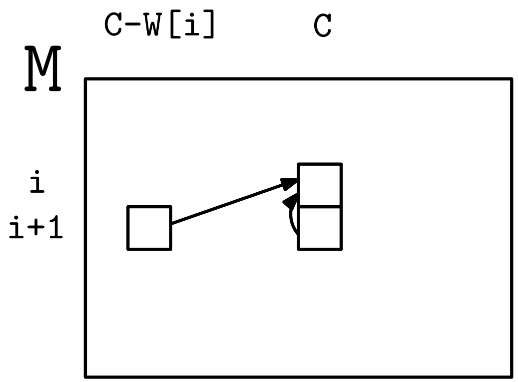

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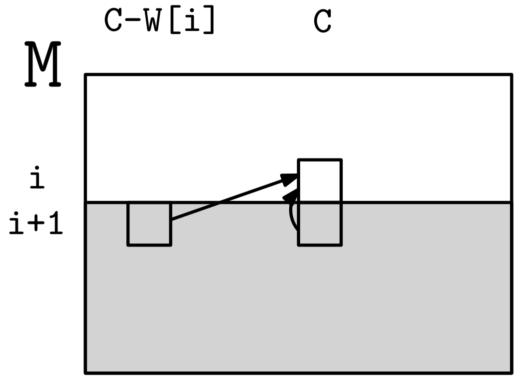

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Pros:

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- Easier to analyze running time
- Sometimes top-down recursion is harder (or impossible) to define

Example

Problem. *In a football game, after every scoring play, the cheerleaders do as many jumps as the total number of points on the scoreboard. For example, if the team first scored a touchdown (7 pts), then a field goal (3 pts), then a safety (2 pts), the cheerleaders did* $7 + 10 + 12 = 29$ *total jumps.*
Given the number n *of total jumps, compute the largest possible number of points scored in the game?*

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You may assume the possible points are given as a set of S of m positive integers, with the largest value at most 20. For example, in regular football rules, $m = 5$ and $S = \{2, 3, 6, 7, 8\}$.

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- Points scored: 7, 3, 2
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$$P(n,x) = \max \left\{ \begin{array}{l} P(n-x, x-S[0]) + S[0] \\ P(n-x, x-S[1]) + S[1] \\ P(n-x, x-S[2]) + S[2] \\ \dots \\ P(n-x, x-S[m-1]) + S[m-1] \end{array} \right\}$$

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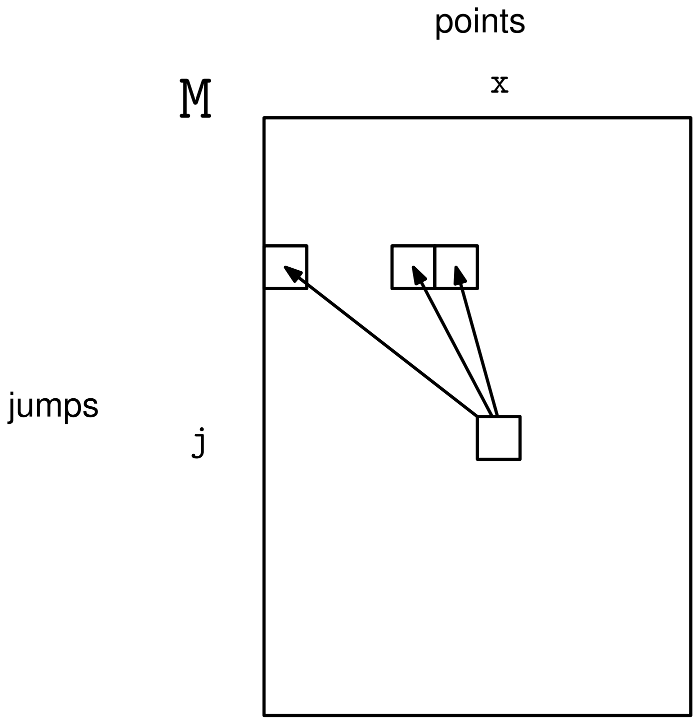
$$x = P(n, x) = \max \left\{ \begin{array}{l} P(n-x, x-S[0]) + S[0] \\ P(n-x, x-S[1]) + S[1] \\ P(n-x, x-S[2]) + S[2] \\ \ldots \\ P(n-x, x-S[m-1]) + S[m-1] \end{array} \right\}$$



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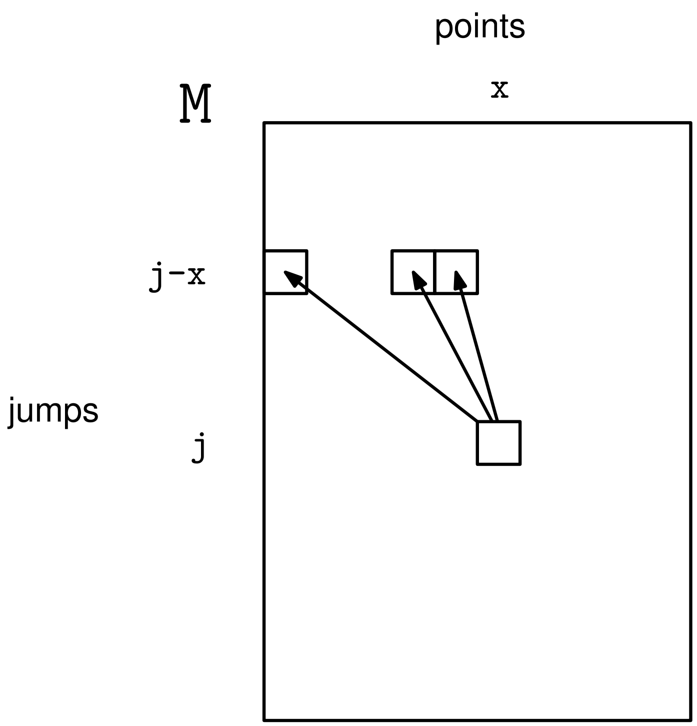

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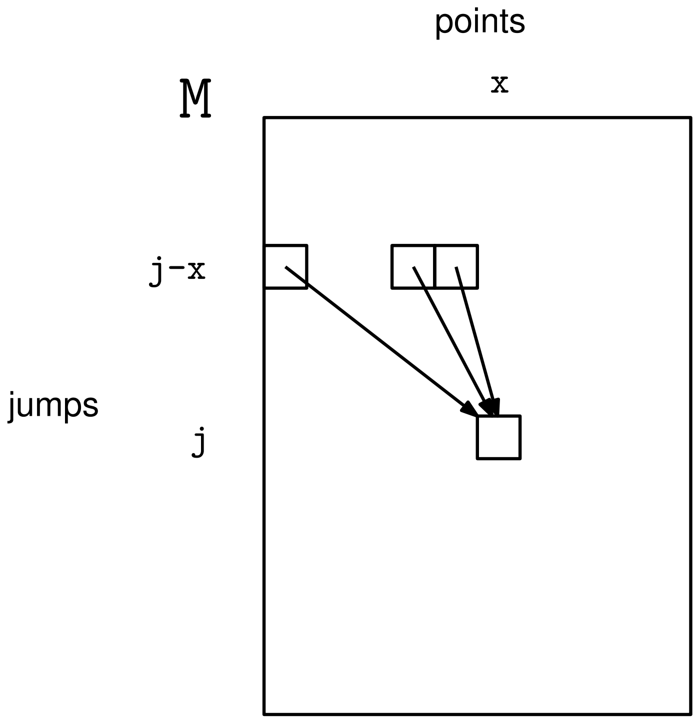

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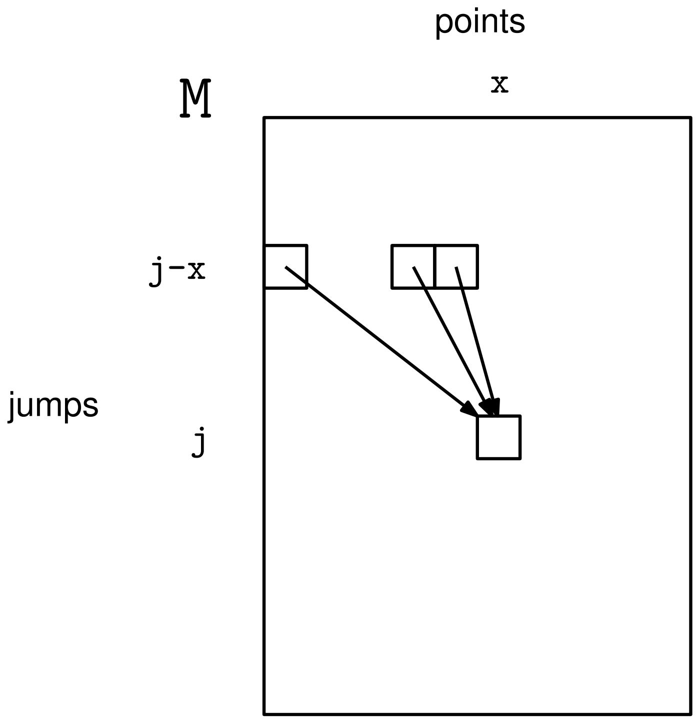
$M[j, x]$ = "Is it possible to have x points for j jumps?"



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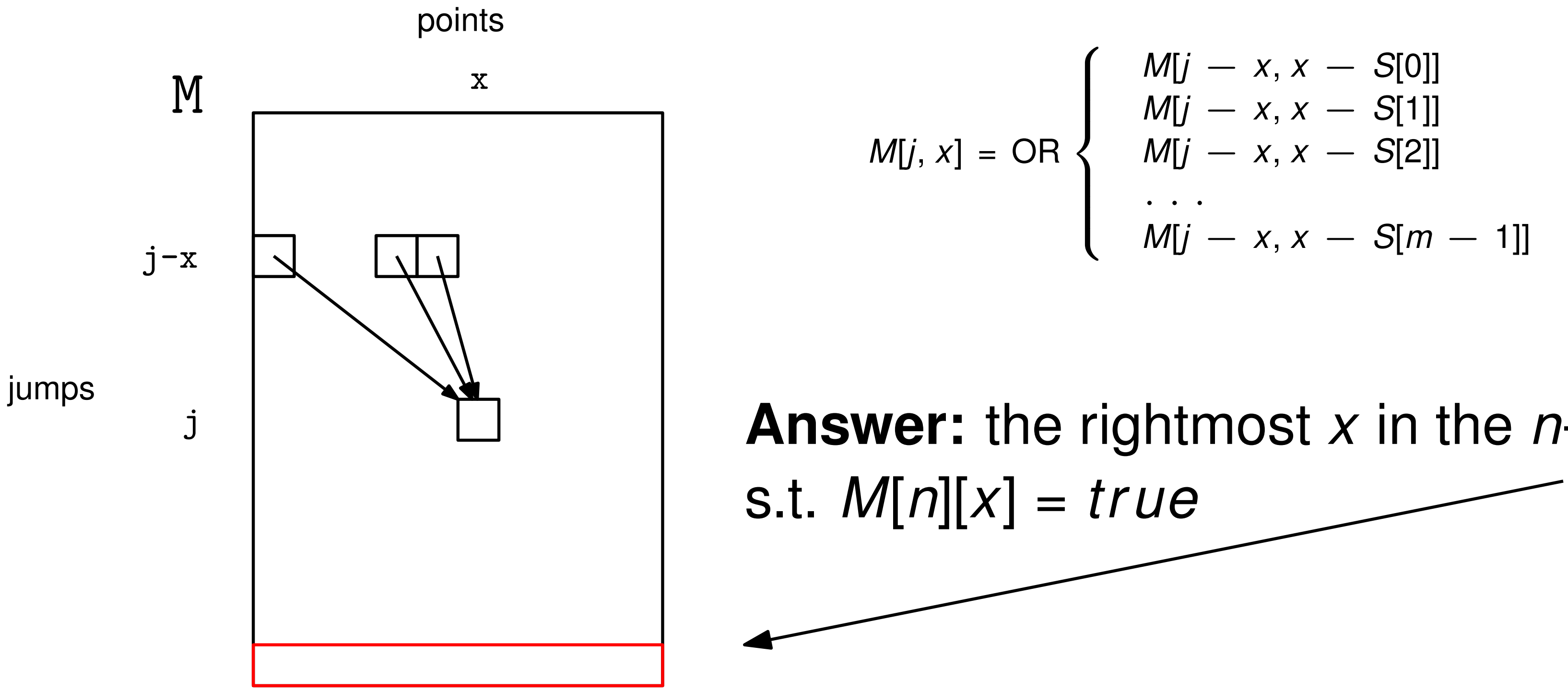

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Answer: the rightmost x in the n-th row, s.t. $M[n][x] = true$

Homework

- Programming assignment (DP)
- Understand Optimal BST problem & solution Ch 15.5