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Enriched Galerkin approach for density-driven flow in coastal aquifer

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Overview

Accurate numerical simulations for density-dependent flow and transport model is one of the crucial keys for successful water resources management in coastal areas and on islands. However, traditional modeling approaches without special treatment may not be able to resolve accurate sharp moving fronts and corresponding groundwater flow velocities due to the numerical instabilities.
In this presentation, we employ the enriched Galerkin finite element methods (EG), which enriches a classical continuous Galerkin finite element methods such as SUTRA with piecewise constant functions to ensure local and global mass conservation. EG has the same bilinear forms as the discontinuous Galerkin (DG) finite element methods but EG has fewer degrees of freedom in comparison with DG. Initial numerical results for an existing benchmark problem show accuracy and efficiency of the proposed method in density-driven flow modeling.

Mathematical Model

Let Ω be a computational domain with a boundary $\partial\Omega$, and $(0,T]$ be a time interval with $T>0$, a final time. The governing conservation system is given as

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

where the velocity $\mathbf{u} : \Omega \times (0,T] \to \mathbb{R}^d$ is Darcy velocity

$$\mathbf{u} := -\kappa \rho_f g \left(\nabla h + \frac{\rho(c) - \rho_f}{\rho_f} \nabla z \right). \tag{2}$$

Here $h : \Omega \times (0,T] \to \mathbb{R}$ is the pressure head, $\kappa := \frac{k}{\mu}$, where k is a permeability, μ is a fluid viscosity, ρ_f is the fresh water density, g is the gravitational force, z is the upward coordinate direction aligned with g, and $\rho(c)$ is the mixed fluid density defined by

$$\rho(c) := \rho_f + \frac{\partial \rho}{\partial c}(c - c_f),$$

where $c : \Omega \times (0,T] \to \mathbb{R}$ is the concentration of salt water and c_f is the concentration of fresh water. The above system is supplemented by the following boundary conditions

$$h = h_D \quad \text{in } \partial\Omega_D \tag{3}$$

$$-\kappa \rho_f g \left(\nabla h + \frac{\rho(c) - \rho_f}{\rho_f} \nabla z \right) \cdot \mathbf{n} = q_0 \quad \text{in } \partial\Omega_N, \tag{4}$$

where $\mathbf{n}$ is outward normal vector of $\partial\Omega$. The transport system for concentration of mass fraction of salt water c is described as

$$\phi \partial_t c + \nabla \cdot (\mathbf{u}c - \phi \mathbf{D}_{\text{eff}} \nabla c) = 0, \tag{5}$$

where ϕ is the porosity and $\mathbf{D}_{\text{eff}}$ is the diffusion dispersion tensor. The boundary of Ω for transport system, denoted by $\partial\Omega$, is decomposed into two parts Γ_{in} and Γ_{out}, the inflow and outflow boundary, respectively. Those are defined as

$$\Gamma_{\text{in}} := \{\mathbf{x} \in \partial\Omega : \mathbf{u}\cdot\mathbf{n} < 0\} \quad \text{and} \quad \Gamma_{\text{out}} := \{\mathbf{x} \in \partial\Omega : \mathbf{u}\cdot\mathbf{n} \geq 0\}. \tag{6}$$

For each boundary, we employ the following boundary conditions

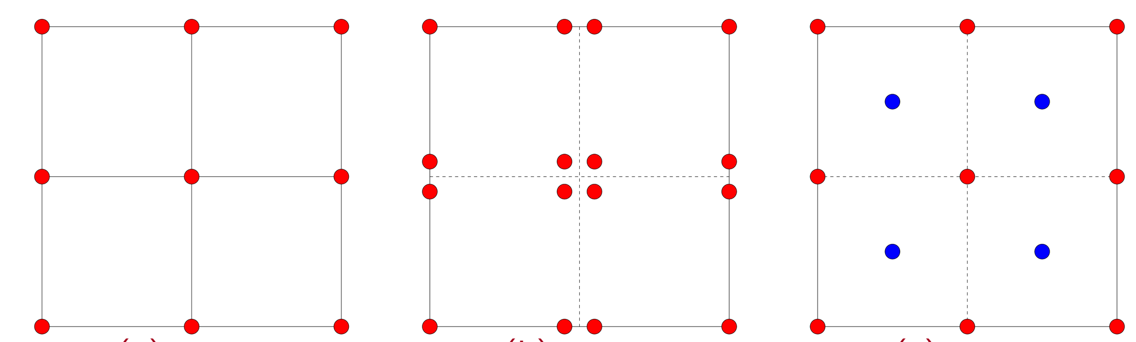
$$(\mathbf{u}c - \varphi \mathbf{D}_{\text{eff}} \nabla c) \cdot \mathbf{n} = c_{\text{in}} \mathbf{u} \cdot \mathbf{n}, \text{ on } \Gamma_{\text{in}} \times (0,T], \tag{7}$$

$$(-\varphi \mathbf{D}_{\text{eff}} \nabla c) \cdot \mathbf{n} = 0, \text{ on } \Gamma_{\text{out}} \times (0,T], \tag{8}$$

where c_{in} is a given inflow boundary value.

Enriched Galerkin Finite Element Method

Enriched Galerkin finite element method (EG) is formulated by enriching piecewise constant functions to the classical continuous Galerkin methods (CG) for each element and it has been employed to several interesting applications [1,2]. Figure 1 illustrates the difference in the degrees of freedom for CG, discontinuous Galerkin (DG), and EG for linear polynomials in quadrilaterals ($\mathbb{Q}_1$). EG has the same bilinear forms as the interior penalty DG schemes, so EG inherits many advantages of DG, for example EG preserves local mass conservation.



(a) CG $\mathbb{Q}_1$ (b) DG $\mathbb{Q}_1$ (c) EG $\mathbb{Q}_1$

Figure: Degrees of freedom for (a) CG, (b) DG, and (c) EG methods on a two dimensional Cartesian grid $\mathbb{Q}$ with a linear polynomial order. Red nodes are the degrees of freedom for CG (DG) and blue circles in the elements denote piecewise constants.

The main advantages of EG

- substantially fewer degrees of freedom in comparison with DG [1,2],
- optimal error convergence rates same as CG and DG [2],
- fast effective solver for elliptic/parabolic problems [2],
- dynamic local mesh adaptivity [3].

Numerical Results: Error Convergence Test

The convergence of the EG methods for flow and transport system (2)-(5) with the gravity and the linear density function. Here we consider the exact solutions given by

$$p = \cos(x - y), \quad c = \sin(t + x - y),$$

in the unit square $\Omega = (0,1)^2$. For each of the flow and transport equations, respectively, five computations on uniform meshes were computed where the mesh size h is divided by two for each cycle. The time discretization is chosen fine enough not to influence the spatial errors and the time step Δt is divided by two for each cycle. Each cycle has 10, 20, 40, 80 and 160 time steps and the errors are computed at the final time $T = 0.1$.

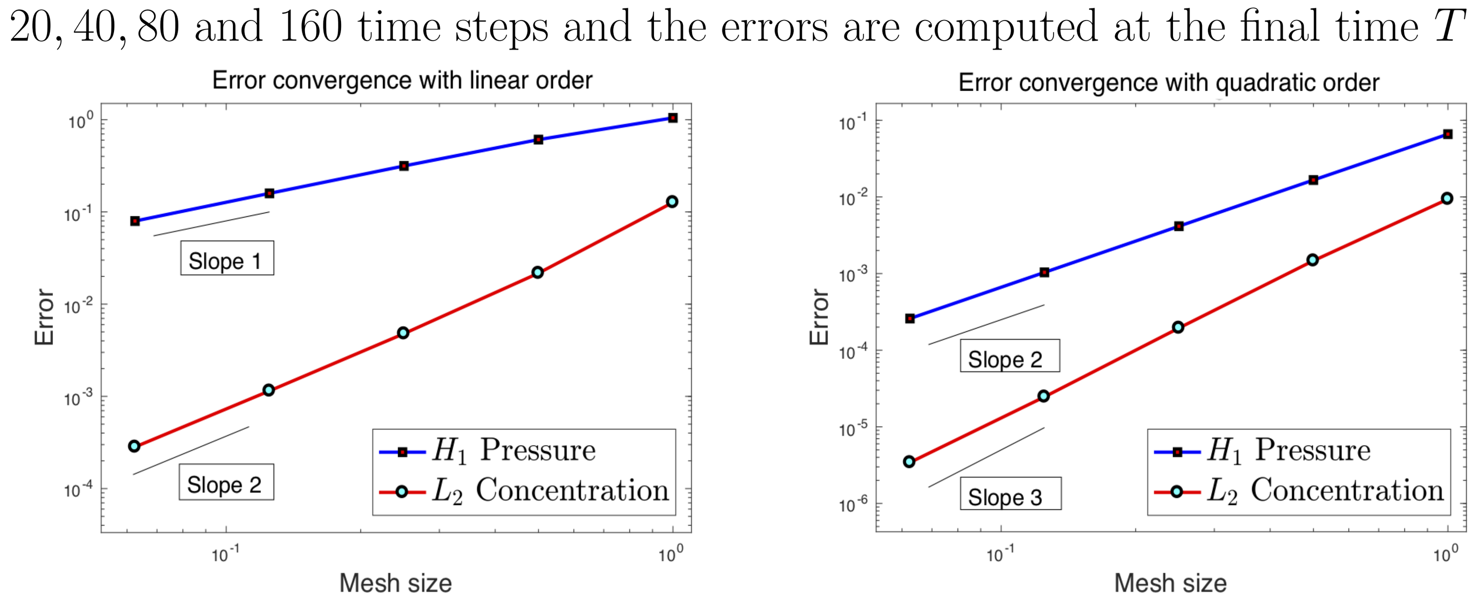


Figure: Error convergence rates for pressure and concentration in semi-H_1 norm and L_2 norm, respectively. Optimal order of convergences are observed for (left) linear and (right) quadratic orders. For quadratic, BDF2 and extrapolation for temporal discretization are employed.

Numerical Results: Henry's Benchmark

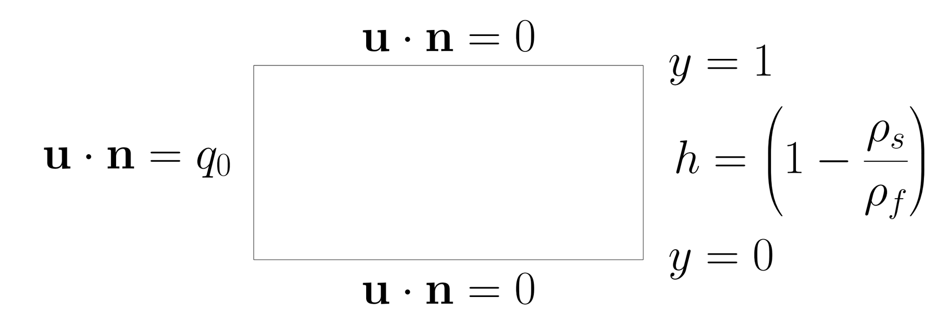


Figure: A domain and setup

To validate our algorithms, we compute the Henry's benchmark problem which considers saltwater intrusion situation. The initial condition $c = 0$ is imposed to indicate that the domain $[0, 2\text{m}] \times [0, 1\text{m}]$ is initially filled with saltwater, and a Dirichlet boundary for concentration $c = 1$ is applied on the right boundary, and $c_{\text{in}} = 0$ on the left boundary. All other physical coefficients are identical as given in the original benchmark problem.

(a) Concentration (b) Fresh water head (c) Pressure

Figure: Concentration (c), freshwater hydraulic head (h), and pressure (p) values at the equilibrium.

Ongoing Works

- Dynamic mesh adaptivity
- Efficient preconditioner constuction
- Coupling with an inverse modeling approach (pyPCGA) to characterize permeability in coastal aquifers [4,5]

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