| ICS 141: Discrete Mathematics for Computer Science I | Spring 2025 |
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| Problem Set 4 | |
| *Kyle Berney* | *Due: Tuesday, March 4, 2025 at 1:30pm* |

All proofs by induction should use strong induction and be written following the boilerplate template shown in the lecture slides. This includes explicitly stating the inductive hypothesis.

## 1 Summation Closed Form (25 pts)

Use strong induction to prove that for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4} .$$

## 2 Gift Certificates (25 pts)

Suppose that a store offers gift certificates in denominations of \$3 and \$7. Use strong induction to prove that any item priced at an integer number of dollars of at least \$12 can be paid for with these gift certificates without the need to use any other form of currency.

## 3 Last Survivor (50 pts)

Suppose there are $n$ people, $p_1, p_2, \ldots, p_n$, sitting in a circle. Starting at the first person, $p_1$, every second person is eliminated until only a single person remains. Let $f(n)$ denote the index of the winner of the last survivor game (i.e., if $p_i$ is the winner, then $f(n) = i$).
For example, if $n = 8$, the sequence of eliminations are: $p_2, p_4, p_6, p_8, p_3, p_7, p_5$. Therefore, $f(8) = 1$.

(a) **(5 pts)** Compute $f(n)$ for $n = 1, 2, 3, \ldots, 7$.

(b) **(15 pts)** Determine a formula for $f(n)$ for all positive integers $n$. *Hint: Consider the cases where $n$ is a power of 2 and not a power of 2, then combine the formulas together into a single expression.*

(c) **(30 pts)** Use stong induction to prove that your formula for $f(n)$ is correct. *Hint: consider how people can be eliminated from the survivor game, such that the game can "restart" with a smaller number of participants.*