Ch 7.2: Probability Theory

ICS 141: Discrete Mathematics for Computer Science I

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Assigning Probabilities

- Let S be a sample space of an experiment with a finite or countable number of outcomes
- Assign a probability $\Pr(s)$ to each outcome s
- There are two requirements:

1. $0 \leq \Pr(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} \Pr(s) = 1$

- The function Pr from the set of all outcomes $s \in S$ is called a probability distribution

Assigning Probabilities

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- Solution: For a fair coin, the probability of heads and tails are equal. Since there are only two possible outcomes, the probability must be $1/2$ each, that is, $\Pr(H) = \Pr(T) = 1/2$.

Assigning Probabilities

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- Solution: For a biased coin we have that

$$\Pr(H) = 2\Pr(T)$$

and

$$\Pr(H) + \Pr(T) = 1$$

Hence,

$$2\Pr(T) + \Pr(H) = 3\Pr(T) = 1$$

and we can conclude that $\Pr(T) = 1/3$ and $\Pr(H) = 2/3$.

Assigning Probabilities

- Definition 1: Let S be a set with n elements. The uniform distribution assigns the probability $1/n$ to each element of S.

- Definition 2: The probability of the event E is the sum of the probabilities of the outcomes in E.

$$\Pr(E) = \sum_{s \in E} \Pr(s)$$

(Note that when E is an infinite set, the above summation is a convergent infinite series.)

Assigning Probabilities

- Ex: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number, but the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

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- Ex: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number, but the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

- Solution: Since 3 appears twice as likely, we have that

$$\Pr(1) = \Pr(2) = \Pr(4) = \Pr(5) = \Pr(6) = 1/7$$

and

$$\Pr(3) = 2/7$$

Hence, the probability of the event $E = \{1, 3, 5\}$ is

$$\Pr(E) = \Pr(1) + \Pr(3) + \Pr(5) = 1/7 + 2/7 + 1/7 = 4/7\ .$$

Probabilities of Complements

- *Recall* from Chapter 7.1 that the probability of the complement $\overline{E}$ is

$$\Pr(\overline{E}) = 1 - \Pr(E)$$

- This equality holds when using Definition 2, since

$$\sum_{s \in S} \Pr(s) = 1 = \Pr(E) + \Pr(\overline{E})$$

Probabilities of Unions

- *Recall* from Chapter 7.1 that the probability of the union of two events E_1 and E_2 is

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$$

- Notice that if E_1 and E_2 are disjoint, then $E_1 \cap E_2 = \emptyset$ and

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$$

- We generalize this observation in the following Theorem

Probabilities of Unions

- Theorem 1: If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$\Pr\left(\bigcup_i E_i\right) = \sum_i \Pr(E_i)$$

Probabilities of Unions

- Theorem 1: If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$\Pr\left(\bigcup_i E_i\right) = \sum_i \Pr(E_i)$$

- Proof: Try it yourself! (In the textbook, this is Exercise 36 and 37.)

Conditional Probability

- Ex: A fair coin is flipped three times and the first flip comes up tails. What is the probability that an odd number of tails appears?

Conditional Probability

- Ex: A fair coin is flipped three times and the first flip comes up tails. What is the probability that an odd number of tails appears?

- Solution: There are four possible outcomes:
  - T, T, T
  - T, T, H
  - T, H, T
  - T, H, H

  An odd number of tails appears only for 2 out of the 4 possible outcomes. Since each of the outcomes are equally likely, the probability is $2/4 = 1/2$.

Conditional Probability

- Let E and F be events with $\Pr(F) > 0$
- Definition 3: The conditional probability of E given F, denoted $\Pr(E \mid F)$, is defined as

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

Conditional Probability

- Ex: A fair coin is flipped three times and the first flip comes up tails. What is the probability that an odd number of tails appears?

Conditional Probability

- Solution: Let E be the event that an odd number of tails occurs and F be the event that the first flip is tails. We have that

$$E = \{HHT, HTH, THH, TTT\}$$
$$F = \{THH, THT, TTH, TTT\}$$
$$\text{and} \quad E \cap F = \{THH, TTT\}$$

Hence, $\Pr(F) = 4/8 = 1/2$ and $\Pr(E \cap F) = 2/8 = 1/4$. Therefore,

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$
$$= \frac{1/4}{1/2} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2}$$

Conditional Probability

- Ex: A bit string of length four is generated with a uniform random distribution. What is the probability that the bit string contains at least two consecutive 0's, given that we know that the most significant bit is a 0?

Conditional Probability

- Solution: Let E be the event that a bit string of length four contains at least two consecutive 0's and F be the event that the most significant bit is a 0. We have that

$$E = \{0000, 0001, 0010, 0011, 0100, 1000, 1001, 1100\}$$
$$F = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$$
$$\text{and} \quad E \cap F = \{0000, 0001, 0010, 0011, 0100\}$$

Since there are $2^4 = 16$ total bit strings of length four, $\Pr(F) = 8/16 = 1/2$ and $\Pr(E \cap F) = 5/16$. Therefore,

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$
$$= \frac{5/16}{1/2} = \frac{5}{16} \cdot \frac{2}{1} = \frac{10}{16} = \frac{5}{8}$$

Independence

- *Recall:* (Slide 11) a previously considered example where a fair coin is flipped three times
  - E is the event that an odd number of tails occurs
  - F is the event that the first flip is tails
- We found that:
  - $\Pr(E) = 4/8 = 1/2$
  - $\Pr(E \mid F) = 1/2$
- In other words, the probability that E occurs is exactly the same as the probability that E occurs given F also occurs
  - We say that E and F are independent events
  - The occurence of one of the events gives no information about the probability that the other event occurs

Independence

$$\Pr(E) = \Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

$$\Rightarrow \Pr(E \cap F) = \Pr(E)\Pr(F)$$

- Definition 4: The events E and F are independent if and only if $\Pr(E \cap F) = \Pr(E)\Pr(F)$.

Independence

- Ex: Let E be the event that a randomly generated bit string of length four begins with a 1 and F be the event that a bit string of length four contains an even number of 1's. Are E and F independent if each of the $2^4 = 16$ bit strings of length four are equally likely?

Independence

- Solution:

$$E = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$$

$$F = \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$$

$$\text{and} \quad E \cap F = \{1001, 1010, 1100, 1111\}$$

Hence,

$$\Pr(E) = \Pr(F) = 8/16 = 1/2$$

$$\Pr(E \cap F) = 4/16 = 1/4$$

$$\text{and} \quad \Pr(E \cap F) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \Pr(E)\Pr(F)$$

Therefore, E and F are independent.

Pairwise and Mutual Independence

- Definition 5: The events $E_1, E_2, \ldots, E_n$ are pairwise independent if and only if
$$\Pr(E_i \cap E_j) = \Pr(E_i)\Pr(E_j)$$
for all pairs of integers i and j such that $1 \leq i < j \leq n$.

- Definition 6: These n events are mutually independent if
$$\Pr(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \Pr(E_{i_1})\Pr(E_{i_2}) \ldots \Pr(E_{i_k})$$
for all integers k such that $2 \leq k \leq n$ and
$1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Pairwise and Mutual Independence

- *Remark:*
  - Every set of n mutually independent events is also pairwise independent
  - However n pairwise independent events are not necessarily mutually independent

Bernoulli Trials and the Binomial Distribution

- Every outcome of an experiment with two possible outcomes is called a Bernoulli trial
  - Generally, an outcome is either a success or a failure

Bernoulli Trials and the Binomial Distribution

- Ex: A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming the flips are independent?

Bernoulli Trials and the Binomial Distribution

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- Solution: There are $2^7 = 128$ possible outcomes when a coin is flipped seven times. The number of ways that four of the flips results in heads is the number of 4-combinations out of 7 items. Since each of the flips are independent, the probability of four heads and three tails is $(2/3)^4(1/3)^3$. Therefore, the probability that exactly four heads occurs is

$$\binom{7}{4}(2/3)^4(1/3)^3 = \frac{35 \cdot 16}{3^7} = \frac{560}{2187}$$

Bernoulli Trials and the Binomial Distribution

- Theorem 2: The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$\binom{n}{k} p^k q^{n-k}$$

Bernoulli Trials and the Binomial Distribution

- Theorem 2: The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$\binom{n}{k} p^k q^{n-k}$$

- Proof: The number of ways to have k successes out of n experiments is equal to the number of k-combinations from n total objects. Since each of the n trials are independent, the probability of exactly k successes and $n - k$ failures is $p^k q^{n-k}$. Therefore, the probability of exactly k successes is

$$\binom{n}{k} p^k q^{n-k}$$

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Bernoulli Trials and the Binomial Distribution

- We denote the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$ as $b(k; n, p)$
  - This function is called the binomial distribution
  - From Theorem 2,

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

Random Variables

- Many problems are concerned with a numerical value associated with the outcome of an experiment
  - Ex: What is the number of times tails occurs when a coin is flipped 20 times?
- Definition: A random variable is a mapping from the sample space of an experiment to the set of real numbers
  - In other words, a random variable assigns a real number to each possible outcome

Random Variables

- Ex: Suppose that a coin is flipped three times. Let X be a random variable that equals the number of heads appears.
  - $X(HHH) = 3$
  - $X(HHT) = 2$
  - $X(HTH) = 2$
  - $X(HTT) = 1$
  - $X(THH) = 2$
  - $X(THT) = 1$
  - $X(TTH) = 1$
  - $X(TTT) = 0$

Random Variables

- Let $\Pr(X = r)$ be the probability that X takes the value r
- Definition: The distribution of a random variable X on a sample space S is the set of pairs $(r, \Pr(X = r))$.

Random Variables

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Random Variables

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  - $X(HTH) = 2$
  - $X(HTT) = 1$
  - $X(THH) = 2$
  - $X(THT) = 1$
  - $X(TTH) = 1$
  - $X(TTT) = 0$
- Hence, the distribution of X is

$$(0, 1/8), (1, 3/8), (2, 3/8), (3, 1/8)$$

Random Variables

- We'll continue studying random variables in Chapter 7.4
  - Indicator random variables
  - Expected value
  - Variance

- Random variables are fundamental to the analysis of randomized algorithms
  - ICS 311: Algorithms
  - Ex: Randomized Quicksort

The Birthday Problem

- Famous puzzle with a surprising answer
- The Birthday Problem: What is the minimum number of people who need to be in the same room so that the probability that at least two of them have the same birthday is greater than 1/2?

The Birthday Problem

- Solution:

  Assumptions:

  - Birthdays of the people in the room are independent
  - Each birthday is equally likely
  - 365 days in a year

The Birthday Problem

- <u>Solution:</u> Let p_n be the probability that n people all have different birthdays.

$$p_1 = 1$$

$$p_2 = \frac{364}{365}$$

$$p_3 = \frac{364}{365} \cdot \frac{363}{365}$$

$$p_4 = \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$$

$$\vdots$$

$$p_n = \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-(n-1)}{365}$$

The Birthday Problem

- Solution: It follows that the probability that out of n people, there are at least two people with the same birthday is

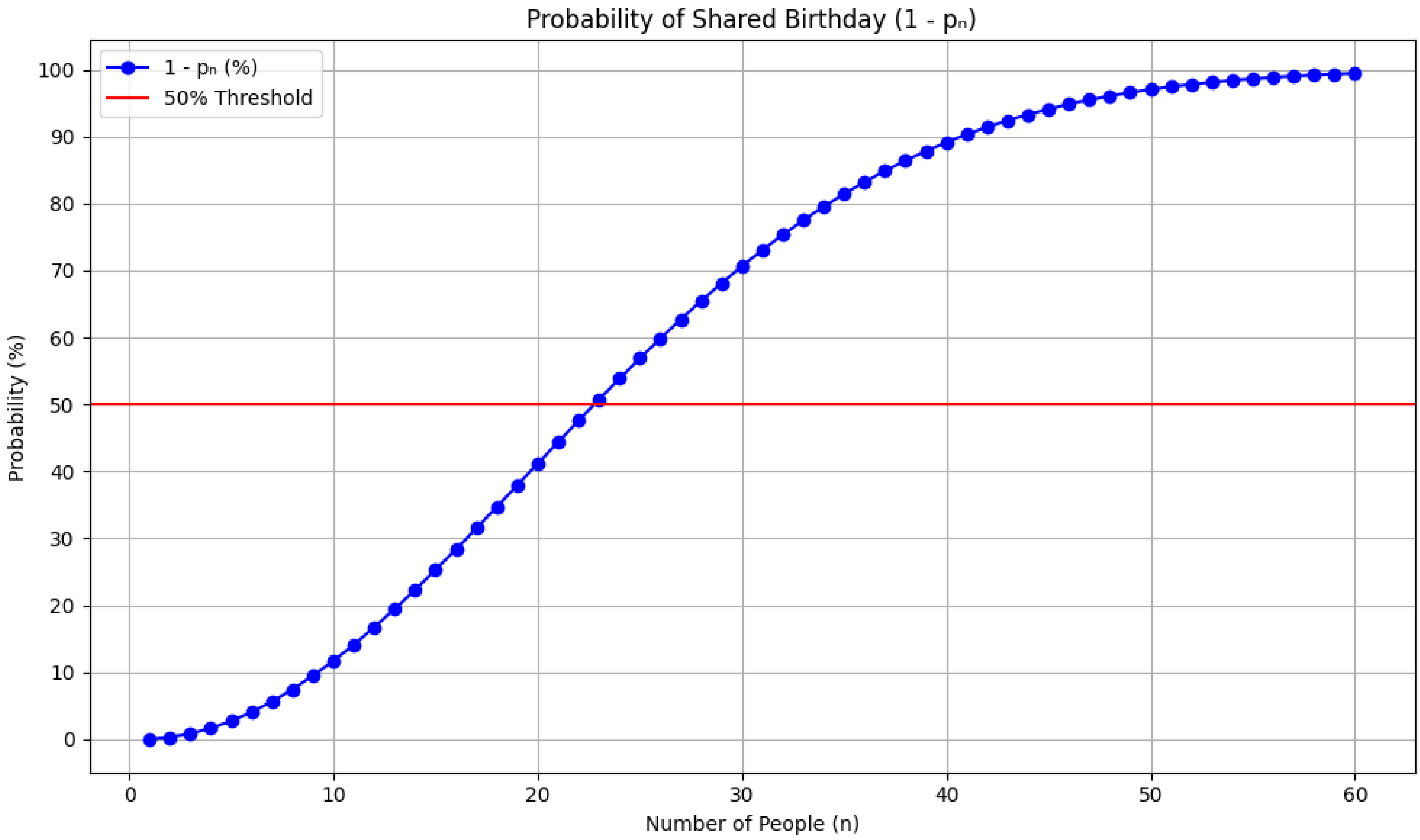
$$1 - p_n = 1 - \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \frac{365 - (n-1)}{365}$$

We compute $1 - p_n$ manually (i.e., since we are computer scientists, we instead write a Python script) and find that

$$1 - p_{22} \approx 0.4756$$
$$1 - p_{23} \approx 0.5072$$

Therefore, the minimum number of people needed so that the probability that at least two people have the same birthday is greater than 1/2 is 23.

The Birthday Problem



$$1 - p_{60} \approx 0.9941$$