Ch 7.1: Introduction to Discete Probability

ICS 141: Discrete Mathematics for Computer Science I

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Terminology

- An experiment is a procedure that yields one of a given set of possible outcomes
- The sample space of the experiment is the set of all possible outcomes
- An event is a subset of the sample space

Finite Probability

- Definition: If S is a finite non-empty sample space of equally likely outcomes and $E \subseteq S$ is an event, then the probability of E is

$$\Pr(E) = \frac{|E|}{|S|}$$

- Also commonly denoted, $p(E)$, $P(E)$, or $\mathbb{P}(E)$

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- *Remark:* It follows from the definition that

$$0 \leq \Pr(E) \leq 1$$

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- <u>Solution:</u> There are 9 possible balls that can be chosen, where 4 of them are blue balls. Hence, the probability that a blue ball is chosen is $4/9$.

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- Ex: What is the probability that when two dice are rolled, the sum of the numbers of the two dice is 7?
- Solution: Using the product rule for counting, there are $6 \cdot 6 = 36$ total outcomes when two dice are rolled. Of those 36 outcomes, there are six rolls that result in a sum of the numbers being 7. Namely, (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), where the values of the first and second dice are represented by an ordered pair. Hence, the probability that the sum of the numbers of the two dice is 7 is $6/36 = 1/6$.

Finite Probability

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- Solution: There is only one winning combiniation of six numbers out of 40. The number of 6-combinations from a set of 40 integers is

$$\binom{40}{6} = \frac{40!}{34!6!} = 3,838,380$$

Therefore, the probability of a winning combination is $1/3,838,380 \approx 0.00000026 = 0.000026\%$

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- <u>Solution:</u> Using the product rule for counting, the number of hands of five cards with a four a kind is the product of the number of ways to pick one kind (out of 13 possible kinds), the number of ways to pick four of this chosen kind, and the number of ways to pick the fifth card in the hand. Since the total number of possible hands is $\binom{52}{5}$, the probability is

$$\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{13 \cdot 1 \cdot 48}{2,598,960} \approx 0.00024 = 0.024\%$$

Finite Probability

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- Solution: Using the product rule for counting, the number of hands of five cards with a full house is the product of the number of ways to pick two kinds (out of 13 possible kinds), the number of ways to pick three of the first kind, and the number of ways two of the second kind. Since the total number of possible hands is $\binom{52}{5}$, the probability is

$$\frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \frac{13 \cdot 12 \cdot 4 \cdot 6}{2,598,960} \approx 0.0014 = 0.14\%$$

Finite Probability

- Ex: What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers 1, 2, . . ., 50 if

  1. the ball selected is not returned to the bin before the next ball is selected
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- Solution: (1.) The number of 5-permutations out of 50 integers is $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254,251,200$. Since we are concerned with picking a single permutation of balls, the probability is $1/254,251,200$.

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- Solution: (2.) The number of 5-permutations with repition out of 50 integers is $50^5 = 312,500,000$. Since we are concerned with picking a single permutation of balls, the probability is $1/312,500,000$.

Probabilities of Complements

- Theorem 1: Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$ is

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- *Remark:* Sometimes it is easier to calculate the probability of the complement in order to determine the probability of the event

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- Solution: Let E be the event that at least one of the 10 bits is 0. Then the complement $\overline{E}$ is the event that all of the bits are 1's. Hence,

$$\Pr(E) = 1 - \Pr(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}}$$

$$= 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Probabilities of Unions

- Theorem 2: Let E_1 and E_2 be events in the sample space S.

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$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$$

- Proof: Using the inclusion-exclusion principle, we know that

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

Hence,

$$\begin{aligned}
\Pr(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} \\
&= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\
&= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\
&= \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) \qquad \blacksquare
\end{aligned}$$

Monty Hall Puzzle

- You are on a game show where you are asked to select one of three doors to open
- A grand pize is behind one of these three doors
- Once you select a door, the game show host does the following:

  1. Regardless of if you chose the correct door or not, they open one of the other two doors that does not have the grand prize behind it
  2. Asks whether you would like to switch doors

- Should you change doors?

Monty Hall Puzzle

- Suppose we chose Door 1

Door 1	Door 2	Door 3	Stay	Switch
L	L	W	L	W
L	W	L	L	W
W	L	L	W	L

- If we stay at Door 1, there is a probability of $1/3$ that we win
- If we switch our choice, there is a probability of $2/3$ that we win
- Therefore, you should always switch your choice