INFORMATION & COMPUTER SCIENCES
UNIVERSITY *of* HAWAIʻI *at* MĀNOA

Ch 6.5: Generalized Permutations and Combinations

ICS 141: Discrete Mathematics for Computer Science I

Kyle Berney
Department of ICS, University of Hawaii at Manoa

Permutations with Repitition

- Theorem 1: The number of r-permutations of a set of n objects with repitition allowed is n^r

Permutations with Repitition

- Theorem 1: The number of r-permutations of a set of n objects with repitition allowed is n^r
- Proof: Let n and r be non-negative integers. There are n ways to select and item for each of the r positions in the r-permutation with repitition allowed. Hence, from the product rule we obtain n^r. ∎

Permutations with Repitition

- Ex: How many strings of length r can be formed using uppercase letters?

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- Solution: Using the product rule, since there are 26 different uppercase letters for each position, there are 26^r different strings of length r.

Combinations with Repitition

- Theorem 2: The number of r-combinations from a set with n elements when repitition is allowed is

$$C(n+r-1, r) = \binom{n+r-1}{r}$$

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$$C(n+r-1, r) = \binom{n+r-1}{r}$$

- Proof: Let n and r be arbitrary non-negative integers. Each r-combination with repitition of a set with n elements can be represented by a list of $(n-1)$ bars and r stars. The $(n-1)$ bars are used to mark off n different cells, with the i-th cell containing stars representing each time the i-th element of the set occurs in the r-combination.

Combinations with Repitition

- <u>Theorem 2:</u> The number of r-combinations from a set with n elements when repitition is allowed is

$$C(n+r-1, r) = \binom{n+r-1}{r}$$

- <u>Proof:</u>
The number of such lists of $(n-1)$ bars and r stars corresponds to an r-combination without repitition of a set with $(n+r-1)$ elements. In other words, there are $(n+r-1)$ positions to place the r stars, and once those positions are chosen the positions of the $(n-1)$ bars are determined. Hence, we obtain

$$C(n+r-1, r) = \binom{n+r-1}{r} \qquad \blacksquare$$

Combinations with Repitition

- Ex: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills. You can assume there are at least five bills of each type.

Combinations with Repitition

- Ex: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills. You can assume there are at least five bills of each type.

- Solution: There are $n = 7$ types of bills and we want to select $r = 5$ of them.

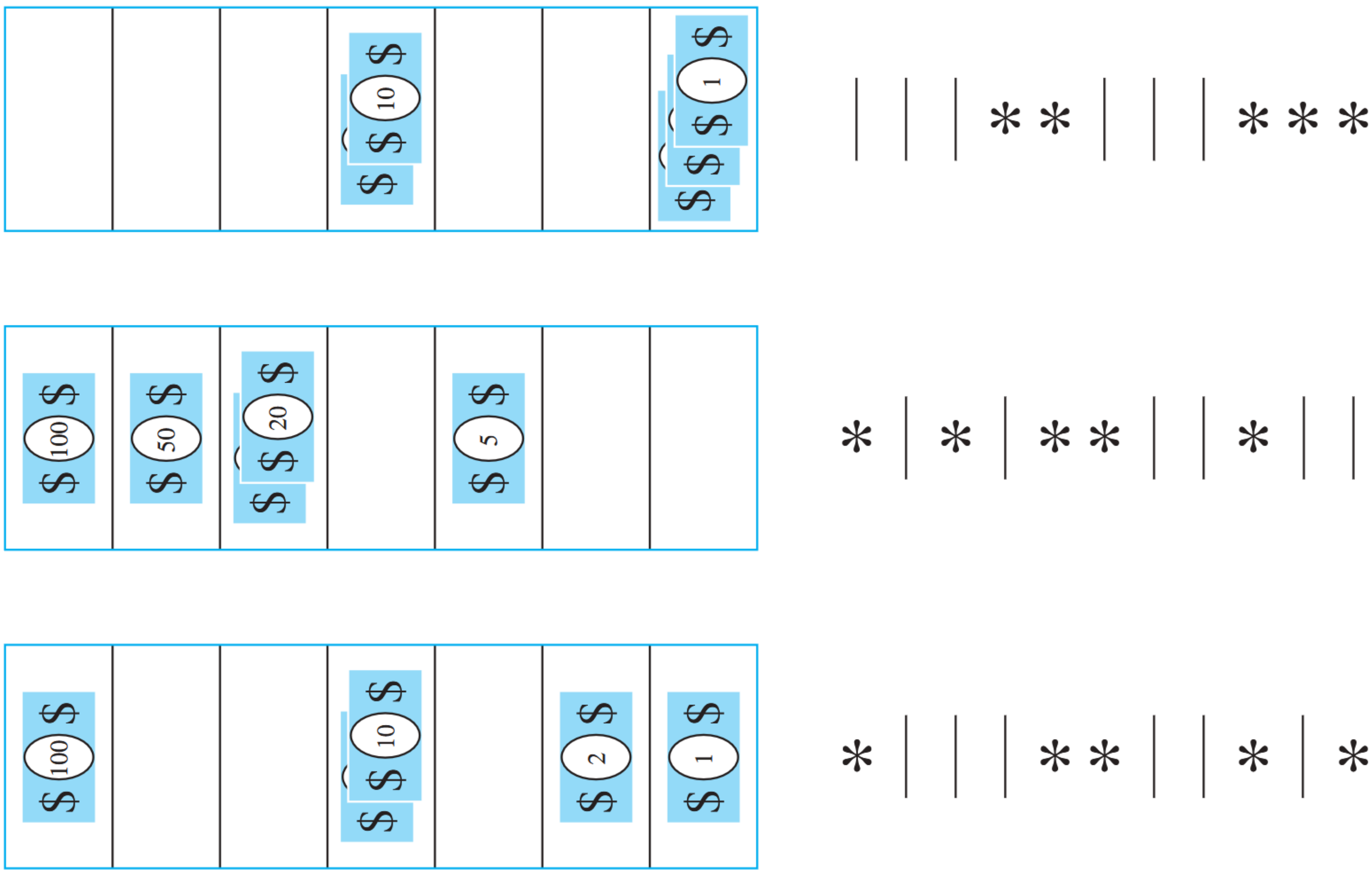
$$\binom{n+r-1}{r} = \binom{7+5-1}{5} = \binom{11}{5}$$

$$= \frac{11!}{5!(11-5)!}$$

$$= \frac{11!}{5!6!} = 462$$

Combinations with Repitition

- *Remark:* We can represent the problem using r = 5 stars and $(n - 1)$ = 6 bars.



Combinations with Repitition

- Ex: How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears? You can assume there are at least four pieces of each fruit in the bowl.

Combinations with Repitition

- <u>Ex:</u> How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears? You can assume there are at least four pieces of each fruit in the bowl.

- <u>Solution:</u> There $n = 3$ different fruit and we want to select $r = 4$ pieces. Hence,

$$\binom{n+r-1}{r} = \binom{3+4-1}{4} = \binom{6}{4}$$
$$= \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!}$$
$$= \frac{6 \cdot 5}{2!} = 15$$

Combinations with Repitition

- Ex: Suppose a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? You can assume the shop has at least six cookies per type.

Combinations with Repitition

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- Solution: There $n = 4$ different types of cookies and we want to choose $r = 6$ cookies. Hence,

$$\begin{aligned} \binom{n+r-1}{r} &= \binom{4+6-1}{6} = \binom{9}{6} \\ &= \frac{9!}{6!(9-6)!} = \frac{9!}{6!3!} \\ &= \frac{9 \cdot 7 \cdot 8}{3!} = 84 \end{aligned}$$

Permutations with Indistinguishable Objects

- In some problems, some objects may be indistinguishable from each other
- Ex: How many different strings can be made by reordering the letters of the word “SUCCESS”?

Permutations with Indistinguishable Objects

- In some problems, some objects may be indistinguishable from each other
- <u>Ex:</u> How many different strings can be made by reordering the letters of the word "SUCCESS"?

- <u>Solution:</u> The word "SUCCESS" contains 3 S's, 2 C's, 1 U, and 1 E. The three S's can be placed in any of the seven positions in $C(7,3)$ ways, leaving four positions free. Then the 2 C's can be placed in any of the remaining four positions in $C(4,2)$ ways, leaving two positions free. Next, the 1 U can be placed in any of the two remaining positions in $C(2,1)$ ways, leaving a single position free. Lastly, the 1 E can only be placed in the remaining position in exactly $C(1,1) = 1$ way.

Permutations with Indistinguishable Objects

- In some problems, some objects may be indistinguishable from each other
- Ex: How many different strings can be made by reordering the letters of the word "SUCCESS"?
- Solution:
  Using the product rule, the number of strings are

$$C(7,3)C(4,2)C(2,1)C(1,1) = \binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}$$

$$= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!}$$

$$= \frac{7!}{3!2!} = 420$$

Permutations with Indistinguishable Objects

- In some problems, some objects may be indistinguishable from each other
- Ex: How many different strings can be made by reordering the letters of the word "SUCCESS"?

- *Remark:* Notice that it does not matter in which order we decide to place the letters. Suppose we decide to place the 2 C's first, then the 1 U, then the 3 S's, then the 1 E.

$$
\begin{aligned}
C(7,2)C(5,1)C(4,3)C(1,1) &= \binom{7}{2}\binom{5}{1}\binom{4}{3}\binom{1}{1} \\
&= \frac{7!}{2!5!}\cdot\frac{5!}{1!4!}\cdot\frac{4!}{3!1!}\cdot\frac{1!}{1!0!} \\
&= \frac{7!}{2!3!} = 420
\end{aligned}
$$

Permutations with Indistinguishable Objects

- Theorem 3: The number of different permutations of n objects, such that there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

Permutations with Indistinguishable Objects

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$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

- Proof: Let n be an arbitrary non-negative integer and for each n_i for $i = 1, 2, \ldots k$ there are n_i indistinguishable objects of type i. The n_1 objects of type 1 can be placed in any of the n positions in $C(n, n_1)$ ways, leaving $(n - n_1)$ free positions remaining. The n_2 objects of type 2 can be placed in any of the $(n - n_1)$ remaining free positions in $C(n - n_1, n_2)$ ways, leaving $(n - n_1 - n_2)$ remaining free positions. We proceed with the objects of type $3, 4, \ldots, k$ in the same manner.

Permutations with Indistinguishable Objects

- Theorem 3: The number of different permutations of n objects, such that there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

- Proof: Using the product rule, we obtain

$$C(n, n_1)C(n - n_1, n_2)\dots C(n - n_1 - \dots - n_{k-1}, n_k)$$

$$= \frac{n!}{n_1!(n - n_1)!} \cdot \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \cdot \dots \cdot \frac{(n - n_1 - \dots - n_{k-1})!}{n_k!0!}$$

$$= \frac{n!}{n_1!n_2!\dots n_k!}$$

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Distributing Objects into Boxes

- Many counting problems can be solved by enumerating the ways objects can be placed into boxes
  - Order of the objects in boxes do not matter

- Objects and boxes can be
  - Distinguishable (i.e., labeled or different from each other)
  - Indistinguishable (i.e., unlabled or identical from each other)

Distinguishable Objects and Distinguishable Boxes

- Ex: How many ways are there to distribute hands of 5 cards to four players from a standard deck of 52 cards?

Distinguishable Objects and Distinguishable Boxes

- Ex: How many ways are there to distribute hands of 5 cards to four players from a standard deck of 52 cards?

- Solution: For this problem, objects are cards and boxes are the hands of each player.
  The first player can be dealt 5 cards in $C(52, 5)$ ways, leaving 47 cards remaining. The second player can be dealt 5 cards in $C(47, 5)$ ways, leaving 42 cards remaining. The third player can be dealt 5 cards in $C(42, 5)$ ways, leaving 37 cards remaining. The fourth player can be dealt 5 cards in $C(37, 5)$ ways.

Distinguishable Objects and Distinguishable Boxes

- Ex: How many ways are there to distribute hands of 5 cards to four players from a standard deck of 52 cards?
- Solution:
  Using the product rule, we obtain

$$
\begin{aligned}
& C(52,5)\,C(47,5)\,C(42,5)\,C(37,5) \\
&= \binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5} \\
&= \frac{52!}{5!47!}\cdot\frac{47!}{5!42!}\cdot\frac{42!}{5!37!}\cdot\frac{37!}{5!32!} \\
&= \frac{52!}{5!5!5!5!32!} = \frac{52!}{(5!)^4 32!}
\end{aligned}
$$

Distinguishable Objects and Distinguishable Boxes

- Theorem 4: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i for $i = 1, 2, \ldots, k$ is

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Distinguishable Objects and Distinguishable Boxes

- Theorem 4: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i for $i = 1, 2, \ldots, k$ is

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

- Proof: Try to do this yourself. Use the product rule.

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

- Solution: Let *A*, *B*, *C*, and *D* be the four employees. There number of ways can be split into four subproblems:

  1. All four employees into a single office
  2. Three employees in one office and one employee in another
  3. Two employees in one office and two employees in another
  4. Two employees in one office, one employee in another, and one employee in another

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

- Solution:
Each of the subproblems can be represented as a way to partition the elements A, B, C, and D into disjoint subsets. For subproblem 1, we can place all four employees in one office in exactly 1 way represented by

$$\{\{A, B, C, D\}\}$$

Distinguishable Objects and Indistinguishable Boxes

- <u>Ex:</u> How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?
- <u>Solution:</u>
  For subproblem 2, we can place three employees in one office and one employee in another in exactly 4 ways represented by

$$\{\{A, B, C\}, \{D\}\}, \{\{A, B, D\}, \{C\}\},$$
$$\{\{A, C, D\}, \{B\}\}, \{\{B, C, D\}, \{A\}\}$$

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

- Solution:
  For subproblem 3, we can place two employees in one office and two employees in another in exactly 3 ways represented by

$$\{\{A, B\}, \{C, D\}\}, \{\{A, C\}, \{B, D\}\},$$
$$\{\{A, D\}, \{B, C\}\}$$

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

- Solution:
  For subproblem 4, we can place two employees in one office, one employee in another, and one employee in another in exactly 6 ways represented by

$$\{\{A, B\}, \{C\}, \{D\}\}, \{\{A, C\}, \{B\}, \{D\}\}, \{\{A, D\}, \{B\}, \{C\}\}$$
$$\{\{B, C\}, \{A\}, \{D\}\}, \{\{B, D\}, \{A\}, \{C\}\}, \{\{C, D\}, \{A\}, \{B\}\}$$

Distinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to put four different employees into three indistinguishable offices, where each office can contain any number of employees?

- Solution:
  Using the sum rule, there are $1 + 4 + 3 + 6 = 14$ ways in total.

Distinguishable Objects and Indistinguishable Boxes

- Unfortunently, there is no simple formula for the number of ways to distribute *n* distinguishable objects into *k* indistinguishable boxes
- However, there is a formula involving summations and Stirling numbers of the second kind
  - Read the textbook if interested

Indistinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Indistinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

- Solution: We enumerate all of the ways to pack the books.

1. 6 books in one box
2. 5 books in one box and 1 book in a second box
3. 4 books in one box and 2 books in a second box
4. 4 books in one box, 1 book in a second box, and 1 book in a third box
5. 3 books in one box and 3 books in a second box
6. 3 books in one box, 2 books in a second box, and 1 book in a third box

Indistinguishable Objects and Indistinguishable Boxes

- Ex: How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

- Solution:

7. 3 books in one box, 1 book in a second box, 1 book in a third box, and 1 book in a fourth box
8. 2 books in one box, 2 books in a second box, and 2 books in a third box
9. 2 books in one box, 2 books in a second box, 1 book in a third box, and 1 book in a fourth box

Therefore, there are 9 total ways to pack the books into boxes.

Indistinguishable Objects and Indistinguishable Boxes

- Unfortunently, there is no simple formula for the number of ways to distribute n indistinguishable objects into k indistinguishable boxes