INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

Ch 6.4: Binomial Coefficients and Identities

ICS 141: Discrete Mathematics for Computer Science I

KYLE BERNEY
DEPARTMENT OF ICS, UNIVERSITY OF HAWAII AT MANOA

Binomial Coefficients

- *Recall:* The number of r-combinations from a set of n elements is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

  which is the binomial coefficient

- Called the binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions

$$(x+y)^n$$

Pascal's Identity

- <u>Theorem 1:</u> (Pascal's Identity) Let n and k be positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Pascal's Identity

- Theorem 1: (Pascal's Identity) Let n and k be positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- Proof: Let n and k be arbitrary positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-1-(k-1)!} + \frac{(n-1)!}{k!(n-1-k)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$

Pascal's Identity

- Theorem 1: (Pascal's Identity) Let n and k be positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- Proof: Let n and k be arbitrary positive integers, such that $k \leq n$.

$$= \frac{(n-1)!}{(k-1)!(n-k)(n-k-1)!} + \frac{(n-1)!}{k(k-1)!(n-k-1)!}$$

$$= \left(\frac{1}{n-k}\right)\left(\frac{(n-1)!}{(k-1)!(n-k-1)!}\right)$$

$$+ \left(\frac{1}{k}\right)\left(\frac{(n-1)!}{(k-1)!(n-k-1)!}\right)$$

Pascal's Identity

- <u>Theorem 1:</u> (Pascal's Identity) Let n and k be positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- <u>Proof:</u> Let n and k be arbitrary positive integers, such that $k \leq n$.

$$= \left(\frac{1}{n-k} + \frac{1}{k}\right)\left(\frac{(n-1)!}{(k-1)!(n-k-1)!}\right)$$

$$= \left(\frac{k}{k(n-k)} + \frac{n-k}{k(n-k)}\right)\left(\frac{(n-1)!}{(k-1)!(n-k-1)!}\right)$$

$$= \left(\frac{n}{k(n-k)}\right)\left(\frac{(n-1)!}{(k-1)!(n-k-1)!}\right)$$

Pascal's Identity

- Theorem 1: (Pascal's Identity) Let n and k be positive integers, such that $k \leq n$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- Proof: Let n and k be arbitrary positive integers, such that $k \leq n$.

$$= \frac{n(n-1)!}{k(k-1)!(n-k)(n-k-1)!}$$

$$= \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}$$

■

Binomial Theorem

- <u>Theorem 2:</u> (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof: Let n be an arbitrary non-negative integer.
  Inductive Hypothesis: Assume inductively that for all integers k, such that $0 \leq k < n$, $P(k)$ is true. In other words,

$$(x+y)^k = \binom{k}{0}x^k + \binom{k}{1}x^{k-1}y + \ldots + \binom{k}{k-1}xy^{k-1} + \binom{k}{k}y^k$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof:
  Base Case: Assume $n = 0$.

$$(x+y)^0 = 1 = \binom{0}{0}y^0 = \binom{n}{n}y^n$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof:
  Inductive Case: Assume $n > 0$.

$$(x+y)^n = (x+y)(x+y)^{n-1}$$

  Since $0 \leq n-1 < n$, from our inductive hypothesis we know that

$$(x+y)^{n-1} = \binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \ldots + \binom{n-1}{n-2}xy^{n-2} + \binom{n-1}{n-1}y^{n-1}$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof: Hence,

$$\begin{aligned}(x+y)^n &= (x+y)(x+y)^{n-1} \\ &= (x+y)\left(\binom{n-1}{0}x^{n-1} + \binom{n-1}{1}x^{n-2}y + \ldots + \binom{n-1}{n-1}y^{n-1}\right) \\ &= \left(\binom{n-1}{0}x^n + \binom{n-1}{1}x^{n-1}y + \ldots + \binom{n-1}{n-1}xy^{n-1}\right) \\ &+ \left(\binom{n-1}{0}x^{n-1}y + \binom{n-1}{1}x^{n-2}y^2 + \ldots + \binom{n-1}{n-1}y^n\right)\end{aligned}$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof:

$$= x^n + x^{n-1}y\left(\binom{n-1}{0} + \binom{n-1}{1}\right) + x^{n-2}y^2\left(\binom{n-1}{1} + \binom{n-1}{2}\right)$$

$$+ \ldots + xy^{n-1}\left(\binom{n-1}{n-2} + \binom{n-1}{n-1}\right) + y^n$$

Binomial Theorem

- Theorem 2: (Binomial Theorem) Let $x, y \in \mathbb{R}$ and let n be a non-negative integer.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof: It follows from Theorem 1 (Pascal's Identity) that

$$= x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + y^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

■

Binomial Coefficients

- Corollary 1: Let n be a non-negative integer.

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

Binomial Coefficients

- Corollary 1: Let n be a non-negative integer.

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

- Proof: Let n be an arbitrary non-negative integer.
  It follows from Theorem 1, that for $x = 1$ and $y = 1$

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} \quad \blacksquare$$

Binomial Coefficients

- Corollary 2: Let n be a positive integer.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

Binomial Coefficients

- Corollary 2: Let n be a positive integer.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

- Proof: Let n be an arbitrary positive integer.
  It follows from Theorem 1 that for $x = -1$ and $y = 1$ that

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Binomial Coefficients

- Corollary 3: Let n be a non-negative integer.

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

Binomial Coefficients

- Corollary 3: Let n be a non-negative integer.

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

- Proof: Let n be an arbitrary non-negative integer.
  It follows from Theorem 1 that for $x = 1$ and $y = 2$ that

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

. . .

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

. . .

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

. . .

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

. . .

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Pascal's Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Vandermone's Identity

- <u>Theorem 3:</u> (Vandermonde's Identity) Let m, n, and r be non-negative integers with $r \leq m$ and $r \leq n$.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Vandermone's Identity

- Theorem 3: (Vandermonde's Identity) Let m, n, and r be non-negative integers with $r \leq m$ and $r \leq n$.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- Proof: Let m, n, and r be arbitrary non-negative integers such that $r \leq m$ and $r \leq n$. Suppose there are two sets where one set has m items and the second set has n items. Then the number of ways to pick r elements from the union of the sets is

$$\binom{m+n}{r}$$

Vandermone's Identity

- Theorem 3: (Vandermonde's Identity) Let m, n, and r be non-negative integers with $r \leq m$ and $r \leq n$.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- Proof:
Another way to pick r elements from the union of the two sets is to pick k elements from the set containing n elements and $(r-k)$ elements from the set containing m elements. Hence, from the product and sum rule we obtain a total of

$$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Vandermone's Identity

- <u>Theorem 3:</u> (Vandermonde's Identity) Let m, n, and r be non-negative integers with $r \leq m$ and $r \leq n$.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- <u>Proof:</u>
  As both of these two expressions represent the number of ways to pick r elements from the union of a set with m elements and a set with n elements, they are equal to each other. ■

Binomial Coefficients

- Corollary 4: Let n be a non-negative integer.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Binomial Coefficients

- Corollary 4: Let n be a non-negative integer.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

- Proof: Let n be an arbitrary non-negative integer.
  We use Vandermonde's identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k}$$

  Since $\binom{n}{k} = \binom{n}{n-k}$, it follows that the above expression is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

■

Binomial Coefficients

- Theorem 4: Let n and r be non-negative integers such that $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Binomial Coefficients

- Theorem 4: Let n and r be non-negative integers such that $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- Proof: Let n and r be arbitrary non-negative integers such that $r \leq n$. The number of bitstrings of length $(n+1)$ containing $(r+1)$ 1's is

$$\binom{n+1}{r+1}$$

Binomial Coefficients

- Theorem 4: Let n and r be non-negative integers such that $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- Proof:
  The last 1 in this bitstring must occur at position $(r+1)$ or $(r+2)$ or $\ldots$ or $(n+1)$. Moreover, if the last 1 in the bitstring is in position k then there must be r 1's in the first $(k-1)$ positions. The number of bitstrings of length $(k-1)$ with r 1's is

$$\binom{k-1}{r}$$

Binomial Coefficients

- <u>Theorem 4:</u> Let n and r be non-negative integers such that $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- <u>Proof:</u>
  Using the sum rule for each of the possible positions of the last 1 in the bitstring of length $(n+1)$, we obtain

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{n}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Binomial Coefficients

- Theorem 4: Let n and r be non-negative integers such that $r \leq n$.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

- Proof:
  Since both of these expressions count the number of bitstrings of length $(n+1)$ containing $(r+1)$ 1's, they are equivalent. ∎