Ch 6.3: Permutations and Combinations

ICS 141: Discrete Mathematics for Computer Science I

KYLE BERNEY
DEPARTMENT OF ICS, UNIVERSITY OF HAWAII AT MANOA

Permutations

- Definition: A permutation of a set is an ordered arrangement of the objects in the set
- Definition: An r-permutation of a set is an ordered arrangement of r elements in the set

Permutations

- Definition: A permutation of a set is an ordered arrangement of the objects in the set
- Definition: An r-permutation of a set is an ordered arrangement of r elements in the set
- Ex: Let $S = \{1, 2, 3\}$
  - Permutations:
    - 1, 2, 3
    - 1, 3, 2
    - 2, 1, 3
    - 2, 3, 1
    - 3, 1, 2
    - 3, 2, 1

Permutations

- Definition: A permutation of a set is an ordered arrangement of the objects in the set
- Definition: An r-permutation of a set is an ordered arrangement of r elements in the set
- Ex: Let $S = \{1, 2, 3\}$
  - 2-permutations:
    - 1, 2
    - 1, 3
    - 2, 1
    - 2, 3
    - 3, 1
    - 3, 2

Permutations

- Theorem 1: Let n be a positive integer. There are

$$n! = n(n-1)(n-2)\ldots 1$$

permutations of a set with n elements.

Permutations

- Theorem 1: Let n be a positive integer. There are

$$n! = n(n-1)(n-2)\ldots 1$$

permutations of a set with n elements.

- Proof: Let n be an arbitrary positive integer. There are n choices for the first position, $(n-1)$ choices for the second position, $(n-2)$ choices for the third position, ..., and 1 choice for the n-th position. Using the product rule, there are

$$n(n-1)(n-2)\ldots 1 = n!$$

total permutations of n elements. ∎

Permutations

- Theorem 2: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$. There are

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n elements

Permutations

- Theorem 2: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$. There are

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

  r-permutations of a set with n elements

- Proof: Let n and r be arbitrary positive integers such that $1 \leq r \leq n$. There are n choices for the first item, $(n-1)$ choices for the second item, $(n-2)$ choices for the third item, $\ldots$, and $(n-r+1)$ for the r-th item. Therefore, using the product rule,

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} \quad \blacksquare$$

Permutations

- Ex: How many ways can we select three students from a group of 5 students to stand in line for a picture? How many ways can we arrange all five of these students in a line for a picture?

Permutations

- Ex: How many ways can we select three students from a group of 5 students to stand in line for a picture? How many ways can we arrange all five of these students in a line for a picture?

- Solution:

$$P(5,3) = 5 \cdot 4 \cdot 3 = 60$$

and

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Therefore, there are 60 ways to arrange three students in a line from a group of 5 students and 120 ways to arrange all five students in a line.

Permutations

- Ex: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations

- Ex: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

- Solution: Since it matters which person wins which prize, this is equivalent to the number of 3-permutations out of a set of 100 elements.

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970200$$

Permutations

- Ex: Suppose that a salesperson has to visit eight different cities. They must begin their trip in a designated city, but can visit the other seven cities in any order. How many possibe orders can the salesperson use when visiting all cities?

Permutations

- Ex: Suppose that a salesperson has to visit eight different cities. They must begin their trip in a designated city, but can visit the other seven cities in any order. How many possibe orders can the salesperson use when visiting all cities?

- Solution: Because the first city is designated, the number of possible paths between the cities is the number of permutations of seven elements.

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

Combinations

- Definition: An r-combination of a set is an unordered arrangement of r elements in the set
- Equivalently, an r-combination is a subset of r elements
- Ex: $S = \{1, 2, 3\}$
  - 3-combinations:
    - $\{1, 2\}$
    - $\{1, 3\}$
    - $\{2, 3\}$

Combinations

- Theorem 3: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$. There are

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

r-combinations of a set with n elements

Combinations

- Proof: Let n and r be arbitrary positive integers such that $1 \leq r \leq n$. For each of the $C(n, r)$ r-combinations, we can reorder them in $P(r, r) = r!$ ways. Hence,

$$\begin{aligned} r! \cdot C(n, r) &= P(n, r) \\ \Rightarrow r! \cdot C(n, r) &= \frac{n!}{(n-r)!} \\ \Rightarrow C(n, r) &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r} \end{aligned}$$

■

Combinations

- Corollary 1: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$.

$$C(n, r) = C(n, n - r)$$

Combinations

- Corollary 1: Let n be a positive integer and r be an integer such that $1 \leq r \leq n$.

$$C(n, r) = C(n, n - r)$$

- Proof: It follows from Theorem 3 that

$$C(n, r) = \binom{n}{r}$$

and $$C(n, n - r) = \binom{n}{n - r}$$

Hence,

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n - r)!} = \binom{n}{n - r} = C(n, n - r)$$

■

Combinations

- Ex. How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Combinations

- Ex. How many poker hands of five cards can be dealt from a standard deck of 52 cards?
- Solution: Because the order of the five cards do not matter, the number of poker hands are

$$
\begin{aligned}
C(52,5) &= \binom{52}{5} \\
&= \frac{52!}{5!47!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{5!47!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 2,598,960
\end{aligned}
$$

Combinations

- Ex: How many ways are there to select a group of 6 people from a class of 30 students?

Combinations

- Ex: How many ways are there to select a group of 6 people from a class of 30 students?

- Solution: Because the order of students in the group does not matter, the number of ways to select a group of 6 from 30 is

$$\begin{aligned} C(30,6) &= \binom{30}{6} \\ &= \frac{30!}{6!24!} \\ &= \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24!}{6!24!} \\ &= \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 593,775 \end{aligned}$$

Combinations

- Ex: Suppose there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee consisting of three faculty members from the mathematics department and four from the computer science department?

Combinations

- Ex: Suppose there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee consisting of three faculty members from the mathematics department and four from the computer science department?

- Solution: Since the order of faculty members do not matter, using the product rule we find that the number of ways to select the committee is

$$\begin{aligned} C(9,3) \cdot C(11,4) &= \binom{9}{3} \cdot \binom{11}{4} \\ &= \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} \\ &= 84 \cdot 330 = 27,720 \end{aligned}$$