INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 6.1: The Basics of Counting

ICS 141: Discrete Mathematics for Computer Science I

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Basic Counting Principles

- Given two tasks, such that
  - n_1 ways of doing the first task
  - n_2 ways of doing the second task
- Product Rule: There are $n_1 n_2$ ways of doing the first task and the second task
- Sum Rule: There are $n_1 + n_2$ ways of doing the first task or the second task
- *Intuitively:*
  - and $\Rightarrow$ multiply
  - or $\Rightarrow$ add

Product Rule

- <u>Ex:</u> The chairs of an auditorium are labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the maximum number of chairs that can be lableled uniquely?

Product Rule

- Ex: The chairs of an auditorium are labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the maximum number of chairs that can be lableled uniquely?

- Solution:
  - 26 different uppercase letters
  - 100 different positive integers not exceeding 100
  - Using the product rule:

$$26 \cdot 100 = 2600$$

  - Therefore, there are a maximum of 2600 chairs that can be labeled uniquely

Product Rule

- Ex: There are 32 computers in a data center. Each computer has 24 ports. How many different computer ports are there in this data center?

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- Solution:
  - 32 computers
  - 24 ports per computer
  - Using the product rule:

$$32 \cdot 24 = 768$$

  - Therefore, there are 768 different computer ports in this data center

Product Rule

- Ex: How many different bit strings of length seven are there?

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- Solution:
  - 2 ways to choose a bit (0 or 1)
  - 7 total bits
  - Using the product rule:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

  - Therefore, there are 128 different bit strings of length seven

Product Rule

- Ex: Given a set S such that $|S| = n$. What is the total number of subsets? In other words, what is the cardinality of $\mathcal{P}(S)$?

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- Solution:
  - Two options for each element, be included in a particular subset or not included
  - n total elements
  - Using the product rule:

$$\underbrace{2 \cdot 2 \ldots \cdot 2}_{n \text{ times}} = 2^n$$

  - Therefore, $|\mathcal{P}(S)| = 2^n$

Sum Rule

- Ex: A computer science representative to a university committe can be either a member of the computer science faculty or a student who is a computer science major. If there are 37 members of the computer science faculty and 83 computer science majors, how many differenent choices are there for this representative?

Sum Rule

- Ex: A computer science representative to a university committe can be either a member of the computer science faculty or a student who is a computer science major. If there are 37 members of the computer science faculty and 83 computer science majors, how many different choices are there for this representative?
- Solution:
  - 37 faculty members
  - 83 students
  - Using the sum rule:

$$37 + 83 = 120$$

  - Therefore, there are 120 different choices for this representative

Sum Rule

- <u>Ex:</u> A student can choose a project by selecting a project from one of three lists. The three lists contain 23, 15, and 19 projects respectively. No project is on more than one list. How many possible projects are there to choose from?

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- Solution:
  - 23 projects on the first list
  - 15 projects on the second list
  - 19 projects on the third list
  - Using the sum rule:

$$23 + 15 + 19 = 57$$

  - Therefore, there are 57 different projects to choose from

More Complex Counting Problems

- Ex: In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An alphanumeric character is either a letter or a number.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

More Complex Counting Problems

- Solution:
  - Let V be the number of different variable names in this version of BASIC
  - Let V_1 be the number of these variable names that are one character long
    - Variables must start with a letter, therefore, $V_1 = 26$
  - Let V_2 be the number of these variable names that are two characters long
    - There are 26 letters for the first character
    - There are 26 + 10 choices for the second character
    - Therefore, excluding the five reserved strings of length two used for programming,
      $V_2 = (26 \cdot (26 + 10)) - 5 = 931$

More Complex Counting Problems

- Solution:
  - $V_1 = 26$
  - $V_2 = 931$
  - Choose to use either a variable of length one or two, therefore there are

$$V = V_1 + V_2 = 26 + 931 = 957$$

  - different names for variables in this version of BASIC

More Complex Counting Problems

- Ex: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or digit. Each password must contain at least one digit. How many possible passwords are there?

More Complex Counting Problems

- Solution:
  - Let P be the total number of possible passwords
  - Let P_6 be the total number of passwords of length 6
  - Let P_7 be the total number of passwords of length 7
  - Let P_8 be the total number of passwords of length 8

More Complex Counting Problems

- Solution:
  - Consider P_6
  - $26 + 10 = 36$ different alphanumeric characters
  - $36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$ strings of length 6 using alphanumeric characters
  - $26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$ strings of length 6 using only letters
  - Therefore, since we require passwords to have at least a single digit we exclude strings of length 6 that use only letters to find that

$$P_6 = 36^6 - 26^6 = 1,867,866,560$$

More Complex Counting Problems

- Solution:
  - Similarly, we have that

$$P_7 = 36^7 - 26^7 = 70,332,353,920$$

$$P_8 = 36^8 - 26^8 = 2,612,282,842,880$$

  - Therefore,

$$P = P_6 + P_7 + P_8 = 2,684,483,063,360$$

Subtraction Rule

- Suppose that a task can be done in one of two ways, but some ways are common to both ways
- If we use the sum rule, then we overcount the total number of ways to do the task since some tasks will be counted twice
- Hence, we must exclude the tasks that are counted twice

- Subtraction Rule: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to the task that are common to the two different ways

Principle of Inclusion-Exclusion

- Subtraction rule is also known as the principle of inclusion-exclusion

- Let A_1 and A_2 be sets

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Subtraction Rule

- Ex: How many bit strings of length 8 either start with a 1 bit or end with two 00 bits?

Subtraction Rule

- Solution:
  - Fix the starting 1 bit, leaving 7 bits
    - 2^7 different bit string of length 8 starting with a 1 bit
  - Fix the ending two 00 bits, leaving 6 bits
    - 2^6 different bit string of length 8 ending with two 00 bits
  - Fix both the starting 1 bit and ending two 00 bits, leaving 5 bits
    - 2^5 different bit string of length 8 starting with a 1 bit and ending with two 00 bits
  - Using the subtraction rule,

$$2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$$

Principle of Inclusion-Exclusion

- Ex: A computer company receives 350 applications for a job. Of these applicants, 220 majored in computer science, 147 majored in business, and 51 majored in both computer science and business. How many of these applicants majored neither in computer science nor business.

Principle of Inclusion-Exclusion

- Solution:
  - Let A_1 be the set of students who majored in computer science
  - Let A_2 be the set of students who majored in business

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316$$

  - Therefore, $350 - 316 = 34$ of these applicants majored neither in computer science nor in business

Division Rule

- Division Rule: There are n/d ways to do a task if it can be done in n ways, and for every way w, exactly d of the n ways correspond to way w

Division Rule

- Ex: Suppose that an automated system has been developed that counts the legs of cows in a pasture. The automated system has determined that in a farmer's pasture, there are exactly 572 legs. How many cows are there in this pasture, assuming that each cow has four legs and there are no other animals present?

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- Solution:
  - Each cow has four legs

$$572/4 = 143$$

  - Therefore, there are 143 cows in the pasture

Division Rule

- <u>Ex:</u> How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and right neighbor?

Division Rule

- Solution:
  - Select an arbitrary seat and label it seat 1
  - Label remainder seats in numerical order, in a clockwise manner
  - There are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ ways to seat four people
  - Each of the four choices for seat 1 leads to the same arrangement (i.e., same left and right neighbors)
  - Using the division rule, there are $24/4 = 6$ different seating arrangements of four people around a circular table