INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 5.1 & Ch 5.2: Mathematical Induction

ICS 141: Discrete Mathematics for Computer Science I

KYLE BERNEY
DEPARTMENT OF ICS, UNIVERSITY OF HAWAII AT MANOA

Proof by Induction

- Induction is a proof method for proving universally quantified proposition
  - Statements about all elements contained in a countable set
  - Ex: $\forall n \in \mathbb{N}\ (P(n))$
- Induction is one of the most useful tools for developing and analyzing algorithms
  - Correspondence between induction and recursive algorithms
  - Every iterative algorithm can be written recursively (and vice versa)

Proof by Induction

- Composed of three main parts:
  - Inductive Hypothesis (I.H.):
    - Assumption that the proposition is true for some subset of values
  - Base Case(s) (or Basis Step(s)):
    - Prove that the proposition is true for “small” values
  - Inductive Case (or Inductive Step):
    - Prove that the proposition is true for all values that are not considered in the base case(s) using the Inductive Hypothesis

Weak vs. Strong Induction

- Given a proposition $P(n)$ for all values of n in a countable set
  - Ex: $n \in \mathbb{Z}^+$
- Weak Induction
  - Inductive Hypothesis: Assume inductively that $P(k)$ is true for $k = n - 1$.
  - $P(n-1) \Rightarrow P(n)$
- Strong Induction
  - Inductive Hypothesis: Assume inductively that $P(k)$ is true for $0 < k < n$.
  - $(P(1) \wedge P(2) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)$

Weak vs. Strong Induction

- Given a proposition $P(n)$ for all values of n in a countable set
  - Ex: $n \in \mathbb{Z}^+$
- Weak Induction
  - Inductive Hypothesis: Assume inductively that $P(k)$ is true for $k = n - 1$.
  - $P(n-1) \Rightarrow P(n)$
- Strong Induction
  - Inductive Hypothesis: Assume inductively that $P(k)$ is true for $0 < k < n$.
  - $(P(1) \wedge P(2) \wedge \ldots \wedge P(n-1)) \Rightarrow P(n)$
- In this course, ALWAYS use strong induction

Axiom of Induction

- Induction is a valid proof technique because of the well-ordering property (Appendix 1)
  - Every non-empty subset of the set of positive integers contains a least element

Axiom of Induction

1. Assume for the sake of contradiction that there exists a positive integer x such that $P(x)$ is false
2. The set of positive integers for which $P(n)$ is false is non-empty
3. By the well-ordering property, there exists a least element y such that $P(y)$ is false
4. We know that $P(1)$ is true, from the base case
5. Hence, $y > 1$ and $P(y - 1)$ must be true
6. Using the inductive hypothesis, we can show that $P(y - 1) \Rightarrow P(y)$
7. $P(y)$ is true which is a contradiction. Therefore, the assumption that $P(x)$ is false is wrong.

Boilerplate Template

- Proposition: $P(n)$ for every $n \in \mathbb{N}$
- Proof: (By induction)
  Let n be an arbitrary integer such that $n \in \mathbb{N}$.
  Inductive Hypothesis (I.H.): Assume inductively that for all integers k, such that $0 \leq k < n$, $P(k)$ is true.
  Case 1: (Base Case)
  Assume that $n = 0$. [Prove base case here]. Therefore $P(0)$ is true.
  Case 2: (Inductive Case)
  Assume that $n > 0$. [Prove inductive case here using the inductive hypothesis]. Therefore $P(n)$ is true. ■

Boilerplate Template

- Important notes:
  - Boilerplate shown only has a single base case
  - Some proofs by induction will have multiple base cases
  - The values of n assumed in the base case(s) and inductive case will change depending on the proposition
- When performing your proof sketch, start with the inductive case in order to figure out what the base cases are
- In this course, <u>ALWAYS</u> structure your proof by induction using the boilerplate

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Proof: Let n be an arbitrary positive integer.
  Inductive Hypothesis: Assume inductively that for all integers k, such that $1 \leq k < n$, $P(k)$ is true. In other words,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Proof:

  Base Case: Assume $n = 1$.

$$\sum_{i=1}^{1} i = 1$$

$$= \frac{1(2)}{2}$$

$$= \frac{n(n+1)}{2}.$$

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Proof:
  Inductive Case: Assume $n > 1$.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-1) + n$$

$$= \left(\sum_{i=1}^{n-1} i \right) + n \, .$$

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Proof:
From our inductive hypothesis, we know that for $1 \leq n-1 < n$,

$$\sum_{i=1}^{n-1} i = \frac{(n-1)(n-1+1)}{2}$$

$$= \frac{n(n-1)}{2} .$$

Exercises

- Proposition: For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Proof:

Hence,

$$\left(\sum_{i=1}^{n-1} i\right) + n = \frac{n(n-1)}{2} + n$$

$$= \frac{n(n-1)}{2} + \frac{2n}{2}$$

$$= \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}. \qquad \blacksquare$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof: Let n be an arbitrary integer such that $n > 23$.
  Inductive Hypothesis: Assume inductively that for all integers k, such that $23 < k < n$, $P(k)$ is true. In other words, we can make a postage of k-cents using an unlimited supply of 5-cent and 7-cent stamps.

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof:
  Case 1: (Base Case) Assume that $n = 24$.
  We use two 7-cent stamps and two 5-cent stamps,

$$7 + 7 + 5 + 5 = 24 .$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.

- Proof:
  Case 2: (Base Case) Assume that $n = 25$.
  We use five 5-cent stamps,

$$5 + 5 + 5 + 5 + 5 = 25 .$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof:
  Case 3: (Base Case) Assume that $n = 26$.
  We use three 7-cent stamps and one 5-cent stamp,

$$7 + 7 + 7 + 5 = 26 \ .$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof:
  Case 4: (Base Case) Assume that $n = 27$.
  We use one 7-cent stamps and four 5-cent stamp,

$$7 + 5 + 5 + 5 + 5 = 27 \; .$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof:
  Case 5: (Base Case) Assume that $n = 28$.
  We use four 7-cent stamps,

$$7 + 7 + 7 + 7 = 28 \; .$$

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- Proof:
  Case 6: (Inductive Case) Assume that $n > 28$.
  We choose to first use a 5-cent stamp, leaving $(n - 5)$ cents remaining. From our inductive hypothesis, we know that for $23 < n - 5 < n$, we can make a postage of $(n - 5)$ cents using an unlimited supply of 7-cent and 5-cent stamps. Therefore, we can make a postage of n cents. ■

Exercises

- Proposition: Given an unlimited supply of 5-cent stamps and 7-cent stamps, we can make any postage larger than 23 cents.
- *Question:* Is there another way to construct this inductive proof?

Exercises

- Proposition: For every integer $n \geq 4$,
$$2^n < n!\,.$$

Exercises

- Proposition: For every integer $n \geq 4$,

$$2^n < n!\ .$$

- Proof: Let n be an arbitrary integer such that $n \geq 4$.
  Inductive Hypothesis: Assume inductively that for all integers k, such that $4 \leq k < n$, $P(k)$ is true. In other words,

$$2^k < k!$$

  Base Case: Assume $n = 4$.
  $2^n = 2^4 = 16$ and $n! = 4! = 24$, therefore,

$$2^n = 16 < 24 = n!$$

Exercises

- Proposition: For every integer $n \geq 4$,
$$2^n < n!\ .$$

- Proof:
Inductive Case: Assume $n > 4$.
From our inductive hypothesis, we know that for $4 \leq n - 1 < n$,
$$2^{n-1} < (n-1)!$$
Hence,
$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &< 2 \cdot (n-1)! \\ &< n \cdot (n-1)! \\ &= n!\ . \end{aligned}$$

■

Exercises

- Proposition: (Bernoulli's Inequality) Let $n \in \mathbb{Z} - \mathbb{Z}^-$. If $h \in \mathbb{R}$ such that $h > -1$, then
$$1 + nh \leq (1 + h)^n .$$

Exercises

- Proposition: (Bernoulli's Inequality) Let $n \in \mathbb{Z} - \mathbb{Z}^{-}$. If $h \in \mathbb{R}$ such that $h > -1$, then
$$1 + nh \leq (1 + h)^n \ .$$

- Proof: Let n be an arbitrary non-negative integer.
Inductive Hypothesis: Assume inductively that for all integers k, such that $0 \leq k < n$, $P(k)$ is true. In other words,
$$1 + kh \leq (1 + h)^k \ .$$
Base Case: Assume $n = 0$.
$1 + nh = 1 + 0 \cdot h = 1$ and $(1 + h)^n = (1 + h)^0 = 1$. Therefore,
$$1 + nh = 1 \leq 1 = (1 + h)^n \ .$$

Exercises

- Proposition: (Bernoulli's Inequality) Let $n \in \mathbb{Z} - \mathbb{Z}^{-}$. If $h \in \mathbb{R}$ such that $h > -1$, then
$$1 + nh \leq (1 + h)^n \, .$$
- Proof:
Inductive Case: Assume $n > 0$.
From our inductive hypothesis, we know that for $0 \leq n - 1 < n$,
$$1 + (n-1)h \leq (1 + h)^{n-1} \, .$$
Hence,
$$\begin{aligned}(1+h)^n &= (1+h)^{n-1}(1+h) \\ &\geq (1 + (n-1)h)(1+h) \\ &= 1 + h + (n-1)h + (n-1)h^2 \\ &= 1 + nh + (n-1)h^2 \\ &\geq 1 + nh \, .\end{aligned}$$
■