INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 2.6: Matrices

ICS 141: Discrete Mathematics for Computer Science I

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Matrix

- Definition: A $m \times n$ matrix is an array of numbers with m rows and n columns
- A matrix with $m = n$ is called a square matrix
- Two matrices are equal if they have
  - The same number of rows and columns
  - All entries are the same
- Ex: 3×2 matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Terminology

- Let $m, n \in \mathbb{Z}^+$ and let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

- The i-th row of A is the $1 \times n$ matrix

$$\begin{bmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{bmatrix}$$

Terminology

- Let $m, n \in \mathbb{Z}^+$ and let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

- The j-th column of A is the $n \times 1$ matrix

$$\begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{m,j} \end{bmatrix}$$

Terminology

- Let $m, n \in \mathbb{Z}^+$ and let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

- The (i, j)-th element or entry is the element $a_{i,j}$ located in the i-th row and j-th column of A.

Matrix Arithmetic

- Definition: The sum of two $m \times n$ matrices A and B, denoted $A + B$, is the $m \times n$ matrix with entries $a_{i,j} + b_{i,j}$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{m,1} & b_{m,2} & \dots & b_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \dots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \dots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \dots & a_{m,n} + b_{m,n} \end{bmatrix}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{aligned} c_{1,1} &= 1 \cdot 2 + 0 \cdot 1 + 4 \cdot 3 \\ &= 14 \end{aligned}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{aligned} c_{1,2} &= 1 \cdot 4 + 0 \cdot 1 + 4 \cdot 0 \\ &= 4 \end{aligned}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{aligned} c_{2,1} &= 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 \\ &= 8 \end{aligned}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ ? & ? \\ ? & ? \end{bmatrix}$$

$$c_{2,2} = 2 \cdot 4 + 1 \cdot 1 + 1 \cdot 0$$
$$= 9$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & ? \\ ? & ? \end{bmatrix}$$

$$\begin{aligned} c_{3,1} &= 3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3 \\ &= 7 \end{aligned}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ ? & ? \end{bmatrix}$$

$$c_{3,2} = 3 \cdot 4 + 1 \cdot 1 + 0 \cdot 0$$
$$= 13$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & ? \end{bmatrix}$$

$$\begin{aligned} c_{4,1} &= 0 \cdot 2 + 2 \cdot 1 + 2 \cdot 3 \\ &= 8 \end{aligned}$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Ex:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

$$c_{4,2} = 0 \cdot 4 + 2 \cdot 1 + 2 \cdot 0$$
$$= 2$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Matrix multiplication is associative

$$(AB)C = A(BC)$$

Matrix Arithmetic

- Definition: The product of an $m \times k$ matrix A and a $k \times n$ matrix B, denoted AB, is the $m \times n$ matrix C with entries

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,k}b_{k,j}$$

- Matrix multiplication is NOT commutative

$$AB \neq BA$$

Identity Matrix

- Definition: The identity matrix of order n is the $n \times n$ matrix I_n with its diagonal entries set to 1 and all other entries set to 0

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

- Let A be an $m \times n$ matrix, then

$$AI_n = I_mA = A$$

Powers of Square Matrices

- Let A be a $n \times n$ square matrix

$$A^0 = I_n$$
$$A^r = \underbrace{AAA \ldots A}_{r \text{ times}}$$

Matrix Transpose

- Let A be an $m \times n$ matrix.
- Definition: The transpose of A, denoted A^T, is an $n \times m$ matrix obtained by swapping the rows and columns of A
  - Entry $a_{i,j}$ in A gets swapped into the j-th row and i-th column in the transpose of A
- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix Transpose

- Let A be an $m \times n$ matrix.
- Definition: The transpose of A, denoted A^T, is an $n \times m$ matrix obtained by swapping the rows and columns of A
  - Entry $a_{i,j}$ in A gets swapped into the j-th row and i-th column in the transpose of A
- Definition: A square matrix A is called symmetric if $A = A^T$
- Ex:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Zero-One Matrices

- Definition: A matrix with all entries being either 0 or 1 is called a zero-one matrix
- We can use logical bit operators:
  - Definition: The join of two zero-one matrices A and B, denoted $A \vee B$, is the zero-one matrix with entries being $a_{i,j} \vee b_{i,j}$
  - Definition: The meet of two zero-one matrices A and B, denoted $A \wedge B$, is the zero-one matrix with entries being $a_{i,j} \wedge b_{i,j}$

Boolean Product

- Let A be a $m \times k$ zero-one matrix
- Let B be a $k \times n$ zero-one matrix
- Definition: The boolean product of A and B, denoted $A \odot B$, is the $m \times n$ matrix with entries

$$c_{i,j} = (a_{i,1} \wedge b_{1,j}) \vee (a_{i,2} \wedge b_{2,j}) \vee \ldots \vee (a_{i,k} \wedge b_{k,j})$$

- Similar to matrix multiplication, except
  - Additions are replaced with $\vee$ (OR)
  - Multiplications are replace with $\wedge$ (AND)

Boolean Powers

- Let A be a square $n \times n$ matrix
- Let $r \in \mathbb{Z}^+$
- Definition: The r-th boolean power of A, denoted $A^{[r]}$, is

$$A^{[0]} = I_n$$

$$A^{[r]} = \underbrace{A \odot A \odot A \odot \ldots \odot A}_{r \text{ times}}$$