INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 2.5: Cardinality of Sets

ICS 141: Discrete Mathematics for Computer Science I

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Cardinality of Finite and Infinite Sets

- Recall that the cardinality of a set A is the number of distinct elements in A
  - We can extend the notion of cardinality to infinite sets using function mappings

Cardinality of Finite and Infinite Sets

- Recall that the cardinality of a set *A* is the number of distinct elements in *A*
  - We can extend the notion of cardinality to infinite sets using function mappings
- Definition: Two sets *A* and *B* have the same cardinality if and only if there is a bijection from *A* to *B*
- Definition: The cardinality of the set *A* is less than or equal to the cardinality of set *B* if there is a injection from *A* to *B*

Countable Sets

- Definition: A set that is either finite or has the same cardinality as the set of positive integers is called countable
    - A set that is not countable is called uncountable
- An infinite set that is countable has cardinality $\aleph_0$ ("aleph null")

Countable Sets

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  $f(n) = 2n - 1$, for $n \in \mathbb{Z}^+$.
  Suppose that $f(n) = f(m)$, then
  $$2n - 1 = 2m - 1$$
  $$\Rightarrow n = m \ .$$
  Hence, f is injective.

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- Proof: To show that the set of positive odd integers is countable, we need to show that there exists a bijection between the set O and $\mathbb{Z}^+$. Consider the function
  $f(n) = 2n - 1$, for $n \in \mathbb{Z}^+$.
  Suppose that $t \in O$, then t is 1 less than some even integer $x = 2k$. Hence, $t = x - 1 = 2k - 1 = f(k)$ and f is a surjection.

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- Proposition: The set of odd integers, O, is countable.

- Proof: To show that the set of positive odd integers is countable, we need to show that there exists a bijection between the set O and $\mathbb{Z}^+$. Consider the function
  $f(n) = 2n - 1$, for $n \in \mathbb{Z}^+$.
  We showed that f is both an injection and surjection and therefore is a bijection. ■

Countable Sets

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers)
  - This defines a bijection from $\mathbb{Z}^+$ to the considered infinite set

Countable Sets

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- Proposition: The set of all integers, $\mathbb{Z}$, is countable.
- Proof: We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers:

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \qquad \blacksquare$$

Uncountable Sets

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Uncountable Sets

- Proposition: The set of real numbers, $\mathbb{R}$, is uncountable.
- Lemma: Every subset of a countable set is also countable.

Uncountable Sets

- Proposition: The set of real numbers, $\mathbb{R}$, is uncountable.

- Proof: (Cantor Diagonalization Argument)
  Assume for the sake of contradiction that $\mathbb{R}$ is countable.
  Consider the open interval $(0, 1)$ which is a subset of $\mathbb{R}$.
  From our lemma, $(0, 1)$ is countable and can be enumerated in some order $r_1, r_2, r_3, \ldots$. Let us list the decimal representations:

Uncountable Sets

- Proposition: The set of real numbers, $\mathbb{R}$, is uncountable.

- Proof: (Cantor Diagonalization Argument)

$$
\begin{aligned}
r_1 &= 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}\ldots \\
r_2 &= 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}\ldots \\
r_3 &= 0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}\ldots \\
r_4 &= 0.d_{4,1}d_{4,2}d_{4,3}d_{4,4}\ldots \\
&\vdots
\end{aligned}
$$

Where $d_{i,j} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

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&\vdots
\end{aligned}
$$

Where $d_{i,j} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We will construct a new real number $r \in (0, 1)$ such that $r \neq r_i$ for all $i \in \mathbb{Z}^+$.

Uncountable Sets

- Proposition: The set of real numbers, $\mathbb{R}$, is uncountable.
- Proof: (Cantor Diagonalization Argument)
  Let $r = 0.d_1 d_2 d_3 d_4 \ldots$ where
  $$d_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{d_{i,i}\}$$
  for all $i \in \mathbb{Z}^+$.

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  Let $r = 0.d_1 d_2 d_3 d_4 \ldots$ where

  $$d_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{d_{i,i}\}$$

  for all $i \in \mathbb{Z}^+$. Notice that r and r_i disagrees at the i-th decimal place. Therefore, $r \in (0, 1)$ and $r \neq r_i$ for all i. A contradiciton, since we assumed that we listed all elements in $(0, 1)$. ■

Results About Cardinality

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  Case 1: Assume that A and B are both finite. Thus, $A \cup B$ is also finite and therefore countable.

Results About Cardinality

- Theorem: If A and B are countable sets, then $A \cup B$ is also countable.
- Proof:
  Case 2: Without loss of generality, assume that A is finite with cardinality $|A| = n$ and B is countably infinite. We list the elements of $A \cup B$ as a sequence $a_1, a_2, a_3, \ldots, a_n, b_1, b_2, \ldots$. Thus, $A \cup B$ is countably infinite.

Results About Cardinality

- Theorem: If A and B are countable sets, then $A \cup B$ is also countable.
- Proof:
  Case 3: Assume that both A and B are countably infinite. We list the elements of $A \cup B$ as a sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$. Hence, $A \cup B$ is countably finite. ■

Results About Cardinality

- Theorem: If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if exists an injection from A to B and from B to A then there exists a bijection between A and B.
- Proof: Out-of-scope of this course.