INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

Ch 2.4: Sequences and Summations

ICS 141: Discrete Mathematics for Computer Science I

KYLE BERNEY
DEPARTMENT OF ICS, UNIVERSITY OF HAWAII AT MANOA

Sequences

- Definition: A sequence is a function from the subset of the set of integers ($\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to a set S.
  - Represents an ordered list
- Denote the n-th term of the sequence as a_n
- Ex:
  - Finite Sequence:
    - 1, 2, 3, 5, 8
  - Infinite Sequence:
    - $a_n = 3^n$
    - $1, 3, 9, 27, 81, \ldots$

Geometric Progression

- Let $a, r \in \mathbb{R}$
- Definition: A geometric progression is a sequence of the form, ar^n:

$$a, ar, ar^2, \ldots$$

- a is the initial term
- r is the common ratio

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- Ex:
  - $b_n = (-1)^n$, $a = 1$, $r = -1$
    $1, -1, 1, -1, 1, \ldots$
  - $c_n = 2 \cdot 5^n$, $a = 2$, $r = 5$
    $2, 10, 50, 250, 1250, \ldots$
  - $d_n = 6 \cdot (1/3)^n$, $a = 6$, $r = 1/3$
    $6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

Arithmetic Progression

- Let $a, d \in \mathbb{R}$
- Definition: A arithmetic progression is a sequence of the form, $a + nd$:

$$a, a + d, a + 2d, \ldots$$

- a is the initial term
- d is the common difference

Arithmetic Progression

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- Ex:
  - $s_n = -1 + 4n, a = -1, d = 4$

    $-1, 3, 7, 11, \ldots$
  - $t_n = 7 - 3n, a = 7, d = 3$

    $2, 4, 1, -2, \ldots$

Strings

- A string of length n is a finite sequence

$$a_1 a_2 a_3 \ldots a_n$$

- The empty string, denoted λ, is a string with no terms
  - Length 0
- Ex:
  - (Recall from Ch. 1) Bitstrings: 01001100
  - Strings: *abcd*

Recurrence Relations

- Method for defining sequences by:
  1. Providing one or more initial terms
  2. Rule for determining subsequent terms from terms that preceed it

- Ex:
  - $a_0 = 2$
  - $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$

    $2, 5, 8, 11, 14, \ldots$
  - $a_0 = 3$
  - $a_1 = 5$
  - $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

    $3, 5, 2, -3, -5, -2, \ldots$

Fibonacci Sequence

- Defined recursively:
  - $F_0 = 0$
  - $F_1 = 1$
  - $F_n = F_{n-1} + F_{n-2}$, for $n = 2, 3, 4, \ldots$
- $0, 1, 2, 3, 5, 8, 13, \ldots$

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- $0, 1, 2, 3, 5, 8, 13, \ldots$
- Famous and useful sequence for many fields:
  - Golden Ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi$$

  - Many artists and architects believe that this ratio is aesthetically pleasing

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- Famous and useful sequence for many fields:
  - Spiral patterns in nature (e.g., flower petals, sunflowers, pinecones, and nautilus shells) follow Fibonacci-like growth (logarithmic spirals)

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- Famous and useful sequence for many fields:
  - Predict population growth (e.g., rabbits, population of major cities)

Fibonacci Sequence

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- Famous and useful sequence for many fields:
  - Used in computer science for Fibonacci Heaps

Closed Formula for Recurrence Relations

- We say that we have solved a recurrence relation when we find an explicit formula, called the closed formula, for the terms of the sequence

- Ex: Closed formula for the Fibionacci Sequence

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

- We can prove the correctness of closed formulas using proof by induction (Chapter 5)

Summations

- The addition of terms in a sequence is denoted as:

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

- In this example,
  - k is the index of summation
  - 0 is the lower limit
  - n is the upper limit
- Summations follow the usual laws of arithmetic

$$\sum_{k=0}^{n} (ax_k + by_k) = a \sum_{k=0}^{n} x_k + b \sum_{k=0}^{n} y_k$$

Nested Summations

- Nested summations arise in many contexts
  - Analysis of nested loops
- Evaluate the summations from the inner-most outwards
- Ex:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

Arithmetic Series

- The arithmetic series is the summation

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n$$

$$= \frac{n(n+1)}{2}$$

Sum of Squares and Cubes

- The sum of squares is

$$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \ldots + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

- The sum of cubes is

$$\sum_{k=1}^{n} k^3 = 1 + 8 + 27 + \ldots + n^3$$

$$= \frac{n^2(n+1)^2}{4}$$

Geometric Series

- For $x \in \mathbb{R}$ such that $x \neq 1$, the geometric (or exponential) series is

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n$$

$$= \frac{x^{n+1} - 1}{x - 1}$$

- When the upper limite is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

Harmonic Series

- For $n \in \mathbb{Z}^+$, the n-th harmonic number is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$
$$= \sum_{k=1}^{n} \frac{1}{k}$$
$$= \ln n + O(1)$$

- Asymptotic notations in Chapter 3

Integrating and Differentiating Series

- Additional formulas can be derived by integrating or differentiating previous formulas
- By differentiating both sides of the infinite geometric series and multiplying by x

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \text{ for } |x| < 1$$

Telescoping Series

- Consider the following series

$$\sum_{k=1}^{n} (a_k - a_{k-1})$$

$$= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$$

$$= a_n - a_0$$

- Each of the terms $a_1, a_2, \ldots, a_{n-1}$ is added in exactly once and subtracted out exactly once
- Called a <u>telescoping series</u>

Telescoping Series

- Ex:

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n-1} \left(\frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} \right)$$

$$= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 - \frac{1}{n}$$

Products

- The product of terms in a sequence is denoted as:

$$\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$$

Telescoping Series of Products

- The denominator and numerator of subsequent terms cancel each other out

$$\prod_{k=1}^{n} \left(\frac{k}{k+1} \right)$$
$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n-1}{n}$$
$$= \frac{1}{n}$$