INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 2.3: Functions

ICS 141: Discrete Mathematics for Computer Science I

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Functions

- Let A and B be nonempty sets
- Definition: A function f from A to B, denoted $f : A \to B$, is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element of a of A
- *Remark:* Functions are also called mappings or transformations
  - We say "f maps A to B"

Terminology

- Let $f : A \to B$
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- B is the codomain of f

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- The range (or image) of f is the set of all images of $a \in A$.
  - *Remark:* The range of f is always a subset of the codomain
    - Codomain is the set of all possible values of f
    - Range is the set of all actual values of f

Equality of Functions

- Two functions f and g are equal, denoted $f = g$, if they have
  - The same domain
  - The same codomain
  - Map each element of their domain to the same element in their codomain
- For $f : A \to B$ and $g : A \to B$,

$$f = g \iff \forall a \in A(f(a) = g(a))$$

Real-valued and Integer-valued Functions

- Definition: A function is called real-valued (respectively integer-valued) if its codomain is $\mathbb{R}$ (respectively $\mathbb{Z}$)

- The term “respectively” (or abbreviated as “resp.”) is commonly used in mathmatical writing
  - Allows the writer to make several statements simultaneously that have the same form, but with a few words different

Real-valued and Integer-valued Functions

- Definition: A function is called real-valued (respectively integer-valued) if its codomain is $\mathbb{R}$ (respectively $\mathbb{Z}$)

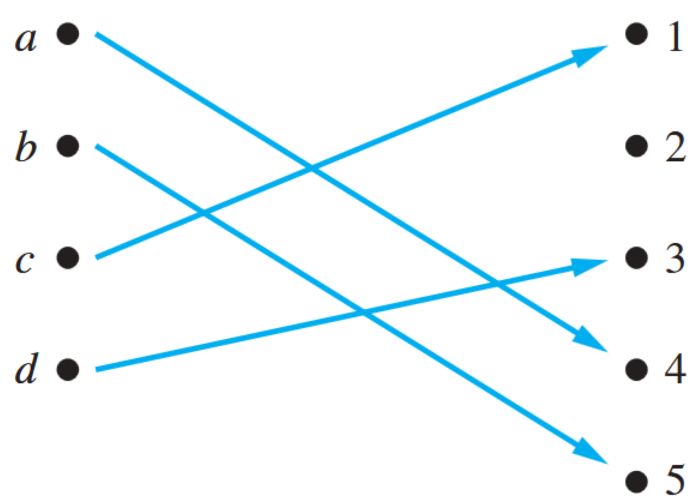
- Real-valued and integer-valued functions with the same domain can be added or multiplied
- Let f_1 and f_2 be functions from A to $\mathbb{R}$ (or $\mathbb{Z}$)
- For all $x \in A$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 \cdot f_2)(x) = f_2(x) \cdot f_2(x)$$

One-to-One

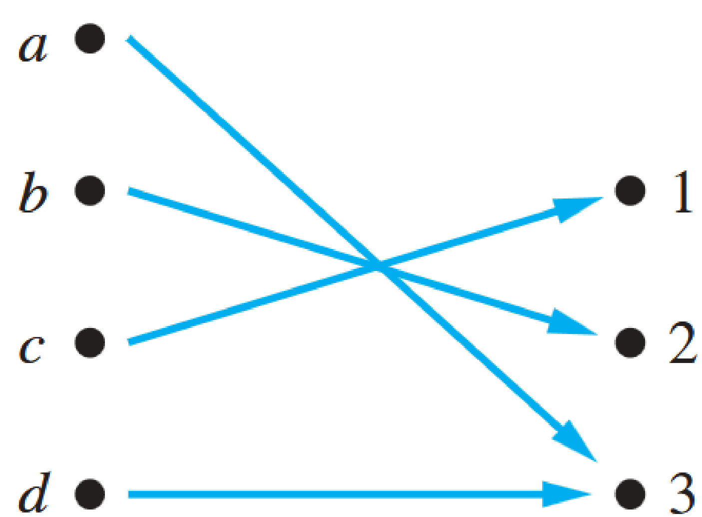
- <u>Definition:</u> A function f is <u>one-to-one</u> if and only if for all values of a and b in the domain of f, $f(a) = f(b) \Rightarrow a = b$.
  - Using the contrapositive, $a \neq b \Rightarrow f(a) \neq f(b)$
- f is one-to-one if it never assigns the same value to two different domain elements
- A function that is one-to-one is also said to be <u>injective</u>



$$f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$$

Onto

- Definition: A function $f : A \to B$ is onto if and only if
$$\forall b \in B\, \exists a \in A(f(a) = b)$$
- f is onto if every element of its codomain is assigned to at least one element of its domain
- A function that is onto is also said to be a surjection



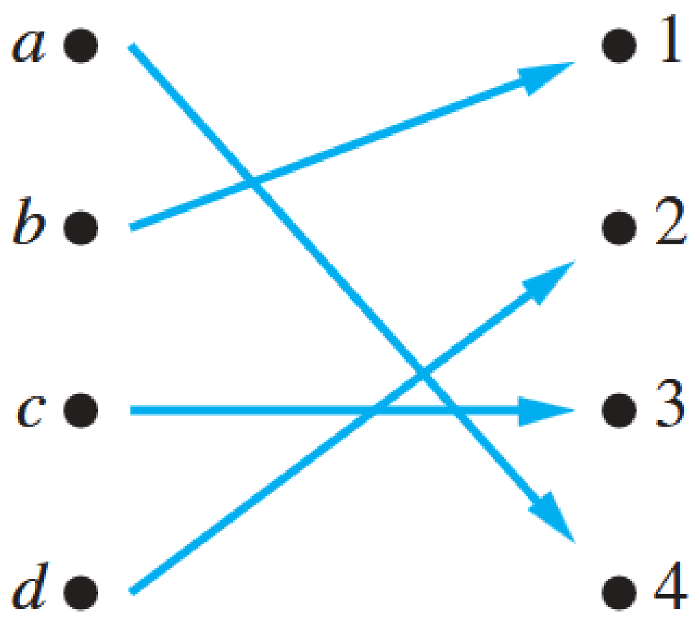
$f : \{a, b, c, d\} \to \{1, 2, 3\}$

Bijection

- Definition: A function f is a bijection if it is both injective and surjective.

$$\forall b \in B\ \exists! a \in A (f(a) = b)$$

- f is a bijection if every element of its codomain is assigned to exactly one element of its domain



$$f : \{a, b, c, d\} \to \{1, 2, 3, 4\}$$

Summary

- Let $f : A \to B$
- To show that f is injective
  - For arbitrary $x, y \in A$, show that $f(x) = f(y) \Rightarrow x = y$
- To show that f is not injective
  - Find $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$
- To show that f is surjective
  - For arbitrary $y \in B$, find $x \in A$ such that $f(x) = y$
- To show that f is not surjective
  - Find $y \in B$ such that $\forall x \in A(f(x) \neq y)$.

Identity Function

- The identity function on a set A is the function $\iota_A : A \to A$, such that for all $x \in A$,

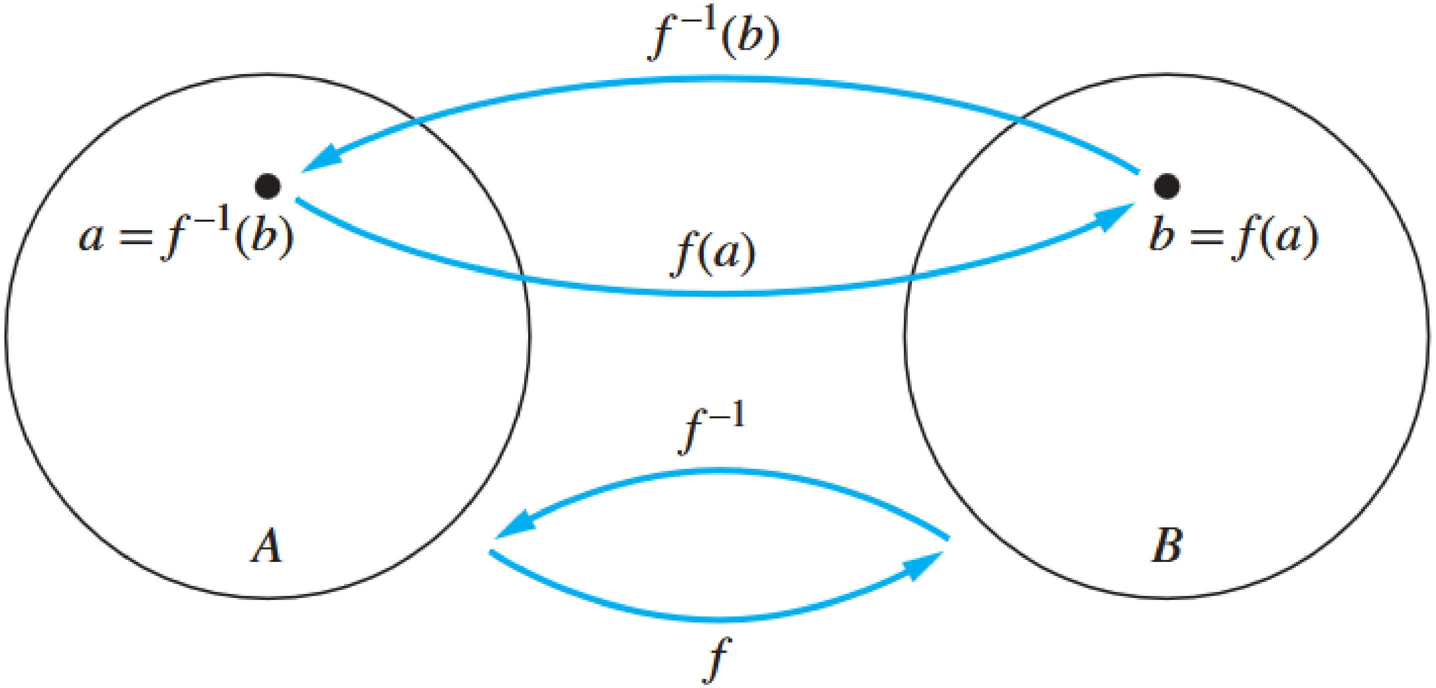
$$\iota_A(x) = x$$

- Assigns each element to itself
- Trivially, ι_A is a bijection
- *Remark:* ι is the Greek letter iota

Inverse Function

- Definition: Let $f : A \to B$ be a bijection. The inverse function, denoted f^{-1}, is the function that assigns an element $b \in B$ to the unique element $a \in A$ such that $f(a) = b$

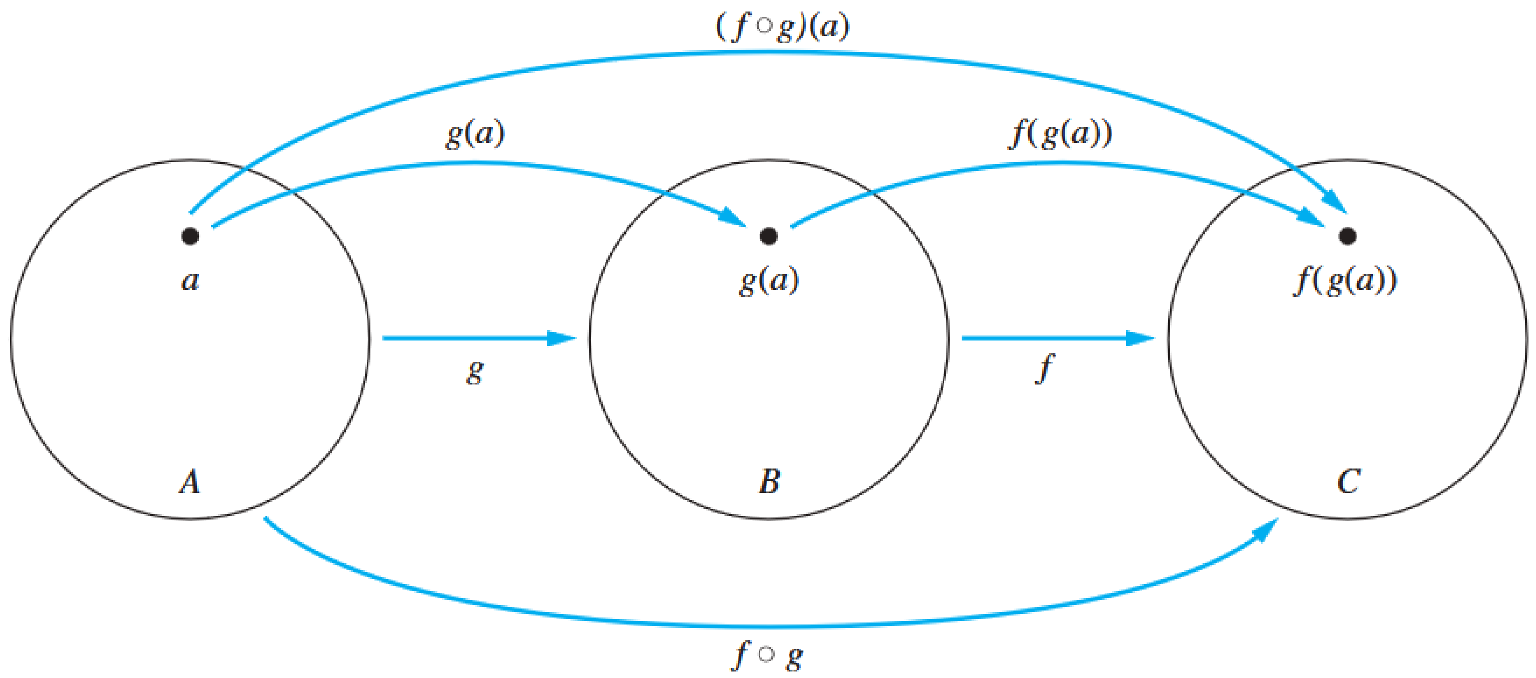
$$f^{-1}(b) = a \iff f(a) = b$$



Compositions of Functions

- Let $g : A \to B$ and $f : B \to C$
- <u>Definition:</u> The <u>composition</u> of the functions f and g, denoted $f \circ g$, is the function from A to C defined as for all $a \in A$

$$(f \circ g)(a) = f(g(a))$$

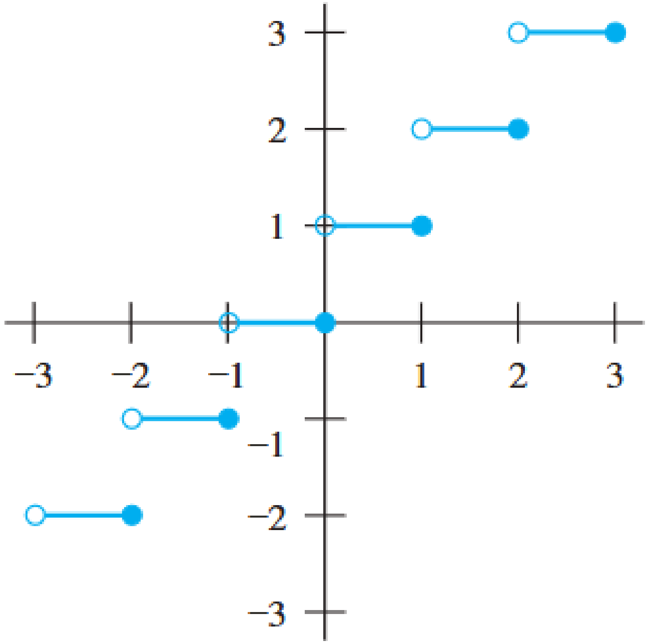


Graph of Functions

- Let $f : A \to B$
- Definition: The graph of the function f is the set of ordered pairs $\{(a, b) : a \in A \text{ and } f(a) = b\}$.
  - Subset of the Cartesian product $A \times B$
- Recall: From geometry, the (x, y) plane is called the Cartesian coordinate system
  - $x, y \in \mathbb{R}$

Ceiling and Floor Functions

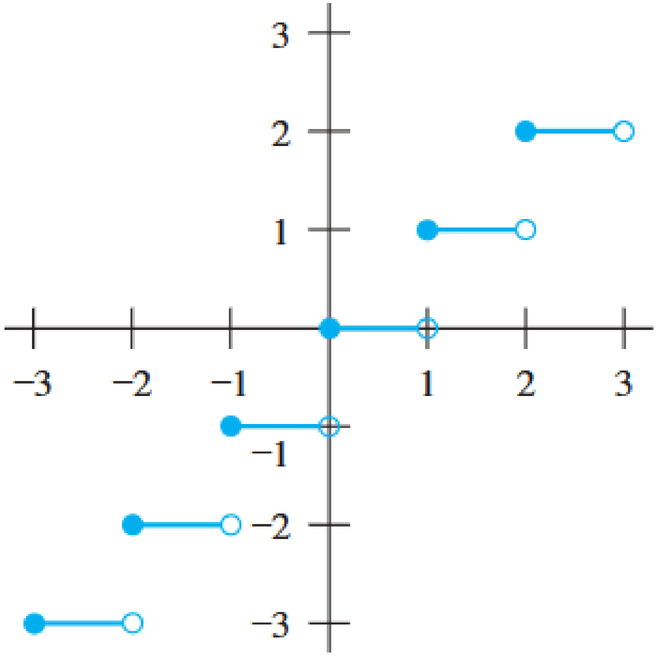
- Let $x \in \mathbb{R}$
- Definition: The ceiling function assigns x to the smallest integer that is greater than or equal to x
  - Denoted $\lceil x \rceil$



$y = \lceil x \rceil$

Ceiling and Floor Functions

- Let $x \in \mathbb{R}$
- Definition: The floor function assigns x to the largest integer that is less than or equal to x
  - Denoted $\lfloor x \rfloor$



$y = \lfloor x \rfloor$

Ceiling and Floor Functions

- Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$
- Useful properties:
  - $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
  - $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
  - $\lceil -x \rceil = -\lfloor x \rfloor$
  - $\lfloor -x \rfloor = -\lceil x \rceil$
  - $\lceil x + n \rceil = \lceil x \rceil + n$
  - $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

Factorial

- Let $n \in \mathbb{Z}$
- Definition: The factorial function is the function $f : \mathbb{N} \to \mathbb{Z}^+$, denoted $f(n) = n!$, is defined as

$$f(n) = \begin{cases} n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

- Will be used heavily in Chapter 6: Counting