Ch 2.2: Set Operations

ICS 141: Discrete Mathematics for Computer Science I

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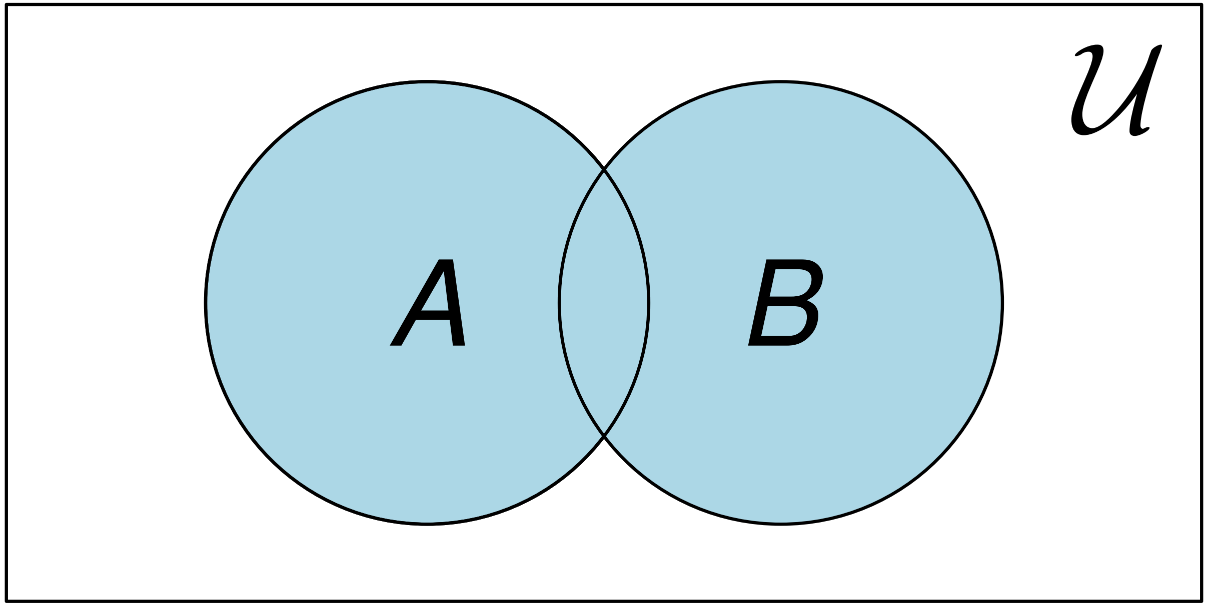
Union

- Definition: The union of two sets A and B, denoted $A \cup B$, is the set that contains elements that are either in A or in B, or in both.

$$A \cup B = \{x : x \in A \vee x \in B\}$$



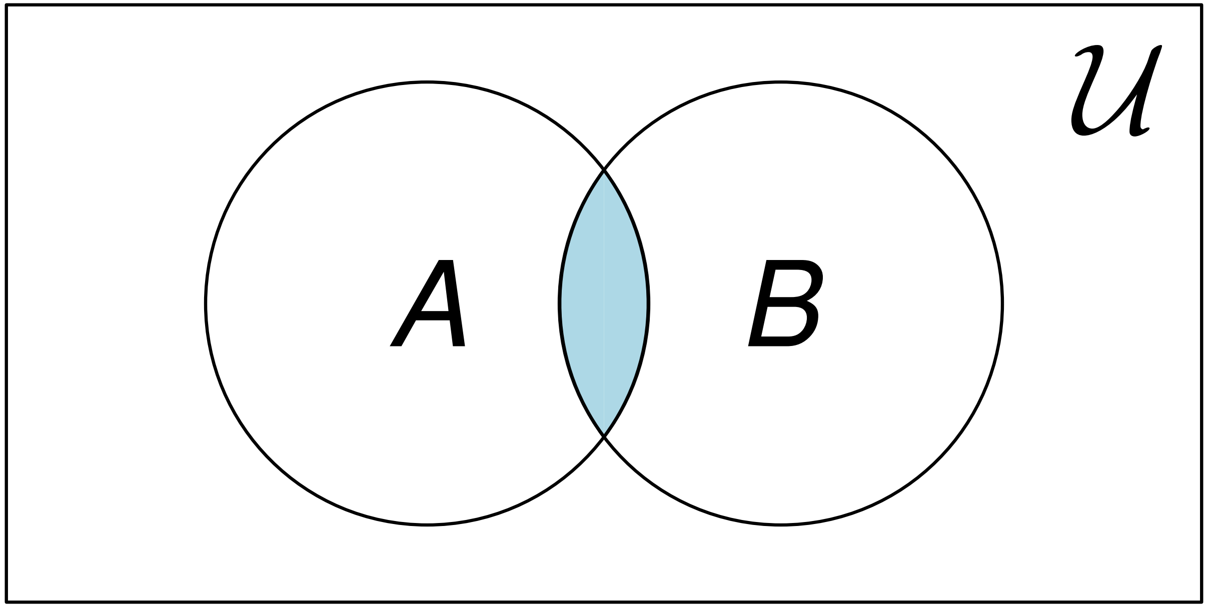
$A \cup B$ is shaded

Intersection

- Definition: The interesection of two sets A and B, denoted $A \cap B$, is the set containing elements in both A and B.

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

- Two sets are called disjoint if their intersection is the empty set, i.e., $A \cap B = \emptyset$



$A \cap B$ is shaded

Cardinality of a Union

- Given two finite sets A and B, the number of elements in $A \cup B$ is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

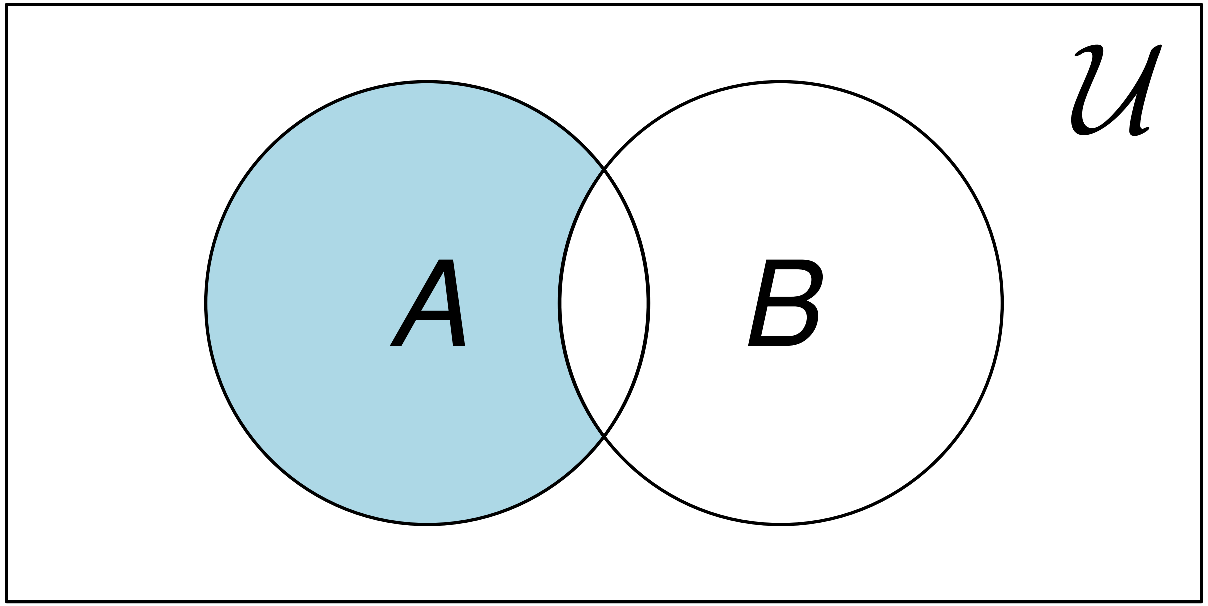
- Known as the principle of inclusion-exclusion
  - More on this in Ch 6: Counting (later this semester) and Ch 8: Advanced Counting Techniques (covered in ICS 241)

Set Difference

- Definition: The difference of two sets A and B, denoted $A - B$, is the set containing elements that are in A, but not in B.

$$A - B = \{x : x \in A \wedge x \notin B\}$$

- Sometimes denoted as $A \setminus B$
- May also be called the complement of B with respect to A

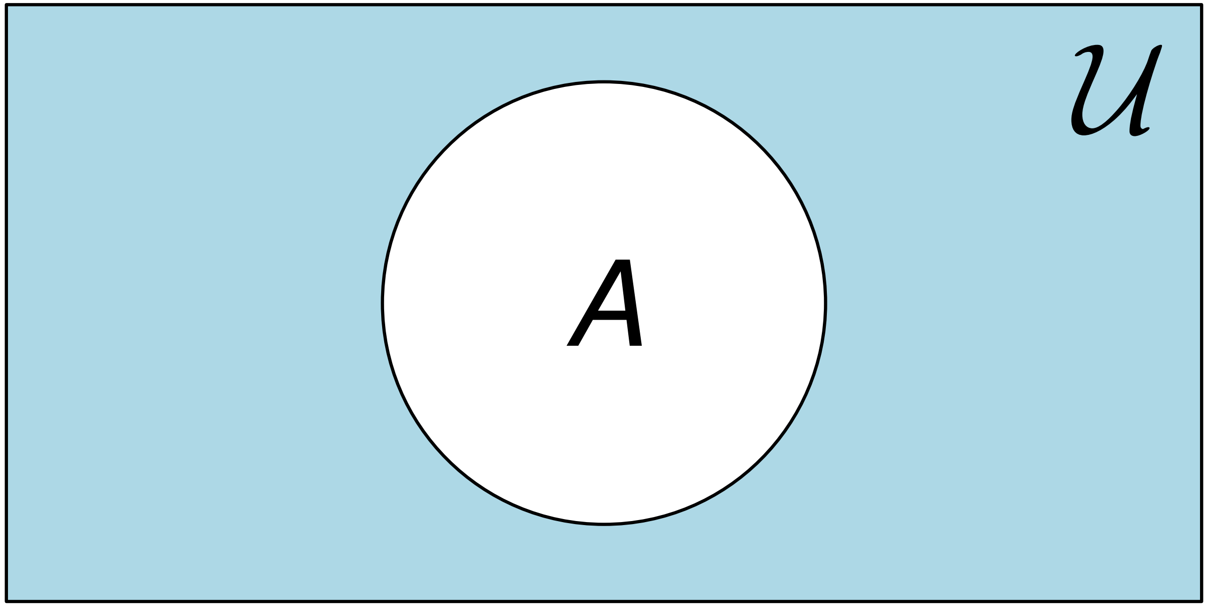


$A - B$ is shaded

Complement

- Definition: The complement of a set A, denoted $\overline{A}$, is the complement of A with respect to $\mathcal{U}$ (the universal set)

$$\overline{A} = \mathcal{U} - A$$
$$= \{x : x \in \mathcal{U} \wedge x \notin A\}$$



$\overline{A}$ is shaded

Exercise

- Draw Venn diagrams for each of these combinations of the sets A, B, and C

  1. $A \cap (B - C)$
  2. $(A \cap B) \cup (A \cap C)$
  3. $(A - B) \cup (A - C) \cup (B - C)$

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- Domination laws
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  - $A \cap \emptyset = \emptyset$
- Idempotent laws
  - $A \cup A = A$
  - $A \cap A = A$

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  - $\overline{(\overline{A})} = A$
- Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative laws
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$

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- Associative laws
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws
  - $A \cup (B \cap C) = (A \cup B) \cap (B \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$

Set Identities

- De Morgan's laws
  - $\overline{A \cap B} = \overline{A} \cup \overline{B}$
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  - $A \cup (A \cap B) = A$
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- Absorption laws
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- Complement laws
  - $A \cup \overline{A} = \mathcal{U}$
  - $A \cap \overline{A} = \emptyset$

Methods for Showing Set Equality

1. Subset method
   - $A \subseteq B$ and $B \subseteq A \iff A = B$
2. Membership table
   - Similar to a truth table
3. Applying set identities

Examples

- Proposition: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

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- Proof: Let A and B be arbitrary sets. We will first show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Assume that $x \in \overline{A \cap B}$. By definition of complement, $x \notin A \cap B$, and by the definition of intersection we know that $\neg((x \in A) \wedge (x \in B))$ is true.

$$\begin{aligned} \neg((x \in A) \wedge (x \in B)) &\equiv (x \notin A) \vee (x \notin B) \\ &\equiv (x \in \overline{A}) \vee (x \in \overline{B}) \\ &\equiv x \in \overline{A} \cup \overline{B}\,. \end{aligned}$$

  Thus, $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

Examples

- Proposition: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- Proof: Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Assume that $x \in \overline{A} \cup \overline{B}$. By definition of union, $x \in \overline{A}$ or $x \in \overline{B}$, and by the definition of complement $x \notin A$ or $x \notin B$. Using De Morgan's laws, the previous statement is equivalent to $\neg((x \in A) \wedge (x \in B))$. It follows from the definition of intersection that $\neg(x \in A \cap B)$. And by the definition of complement, $x \in \overline{A \cap B}$. Hence, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.
  Since we have shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$, we can conclude that $A = B$. ∎

Examples

- Proposition: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Proof: Let A and B be arbitrary sets.

$$\begin{aligned}
\overline{A \cap B} &= \{x : x \notin A \cap B\} \\
&= \{x : \neg(x \in (A \cap B))\} \\
&= \{x : \neg(x \in A \wedge x \in B\} \\
&= \{x : \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x : x \notin A \vee x \notin B\} \\
&= \{x : x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x : x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B} \, . \blacksquare
\end{aligned}$$

Examples

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- Proposition: Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- Proof: Let A, B, and C be arbitrary sets. Consider the membership table

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (B \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

■

Exercise

- Proposition: Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Generalized Unions and Intersections

- Since unions and intersections of sets satisfy associative and commutative laws
  - Union and interesections of multiple sets is well-defined
- <u>Ex:</u>
  - We can write $A \cup B \cup B$ without any ambiguity
- Union of n sets, $A_1, A_2, \ldots, A_n$

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- Intersection of n sets, $A_1, A_2, \ldots, A_n$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$