INFORMATION & COMPUTER SCIENCES
UNIVERSITY *of* HAWAIʻI *at* MĀNOA

Ch 2.1: Sets

ICS 141: Discrete Mathematics for Computer Science I

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Sets

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  - Members of a set are called elements or members of the set.
  - If an element a is contained in the set A, we write $a \in A$ otherwise $a \notin A$.
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  - The set of all vowels $S = \{\text{'a'}, \text{'e'}, \text{'i'}, \text{'o'}, \text{'u'}\}$
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  - $\{\text{'a'}, \text{'e'}, \text{'i'}, \text{'o'}, \text{'u'}\} = \{\text{'a'}, \text{'a'}, \text{'e'}, \text{'i'}, \text{'o'}, \text{'u'}, \text{'i'}, \text{'o'}\}$

Defining Sets

- Demonstrate a pattern
  - Example: The set of all lower case letters

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  - Example: All integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Defining Sets

- Set builder notation
  - Example: Intervals between two values x and y

$$[x, y] = \{z : x \leq z \leq y\}$$
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  - *Note:* It is common to see both : and $|$ used interchangeably
  - Read "such that"

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  - Example: All rational numbers

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, \text{ and } b \neq 0\right\}$$

Notable Sets

- Set of all natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

  - Whether 0 is a natural number or not is a controversial topic (in math)
  - We will follow the convention in the textbook that $0 \in \mathbb{N}$

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$$\mathbb{R}^{+} = \{x \in \mathbb{R} : x > 0\}$$
- Set of all negative real numbers
$$\mathbb{R}^{-} = \{x \in \mathbb{R} : x < 0\}$$
- Set of all complex numbers, $\mathbb{C}$

Remarks

- In programming languages, a <u>datatype</u> is defined as
  - A set of data values
  - A set of operations
- <u>Ex:</u> Boolean datatype
  - Set of values
    - $\{0, 1\} = \{T, F\}$
  - Set of operations
    - $\{$AND (&&), OR (||), NOT (!), XOR (^), $>, <, \geq, \leq,$ == $\}$

Remarks

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- The universal set, denoted $\mathcal{U}$ (or Ω), is the set containing all objects under consideration
  - Depends on the context of the problem

Subsets and Supersets

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- Definition: The set A is a proper subset of set B, denoted $A \subset B$, if and only if every element of A is also an element of B and $A \neq B$.
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- *Intuitively:* Similar to $\leq, <, \geq, >$

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- To show that $A \subset B$, every element of A is an element of B and there exists at least one element of B that is not an element of A

$$\forall x (x \in A \Rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$$

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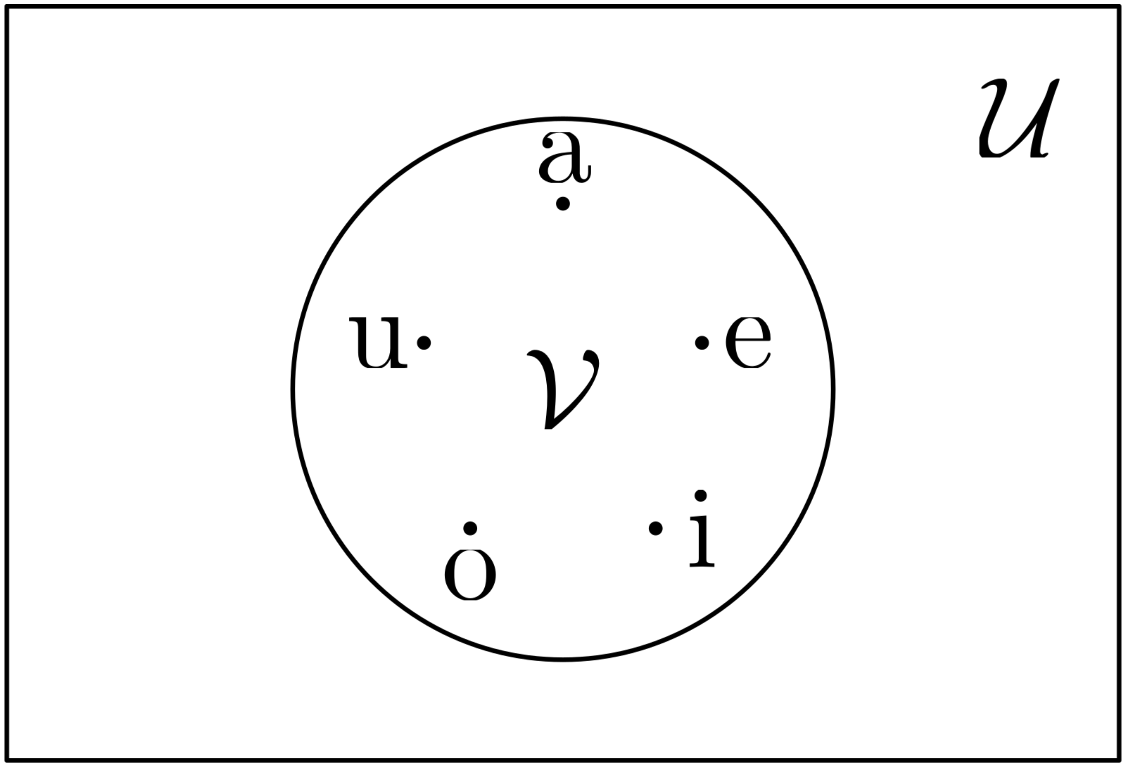
- To show that $A = B$, every element of A is an element of B and vice versa

$$A \subseteq B \land B \subseteq A$$

Graphical Representation of Sets

- Venn diagrams are typically used to represent sets



Cardinality and Power Sets

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- *Remark:* For every set S,
  - $\emptyset \subseteq S$
  - $S \subseteq S$

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- Example: Let $S = \{0, 1, 2\}$

$$\begin{aligned}\mathcal{P}(S) &= \{\emptyset, \{0\}, \{1\}, \{2\},\\ &= \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}\end{aligned}$$

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  - $|S| = 3$ and $|\mathcal{P}(S)| = 8 = 2^3$

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- *Remark:* If a set is not finite, then it is infinite.
  - Example:
    - The set of all integers $\mathbb{Z}$ is infinite
    - The set of all real numbers $\mathbb{R}$ is infinite
  - In Section 2.5, we discuss the cardinality of infinite sets

Cartesian Product

- Definition: An ordered n-tuple, denoted $(a_1, a_2, \ldots, a_n)$, is an ordered collection of elements
- Definition: The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1, 2, \ldots, n$

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Cartesian Product

- Ex:

1. $A = \{1, 2\}$ and $B = \{a, b, c\}$
   - $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
   - $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$
   - *Note:* Order of the sets in the Cartesian product matters!
2. $A = \{1, 2\}$
   - $A \times A = A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
   - $A \times A \times A = A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

Set Notation with Quantifiers

- Recall: Using quantifiers, we can limit the domain by providing additional conditions that the elements of a domain must satisfy.
  - Ex: Domain is all real numbers, $\mathbb{R}$
    - Restrict domain to "all real numbers less than 0."

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- Remark: Typically, instead of stating the domain in english (as we have seen in Chapter 1), we use set notation to state the domain
  - Ex: $\exists x \in \mathbb{Z}(x^2 = 1)$