INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAIʻI at MĀNOA

Ch 1.5: Nested Quantifiers

ICS 141: Discrete Mathematics for Computer Science I

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Nested Quantifiers

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- Allows us to express more complex statements

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- Ex.
  - Domain is all real numbers
  - $\forall x \exists y (x + y = 0)$
  - “For all x, there exists a y such that $x + y = 0$.”
  - Known as the *additive inverse* of x, namely $-x$.

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- Ex.
  - Domain is all real numbers
  - $\forall x \forall y (x + y = y + x)$
  - "For all x and y, $x + y = y + x$."
  - Known as the *commutative law for addition*

Nested Quantifiers

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- Ex.
  - Domain is all real numbers
  - $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$
  - "For all x, y, and z, $(x + (y + z) = (x + y) + z)$."
  - Known as the *associative law for addition*

Thinking of Quantification as Loops

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- Ex. Consider the finite domain $\{0, 1, 2\}$.
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for x = 0 to 2
    for y = 0 to 2
        P(x, y)
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for $x = 0$ to 2
    for $y = 0$ to 2
        $P(x, y)$

- **True** only if $P(x, y)$ is true for all values of (x, y)

$$P(0,0) \wedge P(0,1) \wedge P(0,2)$$
$$\wedge P(1,0) \wedge P(1,1) \wedge P(1,2)$$
$$\wedge P(2,0) \wedge P(2,1) \wedge P(2,2)$$

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- **True** only if for each value of x, $P(x, y)$ is true for at least one value of y

$$
\begin{aligned}
&(P(0,0) \lor P(0,1) \lor P(0,2)) \\
\land &(P(1,0) \lor P(1,1) \lor P(1,2)) \\
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- **True** only if there is a value of x such that for all values of y $P(x, y)$ is true.

$$(P(0,0) \wedge P(0,1) \wedge P(0,2))$$
$$\vee(P(1,0) \wedge P(1,1) \wedge P(1,2))$$
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- **True** only if there exists at least one value of (x, y) such that $P(x, y)$ is true

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\begin{aligned}
&P(0,0) \lor P(0,1) \lor P(0,2) \\
\lor &P(1,0) \lor P(1,1) \lor P(1,2) \\
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- $\forall x \exists y (P(x, y))$
  - "For every x, there exists at least one y such that $P(x, y)$ is true."
- $\exists x \forall y (P(x, y))$
  - "There exists an x, such that for every y, $P(x, y)$ is true."

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- Solution:
  - "For all x and y, there exists at least one value of z such that "$x + y = z$."
  - **True**

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  - "There exists a value for z such that for all values of x and y "$x + y = z$."
  - **False**

Negating Nested Quantifiers

- Recall De Morgan's laws for quantifiers
  - $\neg\forall x(P(x)) \equiv \exists x(\neg P(x))$
  - $\neg\exists x(P(x)) \equiv \forall x(\neg P(x))$
- We can apply these laws to nested quantifiers as well

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  - $\neg\exists x(P(x)) \equiv \forall x(\neg P(x))$
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- Ex.
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- We can apply these laws to nested quantifiers as well
- Ex.
  - Express the negation of $\forall x\exists y(xy = 1)$ so that no negation preceeds a quantifer.
- Solution:

$$\begin{aligned}\neg\forall x\exists y(xy = 1) &\equiv \exists x\neg\exists y(xy = 1)\\ &\equiv \exists x\forall y(\neg(xy = 1))\\ &\equiv \exists x\forall y(xy \neq 1)\ .\end{aligned}$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(a) Everybody loves Jerry
(b) Everybody loves somebody
(c) There is somebody whom everybody loves
(d) Nobody loves everybody
(e) There is somebody whom Lydia does not love
(f) There is somebody whom no one loves
(g) There is exactly one person whom everybody loves

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(a) Everybody loves Jerry

$$\forall x(L(x, \text{Jerry}))$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(b) Everybody loves somebody

$$\forall x \exists y (L(x, y))$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(c) There is somebody whom everybody loves

$$\exists y \forall x (L(x, y))$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(d) Nobody loves everybody

$$\begin{aligned} &\neg\exists x \forall y (L(x, y)) \\ \equiv &\forall x \neg\forall y (L(x, y)) \\ \equiv &\forall x \exists y (\neg L(x, y)) \end{aligned}$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(e) There is somebody whom Lydia does not love

$$\exists x(\neg L(\text{Lydia}, x))$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(f) There is somebody whom no one loves

$$\exists y \neg \exists x (L(x, y))$$
$$\equiv \exists y \forall x (\neg L(x, y))$$

Exercises

- Let $L(x, y)$ be the statement "x loves y", where the domain consists of all people. Use quantifiers to express each of these statements.

(g) There is exactly one person whom everybody loves

$$\begin{aligned} &\exists! y \forall x (L(x, y)) \\ \equiv &\exists y \forall x (L(x, y) \land \forall z (L(x, z) \Rightarrow z = y)) \end{aligned}$$

Exercises

- Let $M(x, y)$ be "x has sent y an email message" and $T(x, y)$ be "x has telephoned y" where the domain consists of all students in the class. Use quantifiers to express each of these statements.

(a) There is a student in your class who has sent everyone else in your class an e-mail message.
(b) There is someone in your class who has either sent an email message or telephoned everyone else in your class.
(c) There is a student in your class who has not received an email message from anyone else in the class and who has not been called by any other student in the class.
(d) There are two different students in your class who between them have sent an email message to or telephoned everyone else in the class.

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(a) There is a student in your class who has sent everyone else in your class an email message.

$$\exists x \forall y (x \neq y \Rightarrow M(x, y))$$

Exercises

- Let $M(x, y)$ be "x has sent y an email message" and $T(x, y)$ be "x has telephoned y" where the domain consists of all students in the class. Use quantifiers to express each of these statements.

(b) There is someone in your class who has either sent an email message or telephoned everyone else in your class.

$$\exists x \forall y (x \neq y \Rightarrow (M(x, y) \vee T(x, y)))$$

Exercises

- Let $M(x, y)$ be "x has sent y an email message" and $T(x, y)$ be "x has telephoned y" where the domain consists of all students in the class. Use quantifiers to express each of these statements.
  (c) There is a student in your class who has not received an email message from anyone else in the class and who has not been called by any other student in the class.

$$\exists x \forall y (x \neq y \Rightarrow (\neg M(x, y) \wedge \neg T(x, y)))$$

Exercises

- Let $M(x, y)$ be "x has sent y an email message" and $T(x, y)$ be "x has telephoned y" where the domain consists of all students in the class. Use quantifiers to express each of these statements.
  (d) There are two different students in your class who between them have sent an email message to or telephoned everyone else in the class.

$$\exists x \exists y (x \neq y \wedge \forall z ((z \neq x \wedge z \neq y) \Rightarrow (M(x, z) \vee M(y, z) \vee T(x, z) \vee T(y, z))))$$