INFORMATION & COMPUTER SCIENCES
UNIVERSITY of HAWAI'I at MĀNOA

Appendix 2: Exponential and Logarithmic Functions

ICS 141: Discrete Mathematics for Computer Science I

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Exponential Functions

- Let $n \in \mathbb{Z}^+$ and b be a fixed positive real number
- Definition: The exponential function to the base b is defined as

$$f_b(n) = b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

Exponential Functions

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$$\begin{aligned} f_b(n) &= b^n \\ &= \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} \end{aligned}$$

- Let $x, y \in \mathbb{R}$

1. $b^{x+y} = b^x b^y$
2. $(b^x)^y = b^{xy}$

Exponential Functions

- Let $n = 0$ and b be a fixed positive real number
- Definition: The exponential function to the base b is defined as

$$\begin{aligned} f_b(0) &= b^0 \\ &= 1 \end{aligned}$$

- Derived from the rule 1.

$$b^0 \cdot b^n = b^{0+n} = b^n$$

$$\Rightarrow b^0 = \frac{b^n}{b^n} = 1$$

Exponential Functions

- Let $n \in \mathbb{Z}$ and b be a fixed positive real number
- Definition: The exponential function to the base b is defined as

$$f_b(-n) = \frac{1}{b^n}$$

- Derived from rule 1.

$$b^{-n} \cdot b^n = b^{-n+n} = b^0 = 1$$

$$\Rightarrow b^{-n} = \frac{1}{b^n}$$

Exponential Functions

- Let $n, m \in \mathbb{Z}$ such that $m > 0$ and b be a fixed positive real number
- Definition: The exponential function to the base b is defined as

$$f_b\left(\frac{n}{m}\right) = \sqrt[m]{b^n}$$

- Derived from rule 2.

$$(b^{\frac{n}{m}})^m = b^n$$

$$\Rightarrow b^{\frac{n}{m}} = \sqrt[m]{b^n}$$

Exponential Functions

- Let $n \in \mathbb{R}$ and b be a fixed real number
- Definition: The exponential function to the base b is defined as

$$f_b(n) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Derived from rule 1.
  - Derivation is out of scope of this course

Exponential Functions

- Let b be a fixed real number and $x, y \in \mathbb{R}$

1. $b^{x+y} = b^x b^y$
2. $(b^x)^y = b^{xy}$

Logarithmic Functions

- If $b \in \mathbb{R}$ such that $b > 1$
  - Then the exponential function from $\mathbb{R} \rightarrow (\mathbb{R} - \mathbb{R}^{-})$ is a bijection
  - Hence, an inverse exists

Logarithmic Functions

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  - Then the exponential function from $\mathbb{R} \rightarrow (\mathbb{R} - \mathbb{R}^{-})$ is a bijection
  - Hence, an inverse exists
- Definition: The logarithmic function to the base b is the inverse of the exponential function to the base b

$$b^{\log_b x} = x \Leftrightarrow \log_b b^x = x$$

Logarithmic Functions

- Let $a, b, x \in \mathbb{R}$ such that $b > 1$
- Definition: The logarithmic function to the base b is the inverse of the exponential function to the base b

$$\log_b a = x \iff b^x = a$$

- Useful identities:
  - $a = b^{\log_b a}$
  - $\log_b a = \frac{\log_c a}{\log_c b}$
  - $\log_c(ab) = \log_c a + \log_c b$
  - $\log_c(a/b) = \log_c a - \log_c b$
  - $\log_b a^n = n \log_b a$
  - $\log_b(1/a) = \log_b(a^{-1}) = -\log_b a$
  - $\log_b a = 1/\log_a b$
  - $a^{\log_b c} = c^{\log_b a}$

Logarithmic Functions

- You will commonly see logarithms expressed without an explicit base, e.g., $\log x$
  - In math, we assume base 10
    - Decimal numbers are base 10
  - In computer science, we assume base 2
    - Binary numbers are base 2
- Natural logarithm (base e) is denoted $\ln x$

Exercises

- Simplify the following expression

$$\frac{\log_5 25 \cdot \log_2 \sqrt{16}}{\log_4 8}$$

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- <u>Solution:</u>

$$\frac{\log_5 25 \cdot \log_2 \sqrt{16}}{\log_4 8} = \frac{\log_5 5^2 \cdot \log_2 16^{\frac{1}{2}}}{\frac{\log_2 8}{\log_2 4}}$$

$$= \frac{2 \cdot \log_5 5 \cdot \frac{1}{2} \cdot \log_2 2^4}{\frac{\log_2 2^3}{\log_2 2^2}}$$

$$= \frac{4 \cdot \log_2 2}{\frac{3 \cdot \log_2 2}{2 \cdot \log_2 2}}$$

$$= 4 \cdot \frac{2}{3} = \frac{8}{3}$$

Exercises

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- <u>Solution:</u>

$$\frac{\log n^3 + \log n^2}{\log \sqrt{n}} = \frac{3 \cdot \log n + 2 \cdot \log n}{\log n^{\frac{1}{2}}}$$

$$= \frac{5 \cdot \log n}{\frac{1}{2} \log n}$$

$$= 10$$

Exercises

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$$\log_5 \sqrt{125} \cdot \frac{\log_4 64 \cdot \log_2 256}{(2^{\log_2 \log_2 16})^2}$$

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- <u>Solution:</u>

$$\log_5 \sqrt{125} \cdot \frac{\log_4 64 \cdot \log_2 256}{(2^{\log_2 \log_2 16})^2} = \frac{1}{2} \cdot \log_5 5^3 \cdot \frac{\log_4 4^3 \cdot \log_2 2^8}{(\log_2 16)^2}$$

$$= \frac{3}{2} \cdot \log_5 5 \cdot \frac{3 \cdot \log_4 4 \cdot 8 \cdot \log_2 2}{(\log_2 2^4)^2}$$

$$= \frac{3}{2} \cdot \frac{3 \cdot 8}{(4 \cdot \log_2 2)^2}$$

$$= \frac{3}{2} \cdot \frac{3 \cdot 8}{4^2} = \frac{9}{4}$$