Lotka-Volterra Model with an Allee Effect on Predator Population

The first time Allee effect was described as a modification to the logistic equation in the chapter 5 of the textbook (Case, 2000: 128-130). The introductory example of an Allee effect was given for the discrete logistic model with unstable equilibrium point at low population densities in Figure 5.46 with very brief explanation. Later we meet again with an Allee effect in chapter 12 when it was added to a Predator-Prey model (Case, 2000: 281-286). By trying to simplify the model the author of the textbook demonstrated an Allee effect only for a prey population that is more common in nature, assuming a linear numerical response for predators. Previously he also described the Predator-Prey model with intraspecific interference among predators with type 2 functional response and showed nonlinear zero-isocline for predator population. However I assume that some predator species might undergo the opposite situation when they need to exceed the particular number of specimens in population to survive (e.g. animals with cooperative hunting). Thus in my short writing paragraph I want to demonstrate the case of Predator-Prey model with an Allee effect on predator.

Assume the presence of an Allee effect on predators, which strength is characterized by the constant $K$. With a larger $K$ an Allee effect will be stronger and with a larger number of predators $P$ it becomes weaker. Then the threshold from an Allee effect for positive per capita growth rate at low density predator population must be described by $\frac{P}{p+K}$, $K > 0$. After adding this expression to the simple Lotka-Volterra differential equation, we obtain:

$$\frac{\partial V}{\partial t} = bV - aVP$$
$$\frac{\partial P}{\partial t} = caVP \frac{P}{P + K} - dP.$$

The nontrivial equilibrium for this Predator-Prey model

$$\frac{\partial V}{\partial t} = 0 \Rightarrow P^* = \frac{b}{a} \quad \text{and} \quad \frac{\partial P}{\partial t} = 0 \Rightarrow V^* = \frac{d(P+K)}{caP} = \frac{db + daK}{abc}$$

shows that equilibrium density for prey population increases with an Allee effect.

Let’s now analyze the stability of the model by Jacobian matrix and obtain the partial derivatives:

$$J = \begin{bmatrix} b - aP & -aV \\ \frac{caP^2}{P + K} & 2caVP - 2dP - dK \end{bmatrix}.$$ 

After substitution of $V$ and $P$ by the equilibrium values $V^*$ and $P^*$ we will get

$$J = \begin{bmatrix} 0 & \frac{db + daK}{b + aK} \\ \frac{cb}{b + aK} & - \frac{db + daK}{b + aK} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & + \\ + & + \end{bmatrix}.$$
Consequently the trace of matrix \( \frac{d a K}{b + a K} \) will be the positive value and the matrix determinant will also have positive real part, which means the unstable equilibrium. The changes in Predator-Prey model caused by an Allee effect in predators are shown on the Figure 1:

![Figure 1](image-url)

**Fig. 1.** Trajectories of predator-prey population densities for simple Lotka-Volterra model (red line) and for an Allee effect on predators (blue line) with the following parameters: \( a = 0.03, b = 0.05, c = 0.33, d = 0.1, V = 50, P = 25, K = 4 \) (redrawn and modified from Zhou et al., 2005).

Thus we can conclude that nontrivial equilibrium in Predator-Prey model will always switch from neutral to unstable equilibrium and from smaller to larger equilibrium value for preys when Allee effect on predator population is added to the model.

I think this study might be also transformed into the homework question with the solution provided above:

“How will the equilibrium of classical Lotka-Volterra model be changed by the adding an Allee effect to predator population?”

The idea of making this small project was taken from the book “Allee effects in ecology and conservation” (Courchamp et al., 2008). All modeling and conclusions were based on the article of Zhou (Zhou et al., 2005) with some additions from the book “Nonlinear dynamics of interacting populations” (Bazykin et al., 1998).
References

