Lesson 3 – Projectile motion

Projectile motion is a good example of motion that takes place in two dimensions. In projectile motion, we try to take account motion in both the horizontal and vertical directions. We can come up with the basic formulas for simple projectile motion by assuming that the acceleration in the x direction is zero, and the acceleration in the y direction is \(-g\) where \(g\) is the acceleration due to gravity. These are both cases where the acceleration is constant, so we can start with the equations for motion with constant acceleration.

\[
\begin{align*}
\text{x direction} \\
v_x &= v_{ox} + a_x t \\
1. \quad v_x &= v_{ox} \\
2. \quad x &= x_o + v_{ox} t \\
\text{y direction} \\
v_y &= v_{oy} + a_y t \\
3. \quad v_y &= v_{oy} - gt \\
y &= y_o + v_{oy} t + \frac{1}{2} a_y t^2 \\
4. \quad y &= y_o + v_{oy} t - \frac{1}{2} gt^2 \\
v_y^2 - v_{oy}^2 &= 2 a_y (y - y_o) \\
5. \quad v_y^2 - v_{oy}^2 &= -2g(y - y_o)
\end{align*}
\]

Try solving the following problem:

A projectile is shot with an initial speed of 30 m/s at an angle of 30°. What is the x position when the y value is 10.3 m above the ground. Assume the initial position for x and y are both zero.

Identify what is known:

\[
\begin{align*}
g &= 9.8 \text{ m/s}^2 \\
v_{ox} &= 30 \cos(30°) = 25.9808 \text{ m/s} \\
v_{oy} &= 30 \sin(30°) = 15.0000 \text{ m/s}
\end{align*}
\]

Identify what you are solving for:

\(x\), when \(y = 10.3 \text{ m}\)

This means you need equation 2, since it contains \(x\). Looking at equation 2, we know everything except for \(t\). So, we need an equation that contains \(t\) and also makes use of the \(y\) value. That would be equation 4 from above. You can certainly solve for \(t\) in equation 4 making use of the quadratic formula. But, let's do this with a spreadsheet instead.

Start by rearranging equation 4.
\[y = y_0 + v_{oy} t - \frac{1}{2} g t^2\]
\[10.3 = 15.0 t - \frac{1}{2} (9.8) t^2\]
\[4.9 t^2 - 15 t + 10.3 = 0\]
\[f(t) = 4.9 t^2 - 15 t + 10.3\]

Let's input this function along with the formula for \( x \).

Use an appropriate increment for time, and solve the problem. (Did you get 27.02 m and 52.51 m?) How hard did you have to work to get answers that are good to 4 significant figures?

**Projectile Motion with Air Drag**

In reality, the gravitational force is not the only force that acts on a projectile. There is also air drag acting on the projectile. The wind affects the relative velocity of the projectile to the air, so the wind also affects drag as well. Since the wind is unpredictable, we will neglect its affects and assume the motion takes place in still air (no wind).

What would the forces look like if air drag is taken into account? Air drag always opposes the motion. If we break the motion up into x and y directions, we would be looking at the following situations:

\( x \) direction

[Diagram of projectile motion with force vectors]
y direction, going up

\[ F_{\text{dy}} \]

\[ mg \]

y direction, going down

\[ F_{\text{dy}} \]

\[ mg \]

So, applying Newton's Second Law in the x direction would give:

\[ \sum F_x = -F_{Dx} = ma_x \]

Assuming inertial drag (a good assumption for air), the drag force is given by:

\[ F_{Dx} = \frac{1}{2} C_D \rho A v_x^2 \]

where \( C_D \) is the drag coefficient
\( \rho \) is the density of air
\( A \) is the cross-sectional area of the projectile
\( v_x \) is the x component of the velocity

We will assume that the projectile is roughly spherical, so \( C_D = 0.4 \.

We will also assume the projectile has a radius of 4 cm (or 0.04 m). This gives a cross-sectional area of:

\[ A = \pi r^2 = \pi (0.04)^2 = 5.02655 \times 10^{-3} \text{ m}^2 \]

The density of air is given by:

\[ \rho = 1.2 \text{ kg/m}^3 \]

Let's rewrite the force equation:
\[ m \, a_x = -\frac{1}{2} C_D \rho A v_x^2 \]
\[ a_x = -\frac{1}{2} \frac{C_D \rho A}{m} v_x^2 \]
\[ a_x = -k \, v_x^2 \], where \( k = \frac{C_D \rho A}{2m} \)

6. \( \frac{dv_x}{dt} = -k \, v_x^2 \)

In addition, we have:

7. \( \frac{dx}{dt} = v_x \)

In the y direction, going up, we would get:

\[ m \, a_y = -mg -\frac{1}{2} C_D \rho A v_y^2 \]
\[ a_y = -g -k \, v_y^2 \]

In the y direction going down, we would get:

\[ m \, a_y = -mg +\frac{1}{2} C_D \rho A v_y^2 \]
\[ a_y = -g +k \, v_y^2 \]

Although the two cases for the y direction differ by a minus sign, this can be taken care of if we use the following equation:

\[ a_y = -g -k \, v_y \, |v_y| \]

where \( |v_y| \) is the absolute value of \( v_y \).

This is because when the projectile goes up, \( v_y \) is positive so that works out to a minus sign in front of \( k \). When the projectile goes down, \( v_y \) is negative, so that works out to a plus sign in front of \( k \).

So, in the y direction we have:

8. \( \frac{dv_y}{dt} = -g -k \, v_y \, |v_y| \)

9. \( \frac{dy}{dt} = v_y \)

Those would be challenging equations to solve analytically, but we can solve them numerically using a Runge-Kutta approximation.

**Euler's method**

To solve equations 6, 7, 8 and 9 numerically, we will start with Euler's method. Euler's method is a little easier to understand that the Runge-Kutta approximation we are going to use. Euler's method is not accurate enough, but it is easier to understand. In addition, seeing
Euler’s method first will make it easier to understand the Runge-Kutta approximation. (We will be using the Runge-Kutta Second order approximation or RK2.)

Consider an equation of motion given by:

\[ a = \frac{dv}{dt} \] where \( f(v, t) \) is a function of time and velocity

Since \( a = \frac{dv}{dt} \), \( a \approx \frac{\Delta v}{\Delta t} = \frac{v - v_o}{\Delta t} \)

This means \( \frac{v - v_o}{\Delta t} \approx f(v, t) \)

\[ v = v_o + f(v, t) \cdot \Delta t \]

So, Euler’s method allows us to approximate a new value for the velocity by choosing a time interval and multiplying the \( f(v, t) \).

In the case of projectile motion with air drag, we would have the following for the x direction:

\[ v_{x new} = v_{x old} - k \cdot (v_{x old}^2) \cdot \Delta t \]

Similarly, we could get the x value using:

\[ x_{new} = x_{old} + v_{x old} \cdot \Delta t \]

The smaller we make \( \Delta t \), the better the approximation.

**Runge-Kutta 2nd order method**

The Runge-Kutta 2nd order method modifies the Euler method. This method, RK2, for short uses the value of the function at an intermediate point, to improve the accuracy of the approximation. This is carried out in two steps.

First an intermediate velocity is obtained:

\[ v_{x inter} = v_{x old} - k \cdot (v_{x old}^2) \cdot \frac{\Delta t}{2} \]

Then, the new value is obtained using the intermediate value:

\[ v_{x new} = v_{x old} - k \cdot (v_{x inter}^2) \cdot \Delta t \]

The new value for x is also obtained using this intermediate value:

\[ x_{new} = x_{old} + v_{x inter} \cdot \Delta t \]

This can be carried out for the y-direction as well making use of equations 8 and 9.

\[ v_{y inter} = v_{y old} + (-g - k \cdot v_{y old} |v_{y old}|) \cdot \frac{\Delta t}{2} \]

\[ v_{y new} = v_{y old} + (-g - k \cdot v_{y inter} |v_{y inter}|) \cdot \Delta t \]

\[ y_{new} = y_{old} + v_{y inter} \cdot \Delta t \]

Now, all we need to do is to setup a spreadsheet to carry out these calculations:

Here is the value for \( k \)
\[ k = \frac{C_D \rho A}{2m} = \frac{(0.4)(1.2)(5.02655 \times 10^{-3})}{(2)(0.5)} = 2.41274 \times 10^{-3} \text{ m}^{-1} \]

Let’s assume an initial speed of 50 m/s at an angle of 30°.

Here are the formulas to get the x direction values:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>k</td>
<td>2.41274E-3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Δt</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( V_0 )</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \theta )</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( V_{0x} )</td>
<td>=B3<em>COS(B4</em>PI()/180)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( V_{0y} )</td>
<td>=B3<em>SIN(B4</em>PI()/180)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>time</td>
<td>( V_x ) inter</td>
<td>( V_x )</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>=B5</td>
<td>=D9</td>
</tr>
</tbody>
</table>

This would evaluate to the following values:

<table>
<thead>
<tr>
<th>A</th>
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<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>k</td>
<td>2.41274E-3</td>
<td></td>
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<tr>
<td>2</td>
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<td>0.25</td>
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</tr>
<tr>
<td>3</td>
<td>( V_0 )</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \theta )</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( V_{0x} )</td>
<td>43.30127</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( V_{0y} )</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>time</td>
<td>( V_x ) inter</td>
<td>( V_x )</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>43.30127</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.25</td>
<td>42.73578</td>
<td>42.199645</td>
</tr>
<tr>
<td>11</td>
<td>0.5</td>
<td>41.66257</td>
<td>41.152655</td>
</tr>
<tr>
<td>12</td>
<td>0.75</td>
<td>40.64190</td>
<td>40.156336</td>
</tr>
</tbody>
</table>

This shows the next couple of rows filled down.
The next screen shot shows the values for the y direction calculated. See if you can get the same answers:

Be careful when entering the formulas. If you don't get the numbers shown above, you need to check your formulas.

**Getting the range**

To get the distance travelled over level ground, or the range, you just note when y goes back down to zero. The x value at that step gives you an estimate of the range.

Possible calculations (use the above values for $k$):

- Assume the initial speed of the projectile is 55 m/s, what angle(s) would result in a horizontal distance traveled (range) of 160 m? What would the corresponding time(s) be for these conditions?
- Assume that the projectile has a range of 180 m when launched at an angle of 39°. Find the initial speed of the projectile.