Math 432 - Real Analysis II
Solutions to Homework due March 11

Question 1. Let \( f(x) = k \) be a constant function for \( k \in \mathbb{R} \).

1. Show that \( f \) is integrable over any \([a, b]\) by using Cauchy’s \( \varepsilon - P \) condition for integrability.

2. Show that \( \int_a^b k \, dx = k(b - a) \).

Solution 2.

(a) Since \( f(x) \) is the constant \( k \) function, then clearly \( M(f, S) = m(f, S) = k \) for any \( S \subset [a, b] \). Thus, for any partition \( P = \{a = t_0 < t_1 < \cdots < t_n = b\} \) of \([a, b]\), we have that
\[
L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} k(t_k - t_{k-1}) = k(b - a).
\]

An identical conversation shows that \( U(f, P) = k(b - a) \) for any partition. Thus for any \( \varepsilon > 0 \), there exists a partition (in fact, any partition) so that \( U(f, P) - L(f, P) = 0 < \varepsilon \). Thus, \( f \) is integrable.

(b) Since \( U(f, P) = k(b - a) = L(f, P) \) for any partition \( P \), then the Darboux integrals are also equal to \( k(b - a) \). Therefore,
\[
\int_a^b k \, dx = U(k) = L(k) = k(b - a).
\]

Question 2. In class, we proved that if \( f \) is integrable on \([a, b]\), then \( |f| \) is also integrable. Show that the converse is not true by finding a function \( f \) that is not integrable on \([a, b]\) but that \( |f| \) is integrable on \([a, b]\).

Solution 2. Consider the function
\[
f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}.
\]

A computation similar to one in a previous HW shows that \( f \) is not integrable. However, \( |f| \) is the constant function 1, which by Question 1 is integrable.

Question 3.

(a) Let \( x, y \in S \). Show that \( |f(x)| - |f(y)| \leq |f(x) - f(y)| \).

(b) Let \( x, y \in S \). Show that \( |f(x) - f(y)| \leq M(f, S) - m(f, S) \).

(c) Use (a) and (b) to show that \( M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) \). Hint: To do this, show that for any \( \varepsilon > 0 \),
\[
M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) + \varepsilon.
\]

Solution 3.
(a) Let \( x, y \in S \). Using the triangle inequality, we have that
\[
|f(x)| = |f(x) - f(y) + f(y)| \leq |f(x) - f(y)| + |f(y)|,
\]
which gives us that \(|f(x)| - |f(y)| \leq |f(x) - f(y)|\).

(b) By definition, if \( x, y \in S \), then the following inequalities are true:
\[
m(f, S) \leq f(x) \leq M(f, S)
\]
\[
m(f, S) \leq f(y) \leq M(f, S).
\]
The second inequality is identical to
\[
-M(f, S) \leq -f(y) \leq m(f, S).
\]
Adding this one to the very first string of inequalities, we get that
\[
-[M(f, S) - m(f, S)] \leq f(x) - f(y) \leq M(f, S) - m(f, S).
\]
This is identical to our desired statement
\[
|f(x) - f(y)| \leq M(f, S) - m(f, S).
\]

(c) We will show that \( M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) \) by showing that \( M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) + \varepsilon \) for every \( \varepsilon > 0 \). So, let \( \varepsilon > 0 \). Since
\[
M(|f|, S) = \inf\{|f(x)| \mid x \in S\},
\]
then by the approximation theorem, there exists an \( x_0 \in S \) such that \( |f(x_0)| > M(|f|, S) - \varepsilon/2 \). Similarly, by the approximation theorem, there exists a \( y_0 \in S \) such that \( |f(y_0)| < m(|f|, S) + \varepsilon/2 \). Putting these together, we get that
\[
|f(x_0)| - |f(y_0)| > M(|f|, S) - m(|f|, S) - \varepsilon/2 - \varepsilon/2 = M(|f|, S) - m(|f|, S) - \varepsilon.
\]
Using the previous parts, we get that
\[
M(|f|, S) - m(|f|, S) - \varepsilon \leq |f(x_0)| - |f(y_0)| \leq |f(x_0) - f(y_0)| < M(|f|, S) - m(|f|, S).
\]
Therefore, for any \( \varepsilon > 0 \),
\[
M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) + \varepsilon.
\]
Since this is true for every \( \varepsilon > 0 \), we have that
\[
M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S).
\]

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**Question 4.** Let \( f \) and \( g \) be integrable functions on \([a, b]\).

(a) Show that \( 4fg = (f + g)^2 - (f - g)^2 \).

(b) Use (a) to show that \( fg \) is also integrable on \([a, b]\).

**Solution 4.**
(a) Beginning with the left-hand side, we get that
\[(f + g)^2 - (f - g)^2 = f^2 + 2fg + g^2 - (f^2 - 2fg + g^2) = 4fg.\]

(b) Since \(f\) and \(g\) are integrable on \([a, b]\), then \(f + g\) and \(f - g\) are integrable. Since squares of integrable functions are integrable, then \((f + g)^2\) and \((f - g)^2\) are integrable. Thus, by (a), \(4fg\) is integrable and \(fg\) is integrable, as desired.

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**Question 5.** Consider the function \(f\) on \([0, 1]\) given by
\[
f(x) = \begin{cases} 
x & \text{if } x \in \mathbb{Q} \\
0 & \text{if } x \notin \mathbb{Q} 
\end{cases}
\]

(a) Let \(P = \{0 = t_0 < \cdots < t_n = 1\}\) be any partition of \([0, 1]\). Show that
\[U(f, P) = U(x, P).\]

(b) Compute the Upper and Lower Darboux sums, \(U(f)\) and \(L(f)\), and use this to decide if \(f\) is integrable.

**Solution 5.**

(a) For any subinterval \([t_{k-1}, t_k]\), there are infinitely many rationals and irrationals. In particular,
\[M(f, [t_{k-1}, t_k]) = \sup\{x | x \in \mathbb{Q}, x \in [t_{k-1}, t_k]\} = t_k.\]

Notice that this is exactly the same values as \(M(x, [t_{k-1}, t_k])\). Thus, \(U(f, P) = U(x, P)\).

(b) Since \(U(x, P) = U(f, P)\) for any partition \(P\), then
\[U(f) = \inf\{U(f, P) | P\ \text{is a partition of } [a, b]\} = \inf\{U(x, P) | P\ \text{is a partition of } [a, b]\} = U(x) = 1/2.\]

For \(L(f)\), since there are infinitely many irrationals in every subinterval \([t_{k-1}, t_k]\) and thus \(M(f, [t_{k-1}, t_k]) = 0\). Thus, for any partition, \(L(f, P) = 0\). So, \(L(f) = 0\). Since \(U(f) \neq L(f)\), then \(f\) is not integrable.

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**Question 6.** Let \(f\) be a bounded function on \([a, b]\). Suppose that there exists a sequence of partitions \(P_n\) on \([a, b]\) such that
\[
\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).
\]

Show that \(f\) is integrable and that
\[
\int_a^b f \, dx = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).
\]

**Solution 6.** Since \(\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n)\), then
\[
\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0.
\]
Let $\varepsilon > 0$. Then, there exists an $N$ such that for all $n > N$,

$$|U(f, P_n) - L(f, P_n) - 0| = U(f, P_n) - L(f, P_n) < \varepsilon.$$ 

Thus, for the partition $P_{N+1}$, the Cauchy integrability is satisfied. So, $f$ is integrable.

Next, we show that $\int_a^b f(x) \, dx = \lim_{n \to \infty} U(f, P_n) = I$. Since this limit is also equal to $\lim_{n \to \infty} L(f, P_n)$ and $L(f, P_k) \leq U(f, P_n)$ for all $k$ and $n$, we must have that $\lim_{n \to \infty} U(f, P_n) \leq U(f, P_N)$ for each $n$. Let $\varepsilon > 0$. Then there exists an $N$ such that for all $n > N$, $|U(f, P_n) - I| = U(f, P_n) - I < \varepsilon$. Thus, we have that

$$\int_a^b f(x) \, dx = U(f) \leq U(f, P_n) < I + \varepsilon = \lim_{n \to \infty} U(f, P_n) + \varepsilon.$$ 

Using $L$’s, we similarly get that

$$\int_a^b f(x) \, dx > \lim_{n \to \infty} L(f, P_n) - \varepsilon = \lim_{n \to \infty} U(f, P_n) - \varepsilon.$$ 

Thus, we get that

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).$$ 

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**Question 7.** Let $f$ and $g$ be bounded functions on $[a, b]$. In what follows, we will show that $\max\{f, g\}$ and $\min\{f, g\}$ are integrable if we know that $f$ and $g$ are individually integrable. Define these functions as

$$\min\{f, g\}(x) = \max\{f(x), g(x)\},$$

and similarly for $\min\{f, g\}$.

(a) Let $a, b \in \mathbb{R}$. Show that

$$\min\{a, b\} = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|.$$

(b) Use (a) to show that if $f$ and $g$ are integrable, then $\min\{f, g\}$ is also integrable.

(c) Find an expression similar to one in (a) for $\max\{a, b\}$. Prove that your expression is correct.

(d) Use (c) to show that if $f$ and $g$ are integrable, then $\max\{f, g\}$ is integrable.

**Solution 7.**

(a) We proceed with cases. Assume that $a \leq b$. Then, $\min\{a, b\} = a$. Since $a \leq b$, then $|a - b| = b - a$.

Thus,

$$\frac{1}{2}(a + b) - \frac{1}{2}|a - b| = \frac{1}{2}(a + b) - \frac{1}{2}(b - a) = a = \min\{a, b\}. $$

In the other case, if $b \leq a$, then $\min\{a, b\} = b$. Since $b \leq a$, then $|a - b| = a - b$. Thus,

$$\frac{1}{2}(a + b) - \frac{1}{2}|a - b| = \frac{1}{2}(a + b) - \frac{1}{2}(a - b) = b = \min\{a, b\}. $$

(b) Since $f$ and $g$ are integrable, then $f + g$ and $f - g$ are integrable. Furthermore, since $f - g$ is integrable, then $|f - g|$ is integrable. Since

$$\min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

is integrable.
(c) We conjecture that
\[ \max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|. \]

Using proofs similar to in (a), this statement is true.

(d) Using an argument similar to that in (b), we get that \( \max\{f, g\} \) is integrable.