Math 432 - Real Analysis II
Solutions to Homework due February 13

In class, we learned that the \( n \)-th remainder for a smooth function \( f(x) \) defined on some open interval containing 0 is given by

\[
R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.
\]

Taylor’s Theorem gives a very helpful expression of this remainder. It says that for some \( c \) between 0 and \( x \),

\[
R_n(x) = f^{(n)}(c) \frac{x^n}{n!}.
\]

A function is called analytic on a set \( S \) if

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k
\]

for all \( x \in S \). In other words, \( f \) is analytic on \( S \) if and only if \( \lim_{n \to \infty} R_n(x) = 0 \) for all \( x \in S \).

Question 1. Use Taylor’s Theorem to prove that all polynomials are analytic on all \( \mathbb{R} \) by showing that \( R_n(x) \to 0 \) for all \( x \in \mathbb{R} \).

Solution 1. Let \( f(x) \) be a polynomial of degree \( m \). Then, for all \( k > m \) derivatives, \( f^{(k)}(x) \) is the constant 0 function. Thus, for any \( x \in \mathbb{R} \), \( R_n(x) \to 0 \). So, the polynomial \( f \) is analytic on all \( \mathbb{R} \).

Question 2. Use Taylor’s Theorem to prove that \( e^x \) is analytic on all \( \mathbb{R} \). by showing that \( R_n(x) \to 0 \) for all \( x \in \mathbb{R} \).

Solution 2. Note that \( f^{(k)}(x) = e^x \) for all \( k \). Fix an \( x \in \mathbb{R} \). Then, by Taylor’s Theorem,

\[
R_n(x) = \frac{e^c x^n}{n!}
\]

for some \( c \) between 0 and \( x \).

We now proceed with two cases. In the first case, assume that \( x > 0 \). Then, \( e^c < e^x \) since \( e^x \) is an increasing function. Thus,

\[
0 \leq R_n(x) = \frac{e^c x^n}{n!} < \frac{e^x x^n}{n!}.
\]

Since \( \frac{e^x x^n}{n!} \to 0 \), then \( R_n(x) \to 0 \) by the Squeeze Theorem.

For the second case, assume that \( x < 0 \). Then, \( e^x < e^c < 1 \). We will show that \( |R_n(x)| \to 0 \), which is equivalent to \( R_n(x) \to 0 \). Notice that since \( 0 < e^x < e^c < 1 \), we have that

\[
\frac{e^x |x|^n}{n!} < \frac{e^c |x|^n}{n!} < \frac{|x|^n}{n!},
\]

where the middle terms is equivalent to \( |R_n(x)| \). Since the two exterior terms limit to 0, by the Squeeze Theorem, the middle term will also tend to 0. So, \( |R_n(x)| \to 0 \) and thus \( R_n(x) \to 0 \).

Thus, for all \( x \in \mathbb{R} \), \( e^x \) is equal to its Taylor series. So, \( e^x \) is analytic on all \( \mathbb{R} \).

Question 3. This next question investigates the relationship between even and odd functions and the powers of their respective Taylor series. Recall that a function is called even if \( f(x) = f(-x) \) for all \( x \in \mathbb{R} \) and is called odd if \( f(-x) = -f(x) \).
(a) Let \( f \) be a differentiable function. Show that if \( f \) is an even function, then \( f' \) is an odd function. Furthermore, show that if \( f \) is an odd function, then \( f' \) is an even function.

(b) Show that if \( f \) is odd, then \( f(0) = 0 \).

(c) Show that if \( f \) is odd, then \( f^{(n)}(0) = 0 \) when \( n \) is even.

(d) Use (c) to show that if \( f \) is odd, then its Taylor series contains only odd powers of \( x \).

**Solution 3.**

(a) If \( f \) is even, then \( f(x) = f(-x) \). Taking derivatives of both sides, we get that \( f'(x) = -f'(-x) \). Thus, \( f' \) is an odd function. If \( f \) is odd, then \( -f(x) = f(-x) \). Taking derivatives of both sides, we get that \( -f'(x) = f'(x) \), which is equivalent to \( f'(x) = f'(-x) \). So, \( f \) is even.

(b) If \( f \) is odd, then \( f(x) = -f(-x) \). Plugging in \( x = 0 \), we get that \( f(0) = -f(0) = -f(0) \). Since \( f(0) = -f(0) \), it must be the case that \( f(0) = 0 \).

(c) For \( n = 0 \), we have that \( f^{(n)}(0) = f(0) = 0 \) by (a). The first derivative of \( f \) is even. Differentiating again, we get that \( f'' \) must be odd since it is the derivative of an even function. Thus \( f''(0) = 0 \). Continuing in this way, we get that \( f^{(k)}(0) = 0 \) if \( k \) is even.

(d) If \( f \) is odd, then its even-powered derivatives at 0 are 0. Thus, when computing the Taylor series, these terms vanish. So, the only remaining terms are odd.

**Question 4.** Consider the function \( f(x) = e^x \) and its Taylor polynomials.

(a) Graph the \( n \)-th degree Taylor polynomials \( \sum_{k=0}^{n} \frac{x^k}{k!} \) for \( n = 1, 2, 3, 4 \) and 5 along with the graph of \( e^x \).

(b) Notice that for \( x > 0 \), the Taylor polynomials for \( e^x \) lie below the graph of \( e^x \) itself. Use Taylor’s Theorem to explain why this is true.

(c) What happens when \( x < 0 \)? As with (b), explain your result using Taylor’s Theorem.

**Solution 4.**

(a) By Taylor’s Theorem, there exists a \( c \in (0, x) \) such that

\[ R_n(x) = \frac{f^{(n)}(c)}{n!} x^n. \]

For \( e^x \), we have that \( f^{(n)}(x) = e^x \). Thus,

\[ R_n(x) = \frac{f^{(n)}(c)}{n!} x^n = \frac{e^c}{n!} x^n. \]

Since \( x > 0 \), all terms are positive and we get that \( R_n(x) > 0 \). Thus, we have that

\[ f(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} > 0 \]

and thus

\[ f(x) > \sum_{k=0}^{n-1} \frac{x^k}{k!} > 0. \]

Thus, for \( x > 0 \), the graph of \( e^x \) is always above any Taylor polynomial approximation.
(b) We observe that when \( n \) is odd, the graph of the Taylor polynomials are below the graph of \( e^x \); conversely, when \( n \) is even, the graph of the Taylor polynomial is above \( e^x \). To understand this, we consider the remainder \( R_{n+1}(x) \). Similar to above, by Taylor’s Theorem, there exists some \( c \in (x, 0) \) such that

\[
R_{n+1}(x) = \frac{e^c}{(n+1)!}x^{n+1}.
\]

Notice that since \( x < 0 \), if \( n \) is odd, then \( x^{n+1} \) is positive and \( R_{n+1}(x) > 0 \). Thus,

\[
f(x) > \sum_{k=0}^{n} \frac{x^k}{k!}.
\]

Conversely, if \( n \) is even, then \( x^{n+1} \) is negative so \( R_{n+1}(x) < 0 \) and thus

\[
f(x) < \sum_{k=0}^{n-1} \frac{x^k}{k!}.
\]

**Question 5.** In class, we used the Taylor series of \( \arctan x \) to approximate \( \pi \). In this question, we will re-visit the normal distribution from Statistics. Recall that the normal distribution function with a mean of \( \mu \) and standard deviation of \( \sigma \) is given by

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\]

In particular, when we specify a mean of \( \mu = 0 \) and a standard deviation of \( \sigma = 1 \), we obtain a more simplified distribution equation:

\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\]

An incredible amount of statistical information can be obtained by understanding definite integrals of this distribution. Unfortunately, we cannot directly use the Fundamental Theorem of Calculus to compute these definite integrals since the integrand has no “nice” antiderivative. Taylor series and their analytic properties, however, give us a way to approximate these integrals.

(a) Use the Taylor series of \( e^x \) to find the Taylor series of \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Discuss the domain of convergence of this Taylor series. Is \( f(x) \) analytic on this domain?

(b) Graph the \( n \)-th degree Taylor polynomials for \( f(x) \) for \( n = 1, 2, 3, 4, 5 \) along with the graph of \( f(x) \) itself.

(c) Use the Taylor series from (a) to approximate

\[
\int_{0}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx.
\]

Using the Taylor polynomial of degree 5 may be wise. This number gives the probability that a randomly chosen sample from a normally distributed data set will fall between the mean and one standard deviation from the mean.

(d) For your approximation of the definite integral using the Taylor polynomial of degree 5, what is the maximum possible error?

(e) Use (c) to find the probability that a randomly chosen sample from a normally distributed data set will fall within one standard deviation of the mean. Check this answer with any number of charts or sheets available online.
Solution 5.

(a) Since $e^x$ is analytic on all $\mathbb{R}$, we get that

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

for all $y \in \mathbb{R}$. Thus, letting $y = -\frac{x^2}{2}$, we get that

$$e^{-x^2/2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!}.$$ 

This equality holds whenever $-x^2/2 \in \mathbb{R}$, which is true for all values of $x$. Thus, $e^{-x^2/2}$ is analytic on all $\mathbb{R}$. So, the Taylor series for $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!}.$$ 

(b) The graphs of the Taylor polynomials of increasing degree get very close to the graph of $f(x)$.

(c) Analytic properties about power series allow us to compute definite integrals in a term-by-term manner. Thus,

$$\int_0^1 f(x) \, dx = \int_0^1 \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} \, dx =$$

$$\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \int_0^1 x^{2k} \, dx = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \frac{1}{2k+1}.$$ 

Thus, for large enough values of $n$, we have that

$$\int_0^1 f(x) \, dx \approx \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!} \frac{1}{2k+1}.$$ 

Computing this for $n = 5$, we get that the integral is approximately .3413.

(d) Since this is an alternating series, we can bound the error of this approximation by the absolute value of the first ignored term, which is

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2^6 \cdot 6!} \cdot (13) = \frac{1}{\sqrt{2\pi}} \cdot \frac{599040}{6} \approx .000000665,$$

which is incredibly small for use of such a small value of $n$.

(e) The question asks us to compute $\int_{-1}^{1} f(x) \, dx$, which can be obtained by doubling the result from (c) since $f(x)$ is an even function. Thus, the probability that a randomly chosen sample from a normally distributed data set will fall within one standard deviation of the mean is approximately

$$2 \cdot .3413 = .6826.$$ 

Question 6. For the following limits, attempt to use L'Hôpital’s Rule straight away and explain why it is fruitless. Then, compute the limit by using an appropriate substitution “$y = \ldots$”, as we did in class, or by some other clever method.

(a) $\lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$
\[
(b) \lim_{x \to \infty} \frac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}}
\]

Solution 6.

(a) Applying L’Hôpital’s Rule straight away will lead to a situation which never simplified. Instead, we can multiply the top and bottom by \(e^{-x}\) and get that
\[
\lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \to \infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + 0}{1 - 0} = 1.
\]

(b) Applying L’Hôpital’s Rule straight away will lead to a situation where the terms become more and more complicated. Instead, we can multiply by \(x^{-1/2}\) on the top and bottom to get
\[
\lim_{x \to \infty} \frac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}} = \lim_{x \to \infty} \frac{1 + x^{-1}}{1 - x^{-1}} = \frac{1 + 0}{1 - 0} = 1.
\]