Question 1. Let $a, b \in \mathbb{R}$.

(a) Show that if $a + b$ is rational, then $a$ is rational or $b$ is irrational.

(b) Use (a) to show that if $a + b$ is rational, then $a$ and $b$ are both rational or both irrational.

Solution 1.

(a) We will instead prove the contrapositive statement, which is “if $a$ is irrational and $b$ is rational, then $a + b$ is irrational. Assume, to the contrary, that $a + b$ is rational. Then, since $b$ is rational, we have that $-b$ is also rational. Since the sum of rational numbers is rational, we get that

$$a = (a + b) - b \in \mathbb{Q}.$$ 

This, of course contradicts that $a$ is irrational. Since we have arrived at a contradiction, then our claim that $a + b$ is rational is false. Thus, $a + b$ is irrational. Having proven the contrapositive, our original statement “if $a + b$ is rational, then $a$ is rational or $b$ is irrational” is true. \(\square\)

(b) Assuming that $a + b$ is rational, (a) tells us that we have two cases: (1) $a$ is rational or (2) $b$ is irrational. For the first case, we assume that $a$ is rational. Thus $-a \in \mathbb{Q}$ and therefore

$$b = (a + b) - a \in \mathbb{Q}.$$ 

Therefore, $b$ is irrational and therefore $a$ and $b$ are both rational. In the second case we have that $b$ is irrational. We wish to show that $a$ is also irrational. Assume, to the contrary, that $a$ is rational. Then, $-a \in \mathbb{Q}$ as well. Thus,

$$b = (a + b) - a \in \mathbb{Q},$$ 

which, of course, contradicts that $b$ is irrational. Thus, $a$ must be irrational. So, $a$ and $b$ are irrational. \(\square\)

In class on Monday, we learned of boundedness, the supremum/infimum, and the Completeness Axiom. Given a bounded set $S \subseteq \mathbb{R}$, a number $b$ is called a supremum or least upper bound for $S$ if the following hold:

(i) $b$ is an upper bound for $S$, and
(ii) if $c$ is an upper bound for $S$, then $b \leq c$.

Similarly, given a bounded set $S \subseteq \mathbb{R}$, a number $b$ is called an infimum or greatest lower bound for $S$ if the following hold:

(i) $b$ is a lower bound for $S$, and
(ii) if $c$ is a lower bound for $S$, then $c \leq b$.

If $b$ is a supremum for $S$, we write that $b = \text{sup} \ S$. If it is an infimum, we write that $b = \text{inf} \ S$.

We were also introduced to our tenth and final axiom, the Completeness Axiom. This axiom states that any non-empty set $S \subseteq \mathbb{R}$ that is bounded above has a supremum; in other words, if $S$ is a non-empty set of real numbers that is bounded above, there exists a $b \in \mathbb{R}$ such that $b = \text{sup} \ S$.

Question 2. Show that if a set $S \subseteq \mathbb{R}$ has a supremum, then it is unique. Thus, we can talk about the supremum of a set, instead of the a supremum of a set.
Solution 2. Let $S$ be a set and assume that $b$ is a supremum for $S$ To show equality, assume as well that $c$ is also a supremum for $S$ and show that $b = c$. Since $c$ is a supremum, it is an upper bound for $S$. Since $b$ is a supremum, then it is the least upper bound and thus $b \leq c$. Similarly, since $b$ is a supremum, it is an upper bound for $S$; since $c$ is a supremum, it is a least upper bound and therefore $c \leq b$. Thus, $c \leq b$ and $b \leq c$, giving us that $b = c$. Thus, a supremum for a set is unique if it exists.

Question 3. Let $S$ be a non-empty subset of $\mathbb{R}$.

(a) Let $-S = \{-x \in \mathbb{R} \mid x \in S\}$. Show that $S$ has a supremum $b$ if and only if $-S$ has an infimum $-b$.

(b) Use (a) to show that if $T$ is a non-empty set that is bounded below, then $T$ has an infimum.

Solution 3.

(a) Assume that $b = \sup S$. Then, $x \leq b$ for all $x \in S$. Multiplying both sides by $-1$, we get that $-b \leq -x$ for all $x \in S$. Thus, $-b$ is a lower bound for the set $S$. Now, assume that $c$ is another lower bound for $-S$; we will show that $c \leq -b$. If not, then $-b < c$. Multiplying by $-1$, this would give us that $-c < b$. Notice that since $c$ is a lower bound for $-S$, then $c \leq y$ for all $y \in -S$. Since $y \in -S$, then $y = -x$ where $x \in S$. So, we have that $c \leq -x$ for all $x \in S$ and therefore $x < -c$ for all $x \in S$. So, $-c$ is an upper bound for $S$. Thus, $-c$ is an upper bound for $S$ and $-c < b$, contradicting that $b$ is a supremum for $S$.

The converse direction is an almost identical argument.

(b) Since $T$ is bounded below, say by $a$, then $a \leq x$ for all $x \in T$. Multiplying by $-1$, we get that $-x \leq -a$ for all $x \in T$. This is equivalence to $y \leq -a$ for all $y \in -T$. Thus, $-T$ is non-empty and bounded above. Thus, by the Completeness Axiom, $-T$ has a supremum $b$. By (a), we have that $-(-T) = T$ has an infimum $-b$, as desired.

Question 4. Prove the following Comparison Theorem: Let $S,T \subset \mathbb{R}$ be non-empty sets such that $s \leq t$ for every $s \in S$ and $t \in T$. If $T$ has a supremum, then so does $S$ and,

$$\sup S \leq \sup T.$$ 

Solution 4. Let $\tau = \sup T$. Since $\tau$ is a supremum for $T$, then $t \leq \tau$ for all $t \in T$. Let $s \in S$ and choose any $t \in T$. Then, since $s \leq t$ and $t \leq \tau$, then $s \leq \tau$. Thus, $\tau$ is an upper bound for $S$. By the Completeness Axiom, $S$ has a supremum, say $\sigma = \sup S$. We will show that $\sigma \leq \tau$. Notice that, by the above, $\tau$ is an upper bound for $S$. Since $\sigma$ is the least upper bound for $S$, then $\sigma \leq \tau$. Therefore,

$$\sup S \leq \sup T.$$

$\square$

Question 5. Consider the set

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}.$$ 

(a) Show that $\max S = 1$. 

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(b) Show that if \(d\) is a lower bound for \(S\), then \(d \leq 0\). [Hint: A proof by contradiction might be helpful, as well as the Archimedean Property.]

(c) Use (b) to show that \(0 = \inf S\).

**Solution 5.**

(a) Let \(x = \frac{1}{n} \in S\), where \(n \geq 1\). Since \(1 \leq n\), we have that \(x = \frac{1}{n} \leq 1\). Thus, for every \(x \in S\), \(x \leq 1\) and 1 is an upper bound. Notice as well that 1 = \(\frac{1}{1}\) \(\in S\). Thus, 1 = \(\max S\).

(b) Let \(d\) be a lower bound for \(S\). Thus, for every \(s \in S\), \(d \leq s\). Assume, to the contrary, that \(d > 0\). Using the Archimedean property, we know that there exists an \(n \in \mathbb{Z}^+\) such that \(1 < dn\). Since \(n > 0\), this gives us that \(1 < d\). But, \(\frac{1}{n} \in S\), and this contradicts the fact that \(d\) is a lower bound for \(S\). Thus, we must conclude that \(d \leq 0\).

(c) Clearly 0 is a lower bound for \(S\) since \(0 \leq \frac{1}{n}\) for all \(n \in \mathbb{Z}^+\). If \(d\) is any other lower bound, then by (b), \(d \leq 0\). Thus, 0 is greatest lower bound and so \(0 = \inf S\).

**Question 6.** Consider the set

\[ T = \left\{ (-1)^n \left(1 - \frac{1}{n}\right) \mid n \in \mathbb{Z}^+ \right\}. \]

(a) Show that 1 is an upper bound for \(T\).

(b) Similar to 5b, show that if \(d\) is an upper bound for \(T\), then \(d \geq 1\).

(c) Use (a) and (b) to show that \(\sup T = 1\).

**Solution 6.**

(a) We will show that for any \(x \in T\), \(x \leq 1\). Since \(x \in T\), then \(x = (-1)^n(1 - 1/n)\) for some \(n \in \mathbb{Z}^+\). Since \(\frac{1}{n} > 0\), then \(1 - \frac{1}{n} < 1\). We argue our desired inequality in two cases. If \(n\) is even, then \(x = (-1)^n(1 - 1/n) = 1 - 1/n < 1\). If \(n\) is odd, then \(x = (-1)^n(1 - 1/n) = 1 - 1/n < 0 < 1\). In either case, \(x \leq 1\) (in fact, \(< 1\)) and 1 is an upper bound for \(T\).

(b) Let \(d\) be an upper bound for \(T\). Thus, \((-1)^n \left(1 - \frac{1}{n}\right) \leq d\) for all \(n \in \mathbb{Z}^+\). Assume, to the contrary that \(d < 1\). Thus, \(1 - d > 0\). By the Archimedean Property, there exists an \(n \in \mathbb{Z}^+\) such that \(1 < (1 - d)n\). Since \(n > 0\), we can re-write this as \(\frac{1}{n} < 1 - d\), which is equivalent to

\[ d < 1 - \frac{1}{n}. \]

If \(n\) is even, then \((-1)^n = 1\) and we have that

\[ d < (-1)^n \left(1 - \frac{1}{n}\right) \in T, \]

contradicting the fact that \(d\) is an upper bound. If \(n\) is odd, then consider instead \(n + 1\), which is even. Then, \((-1)^{n+1} = 1\) and

\[ d < 1 - \frac{1}{n} < (-1)^{n+1} \left(1 - \frac{1}{n+1}\right) \in T. \]

This again contradicts that \(d\) is an upper bound for \(T\). Either way, we reach a contradiction and therefore conclude that \(d \geq 1\).

(c) By (a), 1 is an upper bound for \(T\). By (b), if \(d\) is any other upper bound, then \(1 \leq d\). Thus, \(\sup T = 1\).