In class, we learned that a function \( f : S \to T \) between metric spaces \((S, d_S)\) and \((T, d_T)\) is continuous if and only if the pre-image of every open set in \( T \) is open in \( S \). In other words, \( f \) is continuous if for all open \( U \subset T \), the pre-image \( f^{-1}(U) \subset S \) is open in \( S \).

**Question 1.** Let \( S, T, \) and \( R \) be metric spaces and let \( f : S \to T \) and \( g : T \to R \). We can define the composition function \( g \circ f : S \to R \) by

\[
g \circ f(s) = g(f(s)).
\]

(a) Let \( U \subset R \). Show that 
\[
(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))
\]

(b) Use (a) to show that if \( f \) and \( g \) are continuous, then the composition \( g \circ f \) is also continuous.

**Solution 1.**

(a) We will show that 
\[
(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))
\]

by showing that 
\[
(g \circ f)^{-1}(U) \subset f^{-1}(g^{-1}(U)) \quad \text{and} \quad f^{-1}(g^{-1}(U)) \subset (g \circ f)^{-1}(U).
\]

For the first direction, let \( x \in (g \circ f)^{-1}(U) \). Then, \( g \circ f(x) \in U \). Thus, \( g(f(x)) \in U \). Since \( g(f(x)) \in U \), then \( f(x) \in g^{-1}(U) \). Continuing we get that \( x \in f^{-1}(g^{-1}(U)) \). Thus, 
\[
(g \circ f)^{-1}(U) \subset f^{-1}(g^{-1}(U)).
\]

Conversely, assume that \( x \in f^{-1}(g^{-1}(U)) \). Then, \( f(x) \in g^{-1}(U) \). Furthermore, \( g(f(x)) \in U \). Thus, \( g \circ f(x) \in U \). So, \( x \in (g \circ f)^{-1}(U) \). So, \( f^{-1}(g^{-1}(U)) \subset (g \circ f)^{-1}(U) \).

Thus, 
\[
(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).
\]

(b) Let \( U \) be open in \( R \). Since \( g \) is continuous, then \( g^{-1}(U) \subset T \) is open. Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \) is open. Thus, by (a), \( (g \circ f)^{-1}(U) \) is open. So, \( g \circ f \) is continuous.

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**Question 2.** Let \((S, d_S)\) and \((T, d_T)\) be metric spaces and let \( f : S \to T \).

(a) A function is called **constant** if \( f(s) = t_0 \) for all \( s \in S \). Show that any constant function is continuous.

(b) Show that if \( d_S \) is the discrete metric, then any function \( f \) is continuous.

**Solution 2.**

(a) Let \( U \) be an open set in \( T \). We will show that \( f^{-1}(U) \) is open. We do so in two cases: \( t_0 \in U \) and \( t_0 \notin U \).

If \( t_0 \in U \), then since \( f(s) = t_0 \) for all \( s \in S \), \( f^{-1}(U) = S \), which is always open in \( S \). If \( t_0 \notin U \), then \( f^{-1}(U) = \emptyset \), which is open. In either case, the pre-image of every open set is open. So the constant function \( f \) is continuous.

(b) Recall that in a discrete metric space, every subset is open. Thus, given any open \( U \subset T \), \( f^{-1}(U) \subset S \) is automatically open. Thus, \( f \) is continuous.

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**Question 3.** The **floor function** \( f : \mathbb{R} \to \mathbb{R} \) is given by \( f(x) = \lfloor x \rfloor \), where \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \).
(a) Let $a \notin \mathbb{Z}$. Use an $\varepsilon - \delta$ proof to show that $f(x) = |x|$ is continuous at $a$.

(b) Let $a \in \mathbb{Z}$. Show that $f(x) = |x|$ is not continuous at $a$. To do so, find an $\varepsilon > 0$ such that for any $\delta > 0$, there exists an $x$ with $|x - a| < \delta$ such that $|f(x) - f(a)| \geq \varepsilon$.

Solution 3.

(a) Let $a \notin \mathbb{Z}$. Given $\varepsilon > 0$, let $\delta = \min\{a - |a|, |a + 1| - a\}$. Since $a \notin \mathbb{Z}$, then $a \neq |a|$ and $|a + 1| \neq a$. Thus, $\delta > 0$. Notice that for all $x$ satisfying $|x - a| < \delta$, we have that $f(x) = |x| = |a|$. Thus, $|f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$. Thus, $f$ is continuous at $a$.

(b) Let $a \in \mathbb{Z}$. Then, $f(a) = |a| = a$. Let $\varepsilon = 1/2$. Let $\delta > 0$ and consider $a - \delta/2$. Since $a \in \mathbb{Z}$, then $f(a - \delta/2) < a$. In particular, since $f$ only takes on integral values, $f(a) - f(a - \delta/2) \geq 1$. Thus, $|f(a - \delta/2) - f(a)| \geq 1 > \varepsilon$.

Thus, $f$ is discontinuous at $a$.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Assume that $f(x) \geq 0$ for all $x \in [0, 1]$. Show that if $f(c) > 0$ for some $c \in (0, 1)$, then

$$\int_0^1 f(x) \, dx > 0.$$  

(b) Show that the above is no longer true if the term “continuous” is dropped. That is, given an example of a (necessarily discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0$ and $f(c) > 0$ for some $c \in (0, 1)$, yet

$$\int_0^1 f(x) \, dx = 0.$$ 

Solution 4.

(a) Since $f$ is continuous, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$, then $|f(x) - f(c)| < f(c)/2$. Thus, for $x$ satisfying $|x - c| < \delta$ (which is equivalent to $-\delta < x - c < \delta$, we have that

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}.$$ 

Using the first inequality and adding $f(c)$ to both sides, we get that

$$\frac{f(c)}{2} < f(x)$$

for all $x$ satisfying $-\delta < x - c < \delta$. Since this last pair of inequalities is equivalent to $c - \delta < x < c + \delta$, for these $x$, we have that $\frac{f(c)}{2} < f(x)$. Thus,

$$0 < \frac{f(c)}{2} \cdot 2\delta = \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} \, dx \leq \int_{c-\delta}^{c+\delta} f(x) \, dx \leq \int_0^1 f(x) \, dx.$$
(b) Consider the piecewise function given by

\[ f(x) = \begin{cases} 
0 & \text{if } x \neq 1/2 \\
1 & \text{if } x = 1/2 
\end{cases} \]

Then \( f(x) \geq 0 \) and \( f(1/2) > 0 \), but \( \int_0^1 f(x) = 0 \).

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**Question 5.** Recall that we can equip \( C([0,1]) \), the space of all continuous functions on \([0,1]\), with its \( L^1 \) metric, which is given by

\[ d(f, g) = \int_0^1 |f(x) - g(x)| \, dx. \]

Consider the function \( \varphi : C([0,1]) \to \mathbb{R} \) given by

\[ \varphi(f) = \int_0^1 f(x) \, dx. \]

In this question, we will show that \( \varphi \) is a continuous function.

(a) Show that

\[ \left| \int_0^1 h(x) \, dx \right| \leq \int_0^1 |h(x)| \, dx. \]

Hint: We previously proved that \(-|a| \leq a \leq |a|\) for all \( a \in \mathbb{R} \).

(b) Use the above to give an \( \varepsilon-\delta \) proof that \( \varphi \) is continuous.

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**Solution 5.**

(a) Notice that for all \( x \), \(-|h(x)| \leq h(x) \leq |h(x)|\). Integrating each side, we get that

\[ -\int_0^1 |h(x)| \, dx \leq \int_0^1 h(x) \, dx \leq \int_0^1 |h(x)| \, dx. \]

This is equivalent to

\[ \left| \int_0^1 h(x) \, dx \right| \leq \int_0^1 |h(x)| \, dx. \]

(b) We will show that \( \varphi \) is continuous at any \( f \in C([0,1]) \). Given \( \varepsilon > 0 \), let \( \delta = \varepsilon > 0 \). Then, for all \( g \in C([0,1]) \) satisfying

\[ \int_0^1 |g(x) - f(x)| \, dx < \delta = \varepsilon, \]

we can use the above fact to get that

\[ |\varphi(g) - \varphi(f)| = \left| \int_0^1 g(x) \, dx - \int_0^1 f(x) \, dx \right| = \left| \int_0^1 g(x) - f(x) \, dx \right| \leq \int_0^1 |g(x) - f(x)| \, dx < \varepsilon. \]

Thus, \( |\varphi(g) - \varphi(f)| < \varepsilon \), as desired. So, \( \varphi \) is continuous at any \( f \in C([0,1]) \) and thus \( \varphi \) is a continuous function.
**Question 7.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \not\in \mathbb{Q} \end{cases}$$

We will show that $f$ is continuous only at $a = 0$.

(a) Use an $\varepsilon-\delta$ proof to show that $f(x)$ is continuous at $a = 0$.

(b) Use the theorem relating convergent sequences to continuous functions to show that if $a \neq 0$, then $f(x)$ is not continuous at $a$.

**Solution 7.**

(a) Given $\varepsilon > 0$, let $\delta = \varepsilon$. We will show that for any $x$ satisfying $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. So, let $x$ satisfy $|x| = |x - 0| < \delta = \varepsilon$. We take two cases: $x \in \mathbb{Q}$ or $x \not\in \mathbb{Q}$. If $x \in \mathbb{Q}$, then $f(x) = x$. Thus,

$$|f(x) - f(0)| = |x - 0| < \varepsilon = \delta.$$ 

In the second case, if $x \not\in \mathbb{Q}$, then $f(x) = 0$, so $|f(x) - f(0)| = |0 - 0| < \varepsilon$. In either case, we have that if $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. Thus, $f$ is continuous at $a = 0$.

(b) Let $a \neq 0$. We will consider the two cases: $a \in \mathbb{Q}$ or $a \not\in \mathbb{Q}$. If $a \in \mathbb{Q}$, then, let $x_n$ be a sequence of irrational numbers converging to $a$. If $f$ were continuous at $a$, then $f(x_n) \to f(a)$. However, for all $n$, $f(x_n) = 0$, which converges to 0. However, since $a \in \mathbb{Q}$, $f(a) = a \neq 0$. Thus, $f(x_n) \not\to f(a)$. So, $f$ is discontinuous at $a$. For the second case, assume that $a \not\in \mathbb{Q}$. Then, there exists a sequence of rational numbers $x_n$ such that $x_n \to a$. If $f$ were continuous at $a$, then $f(x_n) \to f(a)$. But $f(x_n) = x_n$ since $x_n \in \mathbb{Q}$. Thus, $f(x_n) = x_n \to a$. However, since $a \not\in \mathbb{Q}$, $f(a) = 0 \neq a$. Thus, $f(x_n) \not\to f(a)$. So, $f$ is discontinuous at $a$. So, at any $a \neq 0$, $f$ is discontinuous at $a$. 
