Math 431 - Real Analysis I
Solutions to Homework due November 21

Question 1. The following questions use the ever-important Mean Value Theorem.

(a) Let \( f(x) \) be any quadratic polynomial \( f(x) = \alpha x^2 + \beta x + \gamma \). Consider the secant line joining the points \((t_1, f(t_1))\) and \((t_2, f(t_2))\). What is the slope of this secant line (in terms of \( \alpha, \beta, \gamma, \) and \( t_i \))? Simplify as much as possible.

(b) For the \( f \) in (a), the Mean Value Theorem guarantees the existence of some \( c \in (t_1, t_2) \) such that \( f'(c) \) is equal to the above slope. For this particular \( f \), what is this point \( c \)?

(c) Use the Mean Value Theorem to deduce the following inequality for all \( x, y \):

\[
|\sin y - \sin x| \leq |y - x|.
\]

Solution 1.

(a) The slope of the second line joining these points is given by

\[
\frac{f(t_1) - f(t_2)}{t_1 - t_2} = \frac{\alpha t_1^2 + \beta t_1 + \gamma - (\alpha t_2^2 + \beta t_2 + \gamma)}{t_1 - t_2} = \frac{\alpha(t_1^2 - t_2^2) + \beta(t_1 - t_2)}{t_1 - t_2} = \frac{(t_1 - t_2)[\alpha(t_1 + t_2) + \beta]}{t_1 - t_2} = \alpha(t_1 + t_2) + \beta.
\]

(b) Taking the derivative, we have that

\[
f'(x) = 2\alpha x + \beta.
\]

This will equal our mean slope when

\[
2\alpha x + \beta = \alpha(t_1 + t_2) + \beta,
\]

which occurs when \( x = \frac{t_1 + t_2}{2} \), the midpoint of \( t_1 \) and \( t_2 \).

(c) Consider \( f(x) = \sin x \). For any \( x < y \in \mathbb{R} \), we have that there exists some \( c \) such that \( x < c < y \) such that

\[
f'(c) = \frac{\sin y - \sin x}{y - x}.
\]

Since the derivative of \( \sin x \) is \( \cos x \), then \( |f'(c)| \leq 1 \). Thus, we have that

\[
\left| \frac{\sin y - \sin x}{y - x} \right| \leq 1.
\]

Cross-multiplying, we get that \( |\sin y - \sin x| \leq |y - x| \).

Question 2. Let \( f \) be a function that is continuous on \([a, b]\) and second differentiable (i.e., \( f'' \) exists) on \((a, b)\). Assume that the line segment joining the points \( A = (a, f(a)) \) and \( B = (b, f(b)) \) intersect the graph of \( f \) in a third point different from \( A \) and \( B \). Show that \( f''(c) = 0 \) for some \( c \in (a, b) \).

Solution 2. We will use the MVT thrice. First, label the point where the secant line intersect the graph as \( D = (d, f(d)) \). Then, notice that the slope of the secant line from \( A \) to \( B \) is the same as the slope of the secant line from \( A \) to \( D \) and from \( D \) to \( B \). Thus,

\[
\frac{f(b) - f(a)}{b - a} = \frac{f(d) - f(a)}{d - a} = \frac{f(b) - f(d)}{b - d}.
\]
Using the MVT on \([a, d]\), we get that there exists some \(\alpha \in (a, d)\) such that
\[
f'(\alpha) = \frac{f(d) - f(a)}{d - a}
\]
Similarly, using the MVT on \([d, b]\), there exists some \(\beta \in [d, b]\) such that
\[
f'(\beta) = \frac{f(d) - f(b)}{d - b}.
\]
Because these two secant slopes are equal, we have that
\[
f'(\alpha) = f'(\beta).
\]
Now, we can use MVT on the interval \([\alpha, \beta]\) with the differentiable function \(f'(x)\). Doing so, we get that there exists some \(c \in [\alpha, \beta]\) such that
\[
f''(c) = \frac{f'(\alpha) - f'(\beta)}{\alpha - \beta} = 0.
\]
Thus, \(f''(c) = 0\) as desired.

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**Question 3.** Let \(f\) and \(g\) be differentiable functions. Show that if \(f'(x) = g'(x)\) for all \(x\), then \(f(x) = g(x) + k\) where \(k \in \mathbb{R}\).

**Solution 3.** Consider the function \(f(x) - g(x)\), which is also differentiable. Notice that its derivative is \(f'(x) - g'(x) = 0\). Thus, by a theorem in class, \(f(x) - g(x)\) is constant. So, \(f(x) - g(x) = k\) for some \(k \in \mathbb{R}\). Thus, \(f(x) = g(x) + k\).

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**Question 4.** The hypotheses of the Mean Value Theorem are each quite important. They state that \(f\) must be continuous on \([a, b]\) and differentiable on \((a, b)\).

(a) Find a counterexample to the MVT if the hypothesis “\(f\) is differentiable on \((a, b)\)” is dropped. To do this, find a function that is continuous on \([a, b]\) but not differentiable on \((a, b)\) where
\[
f'(c) \neq \frac{f(b) - f(a)}{b - a}
\]
for all \(c\).

(b) Find a counterexample to the MVT if the hypothesis “\(f\) is continuous on \([a, b]\)” is dropped. To do this, find a function that is not continuous on all of \([a, b]\) but \(f\) is differentiable on \((a, b)\) where
\[
f'(c) \neq \frac{f(b) - f(a)}{b - a}
\]
for all \(c\).

**Solution 4.**

(a) Consider the function \(f(x) = |x|\), which is continuous on the closed interval \([-1, 1]\), but is not differentiable at 0. Notice that
\[
\frac{f(-1) - f(1)}{-1 - 1} = 0.
\]
However, at any \(c \in [-1, 1]\) where \(f'\) does exist, the derivative is always \(\pm 1\), but never 0.
(b) Consider the function defined on \([0, 1]\) given by

\[
f(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } 0 < x \leq 1
\end{cases}
\]

Notice that \(f\) is discontinuous only at \(x = 0\). On \((0, 1)\), \(f'(x) = 0\) since it is constant on that open interval. However, the mean slope is given by

\[
\frac{f(1) - f(1)}{1 - 0} = -1.
\]

Thus, there is no \(c\) such that \(f'(c)\) is equal to the mean slope.

**Question 5.** Let \(a, r \in \mathbb{R}\) with \(r \neq 1\). Use induction to show that

\[
\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}
\]

for all \(n \geq 0\).

**Solution 5.** Let \(A(n)\) be the statement that

\[
\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}.
\]

We will show that \(A(n)\) is true for all \(n \geq 0\). For the base case, notice that

\[
\sum_{k=0}^{0} ar^k = ar^0 = a = a \cdot \frac{1 - r}{1 - r} = \frac{a - ar^0}{1 - r}.
\]

Thus, \(A(0)\) holds.

Now, assume that \(A(n)\) hold for some \(n \geq 0\). We will show that \(A(n+1)\) also holds. Starting with the left-hand side of the \(A(n+1)\) expression, we have that

\[
\sum_{k=0}^{n+1} ar^k = \sum_{k=0}^{n} ar^k + ar^{n+1} = \frac{a - ar^{n+1}}{1 - r} + ar^{n+1} = \frac{a - ar^{n+1} - ar^{n+2}}{1 - r} = \frac{a - ar^{(n+1)+1}}{1 - r}.
\]

Thus, \(A(n+1)\) is true. So, by induction, \(A(n)\) holds for all \(n \geq 0\).

**Question 6.** In this question, we will show that if \(|r| < 1\), then \(r^n \to 0\).

(a) State the binomial theorem. Use it to show that if \(b > 0\), then \((1 + b)^n > nb\).

(b) Prove that if \(|r| < 1\), then \(r^n \to 0\) using an \(\varepsilon - N\) proof. To do so, it would be wise to note that if \(|r| < 1\), then

\[
|r| = \frac{1}{1 + b}
\]

for some \(b > 0\).
Solution 6.

(a) The binomial theorem states that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\]

Thus, for \((1 + b)^n\), we have that

\[(1 + b)^n = 1 + nb + \binom{n}{2} b^2 + \cdots + b^n.\]

Since \(b > 0\), all the terms are positive. Thus, the sum is greater than or equal to each individual term and thus \((1 + b)^n > nb\).

(b) Let \(\varepsilon > 0\). Since \(|r| < 1\), then we can write it as

\[|r| = \frac{1}{1+b}\]

for some \(b > 0\). Consider \(N = \frac{1}{\varepsilon b} > 0\). Assume that \(n > \varepsilon = \frac{1}{\varepsilon b}\). Then, we have that

\[|r^n - 0| = |r|^n = \frac{1}{(1+b)^n}.\]

Since \((1+b)^n > nb\), we have that

\[\frac{1}{(1+b)^n} < \frac{1}{nb}\]

Since \(n > \frac{1}{\varepsilon b}\), we have that

\[\frac{1}{nb} < \varepsilon.\]

Thus, \(|r^n - 0| < \varepsilon\). So, we have that \(r^n \rightarrow 0\).