Question 1.

(a) Let $a_k, b_k \geq 0$ for all $k$. Show that if $\sum_{k=0}^{\infty} a_k$ converges and $b_k$ is a bounded sequence, then $\sum_{k=0}^{\infty} a_k b_k$ converges as well.

(b) Find a counterexample to above statement if the hypothesis “$a_k, b_k \geq 0$” is removed.

Solution 1.

(a) Since $b_k$ is a bounded sequence, we have that $|b_k| \leq M$ for some $M$. Since $b_k \geq 0$, we have that $b_k \leq M$ for all $k$. Since $a_k \geq 0$, we have that $0 \leq a_k b_k \leq M \cdot a_k$.

Notice that since $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} M \cdot a_k$ converges as well. Thus, by the Comparison Test, we have that $\sum_{k=0}^{\infty} a_k b_k$ converges as well.

(b) Consider the convergent sequence $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$

and the bounded sequence $b_k = (-1)^k$. Notice that the sequence $a_k b_k = \frac{1}{k}$, but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Question 2. Show that

$$\sum_{k=2}^{\infty} \frac{1}{k(ln k)^p}$$

converges if and only if $p > 1$.

Solution 2. We will show that $\sum_{k=2}^{\infty} \frac{1}{k(ln k)^p}$ converges if and only if $p > 1$ by using the integral test. Consider the function

$$f(x) = \frac{1}{x(ln x)^p}$$

defined for $x \geq 2$. Notice that $f(x) \geq 0$ for all $x \geq 2$ and that

$$0 \leq \frac{1}{x(ln x)^p} \leq \frac{1}{x}.$$

Thus, by the Squeeze Theorem, we have that $\lim_{x \to \infty} f(x) = 0$. Furthermore, notice that if $x \leq y$, then $(ln(x))^p \leq (ln(y))^p$. Thus, we get that $x(ln x)^p \leq y(ln y)^p$. Thus,

$$f(y) = \frac{1}{y(ln y)^p} \leq \frac{1}{x(ln x)^p} = f(x).$$
Thus, $f$ is a decreasing function. So, we can apply the integral test. Notice that letting $u = \ln x$ and $du = \frac{dx}{x}$, we have that
\[
\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{b \to \infty} \int_2^b \frac{dx}{x(\ln x)^p} = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^p}.
\]

If $p = 1$, then the above integral evaluates to
\[
\lim_{b \to \infty} \ln |\ln b| - \ln 2 = \infty.
\]

If $p \neq 1$, then the above integral evaluates to
\[
\lim_{b \to \infty} \frac{b^{1-p}}{1-p} + \frac{1}{p-1}.
\]
This will converge if and only if $p < 1$.

Thus, by the integral test, our series converges if and only if $p > 1$.

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**Question 3.**

(a) Give an example of a divergent series $\sum_{k=1}^\infty a_k$ for which $\sum_{k=1}^\infty a_k^2$ converges.

(b) Prove that if $a_k \geq 0$ and $\sum_{k=1}^\infty a_k$ converges, then $\sum_{k=1}^\infty a_k^2$ converges as well.

(c) Find a counterexample to the above statement if the hypothesis “$a_k \geq 0$” is removed.

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**Solution 3.**

(a) Consider the divergent series
\[
\sum_{k=1}^\infty \frac{1}{k^2},
\]

Notice that
\[
\sum_{k=1}^\infty \frac{1}{k^2}
\]
converges.

(b) Since $\sum_{k=1}^\infty a_k$ converges, then $a_k \to 0$ by the Divergence Test. Thus, there exists some $N$ such that for all $n > N$, $|a_n - 0| < 1$, and thus $a_n < 1$. So, for all $n > N$, $a_n < 1$ and thus $a_n^2 < a_n$. Thus, by the Generalized Comparison Test, since $\sum_{k=1}^\infty a_k$ converges, then $\sum_{k=1}^\infty a_k^2$ converges as well.

(c) Consider the alternating series
\[
\sum_{k=1}^\infty \frac{(-1)^k}{\sqrt{k}},
\]
which converges by the Alternating Series Test. Squaring the terms, we get that
\[
\sum_{k=1}^\infty \frac{1}{k}
\]
diverges.
**Question 4.** In this question, we will show that if both

\[ \sum_{k=0}^{\infty} a_k^2 \text{ and } \sum_{k=0}^{\infty} b_k^2 \]

converges, then

\[ \sum_{k=0}^{\infty} a_k b_k \]

converges absolutely.

(a) Show that \(2|a_k b_k| \leq a_k^2 + b_k^2\) for all \(k\).

(b) Use (a) to show that if \(\sum_{k=0}^{\infty} a_k^2\) and \(\sum_{k=0}^{\infty} b_k^2\) converges, then \(\sum_{k=0}^{\infty} a_k b_k\) converges absolutely.

**Solution 4.**

(a) Notice that \((a_k - b_k)^2 \geq 0\). Thus, we have that \(a_k^2 - 2a_k b_k + b_k^2 \geq 0\). Rewriting, we get that

\[2a_k b_k \leq a_k^2 + b_k^2.\]

Similarly, consider \((a_k + b_k)^2 \geq 0\). Thus, we have that \(a_k^2 + 2a_k b_k + b_k^2 \geq 0\). Re-writing, we have that

\[-(a_k^2 + b_k)^2 \leq 2a_k b_k.\]

Putting these two inequalities together, we have that

\[-(a_k^2 + b_k)^2 \leq 2a_k b_k \leq a_k^2 + b_k^2,\]

which is equivalent to our desired result.

(b) Since \(\sum_{k=0}^{\infty} a_k^2\) and \(\sum_{k=0}^{\infty} b_k^2\) converges, their sum also converges and is, in fact, equal to \(\sum_{k=0}^{\infty} a_k^2 + b_k^2\). Since \(0 < 2|a_k b_k| \leq a_k^2 + b_k^2\), by the Comparison Theorem, we have that

\[\sum_{k=0}^{\infty} 2|a_k b_k|\]

converges, and thus

\[\sum_{k=0}^{\infty} |a_k b_k|\]

converges. Thus,

\[\sum_{k=0}^{\infty} a_k b_k\]

converges absolutely.

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**Question 5.** Let \(a_n\) be a sequence of non-zero real numbers such that the sequence \(\frac{a_{n+1}}{a_n}\) of ratios is a constant sequence. Show that \(\sum_{k=1}^{\infty} a_k\) is a geometric series.

**Solution 5.** Let’s assume that \(\frac{a_{n+1}}{a_n}\) is equal to the constant sequence \(r\). Then, we have that for all \(n\),

\[\frac{a_{n+1}}{a_n} = r\]
and thus \(a_{n+1} = ra_n\). This gives a recursive definition of \(a_n\). So, assume that \(a_0 = c\). Then, \(a_1 = ra_0 = c \cdot r\). Similarly \(a_2 = r \cdot a_1 = c \cdot r^2\). Continuing in this way (using induction), one can see that \(a_n = cr^n\). Thus, 
\[
\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} cr^k, \text{ a geometric series.}
\]

In class, we learned of the following definition for the limit superior and limit inferior of a sequence \(x_n\).

Suppose a number \(U\) has the following two properties:

- For all \(\varepsilon > 0\), there exists an \(N\) such that for all \(n > N\), \(x_n < U + \varepsilon\); and
- For all \(\varepsilon > 0\) and \(m > 0\), there exists an \(n > m\) such that \(x_n > U - \varepsilon\).

Then, \(\lim \sup x_n = U\).

Similarly, suppose that a number \(L\) has the following two properties:

- For all \(\varepsilon > 0\), there exists an \(N\) such that for all \(n > N\), \(x_n > L - \varepsilon\); and
- For all \(\varepsilon > 0\) and \(m > 0\), there exists an \(n > m\) such that \(x_n < L + \varepsilon\).

Then \(\lim \inf x_n = L\).

**Question 6.** Let \(x_n\) be a sequence. If \(\lim \inf x_n = A = \lim \sup x_n\), show that \(x_n\) converges and that, in fact, \(x_n \to A\).

**Solution 6.** Let \(\varepsilon > 0\). Since \(\lim \inf x_n = A\), then there exists an \(N_−\) such that for all \(n > N_−\), \(x_n > A - \varepsilon\). Since \(\lim \sup x_n = A\), then there exists an \(N_+\) such that for all \(n > N_+\), \(x_n < A + \varepsilon\). Choose \(N = \max\{N_−, N_+\}\). Then, for all \(n > N\), we have that \(A - \varepsilon < x_n < A + \varepsilon\), which is equivalent to \(|x_n - A| < \varepsilon\). So, \(x_n \to A\).

**Question 7.** Consider the series
\[
\sum_{k=1}^{\infty} \sqrt{k+1} - \sqrt{k}.
\]

(a) Show that
\[
\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.
\]

(b) Use (a) to decide if the series converges or diverges.

**Solution 7.**

(a) If we take our term and multiply by a special form of 1, we get
\[
\sqrt{k+1} - \sqrt{k} \cdot \frac{\sqrt{k+1} + \sqrt{k}}{\sqrt{k+1} + \sqrt{k}} = \frac{k + 1 - k}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.
\]
(b) We can now limit compare our terms to $\frac{1}{\sqrt{k}}$ to get

$$\frac{\frac{1}{\sqrt{k+1} + \sqrt{k}}}{\frac{1}{\sqrt{k}}} = \sqrt{k} = \frac{1}{\sqrt{k+1} + 1} = \frac{1}{2}.$$ 

Thus, since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges by the $p$-series test, we have that our original series diverges as well.

**Question 8.** Consider the series $\sum_{k=1}^{\infty} a_k$ where

$$a_k = \left(\frac{2}{(-1)^k - 3}\right)^k.$$ 

(a) Compute $\frac{a_{k+1}}{a_k}$ for general $k$. Use this to compute

$$\limsup \frac{a_{k+1}}{a_k} \quad \text{and} \quad \liminf \frac{a_{k+1}}{a_k}.$$ 

Can you use the Ratio Test to reach a conclusion about the convergence or divergence of the series?

(b) Compute $|a_k|^{1/k}$ for general $k$. Use this to compute

$$\limsup |a_k|^{1/k}.$$ 

Can you use the Root Test to reach a conclusion about the convergence or divergence of the series?

(c) In spite of the powerful Ratio Test and Root Tests, at times the more rudimentary tests are more helpful. Show that our series diverges using the Divergence Test.

**Solution 8.**

(a) Consider $\frac{a_{k+1}}{a_k}$. Performing this division, we get

$$\frac{a_{k+1}}{a_k} = \frac{2}{(-1)^{k+1} - 3} \frac{(-1)^{k+1} - 3}{(-1)^k - 3} = \frac{2}{(-1)^k - 3} \frac{(-1)^k - 3}{(-1)^{k+1} - 3} = 2.$$ 

In absolute values, when $k$ is even, this gives

$$\left|\frac{2}{(-4)^{k+1}}\right| = 2^k \frac{1}{2^{k+1}} = \left(\frac{1}{2}\right)^{k+1}.$$ 

In absolute values, when $k$ is odd, this gives

$$\left|\frac{2}{(-2)^{k+1}}\right| = \left(\frac{4}{2}\right)^k = 2^k.$$ 

Thus, the sequence of ratios alternates between $\frac{1}{2^{k+1}}$ and $2^k$. Thus, the limit superior is $\infty$ and the limit inferior is $0$. So, the Ratio Test does not yield an answer.
(b) Consider $|a_k|^{1/k}$ to get

$$|a_k|^{1/k} = \left| \frac{2}{(-1)^k - 3} \right|.$$ 

This sequence of ratios is equal to 1 when $k$ is even and $\frac{1}{2}$ when $k$ is odd. Thus, the limit superior of this sequence is 1, in which case the Root Test also does not yield an answer.

(c) Notice that when $k$ is even, we get that $|a_k| = 1$. Thus, there is no way that $a_k \to 0$. So, by the Divergence Test, the series diverges.