Question 1. Let \( a, b \in \mathbb{R} \).

(a) Show that if \( a + b \) is rational, then \( a \) is rational or \( b \) is irrational.

(b) Use (a) to show that if \( a + b \) is rational, then \( a \) and \( b \) are both rational or both irrational.

In class on Monday, we learned of boundedness, the supremum/infimum, and the Completeness Axiom.

Given a bounded set \( S \subset \mathbb{R} \), a number \( b \) is called a \textit{supremum} or \textit{least upper bound} for \( S \) if the following hold:

(i) \( b \) is an upper bound for \( S \), and

(ii) if \( c \) is an upper bound for \( S \), then \( b \leq c \).

Similarly, given a bounded set \( S \subset \mathbb{R} \), a number \( b \) is called an \textit{infimum} or \textit{greatest lower bound} for \( S \) if the following hold:

(i) \( b \) is a lower bound for \( S \), and

(ii) if \( c \) is a lower bound for \( S \), then \( c \leq b \).

If \( b \) is a supremum for \( S \), we write that \( b = \sup S \). If it is an infimum, we write that \( b = \inf S \).

We were also introduced to our tenth and final axiom, the \textit{Completeness Axiom}. This axiom states that any non-empty set \( S \subset \mathbb{R} \) that is bounded above has a supremum; in other words, if \( S \) is a non-empty set of real numbers that is bounded above, there exists a \( b \in \mathbb{R} \) such that \( b = \sup S \).

Question 2. Show that if a set \( S \subset \mathbb{R} \) has a supremum, then it is unique. Thus, we can talk about \textit{the} supremum of a set, instead of the \textit{a} supremum of a set.

Question 3. Let \( S \) be a non-empty subset of \( \mathbb{R} \).

(a) Let \( -S = \{ -x \in \mathbb{R} \mid x \in S \} \). Show that \( S \) has a supremum \( b \) if and only if \( -S \) has an infimum \( -b \).

(b) Use (a) to show that if \( T \) is a non-empty set that is bounded below, then \( T \) has an infimum.

Question 4. Prove the following \textit{Comparison Theorem}: Let \( S, T \subset \mathbb{R} \) be non-empty sets such that \( s \leq t \) for every \( s \in S \) and \( t \in T \). If \( T \) has a supremum, then so does \( S \) and,

\[ \sup S \leq \sup T. \]

Question 5. Consider the set

\[ S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}. \]

(a) Show that max \( S = 1 \).

(b) Show that if \( d \) is a lower bound for \( S \), then \( d \leq 0 \). [Hint: A proof by contradiction might be helpful, as well as the Archimedean Property.]

(c) Use (b) to show that \( 0 = \inf S \).

Question 6. Consider the set

\[ T = \left\{ (-1)^n \left( 1 - \frac{1}{n} \right) \mid n \in \mathbb{Z}_+ \right\}. \]

(a) Show that 1 is an upper bound for \( T \).

(b) Similar to 5b, show that if \( d \) is an upper bound for \( T \), then \( d \geq 1 \).

(c) Use (a) and (b) to show that supp \( T = 1 \).