Math 232 - Calculus IV
Homework due May 4

Given a vector field \( \mathbf{F}(x, y, z) \) on a surface \( S \), we can find its surface integral

\[
\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}.
\]

To do so, we first parameterize \( S \) via

\[
T(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,
\]
defined on \( R \) and then compute

\[
\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S} = \iint_R \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot (T_u \times T_v) \ dudv.
\]

The choice of normal vector \( T_u \times T_v \) versus \( T_v \times T_u \) give the two different orientations on an orientable surface \( S \).

**Question 1.** Compute the following surface integrals of vector fields.

(a) \( \iint_S \langle x, y, z \rangle \cdot d\mathbf{S} \) across the sphere \( S \) of radius \( a \) given by \( x^2 + y^2 + z^2 = a^2 \), oriented with outward pointing normal vector.

(b) \( \iint_S \langle 0, 0, z \rangle \cdot d\mathbf{S} \) across the sphere of radius \( a \) in the first octant \( (x, y, z \geq 0) \) oriented with outward pointing normal vector.

(c) \( \iint_S \langle x, y, z \rangle \cdot d\mathbf{S} \) across the cylinder \( x^2 + y^2 = 1 \) cut by the planes \( z = 0 \) and \( z = a \) with outward pointing normal vector.

(d) \( \iint_S \langle xy, 0, -z \rangle \cdot d\mathbf{S} \) across the cone \( z = 2\sqrt{x^2 + y^2}, 0 \leq z \leq 2 \) with normal vector pointing away from the \( z \)-axis.

We can relate a line integral of a vector field \( \mathbf{F} \) on a closed curve \( C \) with a surface integral on a surface \( S \) by Stokes’ Theorem. If \( S \) is an oriented surface with boundary \( C \), oriented according to the right-hand rule, then

\[
\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.
\]
**Question 2.** Compute the line integral

\[ \oint_C \left< \sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz \right> \cdot ds, \]

where \( C \) is the circle of radius 1 on the plane \( z = 1 \) centered about the \( z \)-axis, oriented counterclockwise when viewed from above. To do this, use Stokes’ Theorem to transform this into a surface integral on the flat disk of radius 1 whose boundary is \( C \).

**Question 3.** Use Stokes’ Theorem to compute

\[ \oint_C \left< 2y, 3x, -z^2 \right> \cdot ds, \]

where \( C \) is the circle \( x^2 + y^2 = 9 \) in the \( xy \)-plane, oriented counterclockwise when viewed from above.

Similar to Stokes’ Theorem, we can relate a surface integral of a vector field with a triple integral. Let \( W \) be a three-dimensional subset of \( \mathbb{R}^3 \) with boundary \( S \), oriented outward. Then,

\[ \iint_S \mathbf{F} \cdot dS = \iiint_W \nabla \cdot \mathbf{F} \, dV. \]

**Question 4.** Use the Gauss Divergence Theorem to compute the following surface integrals of vector fields.

(a) \[ \iint_S \left< x^2, y^2, z^2 \right> \cdot dS, \] where \( S \) is the cylinder \( x^2 + y^2 = 1 \), capped off at \( z = 0 \) and \( z = 4 \), oriented outward.

(b) \[ \iint_S \left< y, xy, -z \right> \cdot dS, \] where \( S \) is the cylinder \( x^2 + y^2 = 4 \), capped off on the bottom at \( z = 0 \) and on top at the paraboloid \( z = x^2 + y^2 \), oriented outward.

(c) \[ \iint_S \left< x^3, y^3, z^3 \right> \cdot dS, \] where \( S \) is the sphere of radius \( a \), oriented outward.

**Question 5.** Let \( S \) be a closed surface oriented outward, and let \( \mathbf{F} \) be any vector field. Use the Gauss Divergence Theorem to show that

\[ \iint_S \nabla \times \mathbf{F} \cdot dS = 0. \]