Unit impulse and unit step functions

- Used as building blocks to construct and represent other signals.

Discrete-time unit impulse and unit step functions:

\[
\delta[n] = \begin{cases} 
0, & n = 0 \\
1, & n \neq 0
\end{cases}, \quad \text{and} \quad u[n] = \begin{cases} 
0, & n < 0 \\
1, & n \geq 0
\end{cases}.
\]

- \(\delta[n] = u[n] - u[n-1]\).

- \(u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k]\).

- To pick the \(n_0\)-th sample of a signal \(x[n]\), multiply \(x[t]\) by \(\delta[n-n_0]\) because

\[
x[n] \delta[n-n_0] = x[n_0] \delta[n-n_0].
\]
Continuous-time unit impulse and unit step functions:

\[ u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \]

and \( \delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t) \) where

\[ \delta_{\Delta}(t) = \frac{u(t) - u(t - \Delta)}{\Delta}. \]

We use the following graphical notations for \( \delta(t) \) and \( k\delta(t) \):

- \( \delta(t) = \frac{d}{dt} u(t) \).
- \( u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \int_{0}^{\infty} \delta(t - \tau) d\tau \).
- To pick the value of a signal \( x(t) \) at \( t = t_0 \), multiply \( x(t) \) by \( \delta(t - t_0) \) because

\[ x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \quad \Rightarrow \quad \int_{-\infty}^{\infty} x(t)\delta(t - t_0) dt = x(t_0). \]
Example:

Consider a unit mass with initial velocity $v(0)$.

If we apply the force $f(t) = k\delta(t)$, $v(t)$ will be

$$v(t) = v(0) + k\int_0^t \delta(t) d\tau, \quad \text{for } t \geq 0.$$ 

As $\Delta \downarrow 0$, the velocity transfer from $v(0)$ to $v(0) + k$ will be faster.

If we apply the idealized force $f(t) = k\delta(t)$, $v(t)$ will be

$$v(t) = v(0) + k\int_0^t \delta(t) d\tau = v(0) + ku(t), \quad \text{for } t \geq 0.$$ 

In other words, the velocity will jump from $v(0)$ to $v(0) + k$ instantaneously.