Anatomy of a Cobb-Douglas Type Production/Utility Function in Three Dimensions

(A Visual Guide for Econ Majors)

Peter Fuleky
Department of Economics, University of Washington

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Decreasing returns to scale (Strongly concave y/U)

production/utility function = a x_1^a x_2^b
a=1, \alpha=\rho=0.25 - decreasing returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

diminishing marginal product/utility

increasing both inputs/goods by the same factor \( x_1^3, x_2^2 \)

slice along a ray through the origin

diminishing returns to scale
Constant returns to scale (Weakly concave y/U)

production/utility function = \( a x_1^a x_2^b \)
\( a=1, \alpha=0.5 \) - constant returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

diminishing marginal product/utility

increasing both inputs/goods by the same factor \( \{x_1, x_2\} \)

slice along a ray through the origin

constant returns to scale
Increasing returns to scale with diminishing marginal product/utility (Quasiconcave $y/U$)

production/utility function $= a x_1^\alpha x_2^\beta$
$a=1, \alpha=\beta=0.75$ - increasing returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

diminishing marginal product/utility

increasing both inputs/goods by the same factor ($x_1^3, x_2^2$)

slice along a ray through the origin

increasing returns to scale
Increasing returns to scale with linear marginal product/utility (Quasiconcave y/U)

production/utility function $\alpha x_1^\alpha x_2^\beta$  
$\alpha=1, \beta=1$ - increasing returns to scale

holding output/utility fixed

convex isoquants/indifference curves

holding one input/good fixed

slice parallel to axis

constant marginal product/utility

increasing both inputs/goods by the same factor $[x_1^3, x_2^2]$

slice along a ray through the origin

increasing returns to scale
Increasing returns to scale with increasing marginal product/utility (Quasiconcave y/U)
Non-symmetric production/utility function with constant returns to scale

production/utility function: \( a x_1^{a-1} x_2 \)

\( a=1, \alpha=0.25, \rho=0.75 \)

isoquants/indifference curves

marginal properties (intensive margin)

meshgrid

scale properties (extensive margin)
Non-symmetric production/utility function with increasing returns to scale

production/utility function: $a x_1^x x_2^\alpha$
$a = 1, \alpha = 1.75, \beta = 0.25$

isoquants/inferrence curves

marginal properties (intensive margin)

meshgrid

scale properties (extensive margin)
Profit maximization: production function with decreasing returns to scale

production function: \( y = a x_1^\alpha x_2^\beta \) 
\( a=1, \alpha=\beta=0.25 \) - decreasing returns to scale

\[
c = w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad w_1 - w_2 = 1
\]

the two functions combined

the vertical difference: value of output - cost = profit
only profit>0 displayed

slice along a ray

with decreasing returns to scale,
an interior solution for maximum profit can be achieved
Profit maximization: production function with constant returns to scale

production function: \( y = a x_1^\alpha x_2^\beta \)
\( a=1, \alpha=\beta=0.5 \) - constant returns to scale

cost function: \( C = w_1 x_1 + w_2 x_2 \)
\( w_1=2, w_2=3 \)

In a competitive market, price and wage adjust:
\( p = 2 \sqrt{w_1 w_2} = 2 \sqrt{6} \)
profit = value of output - cost = zero

In a non-competitive market, we don't have to impose zero profit

Profit in a non-competitive market
profit = value of output - cost

In a non-comp. mkt., profit could be infinite
(but with increasing production, decreasing returns to scale eventually kick in)
Utility maximization: utility function with decreasing returns to scale

utility function: \( U = \alpha x_1^\alpha x_2^\beta \)
\( \alpha = 1, \beta = 0.25 \) - decreasing returns to scale

expenditure held constant at: \( m = p_1 x_1 + p_2 x_2 \)
\( p_1 = 2, p_2 = 3 \)

maximize utility subject to expenditure constraint

constrained maximum utility:
the tangency point of the constraint
and the indifference curve with the maximum value
Utility maximization: utility function with increasing returns to scale

utility function: \( U = \alpha x_1^\alpha x_2^\beta \)
\( \alpha = 1, \beta = 1.25 \) - increasing returns to scale

expenditure held constant: \( M^0 = p_1 x_1 + p_2 x_2 \)
\( p_1 = 2, p_2 = 3 \)

maximize utility subject to expenditure constraint

constrained maximum utility:
the tangency point of the constraint
and the indifference curve with the maximum value
Expenditure minimization: utility function with constant returns to scale

Expenditure function: $M = p_1 x_1 + p_2 x_2$
$p_1 = 2$, $p_2 = 3$

Utility held constant at: $U^g = \alpha x_1^g x_2^g$
$\alpha = 1$, $\alpha = 0.5$ - constant returns to scale

Minimize expenditure subject to utility constraint

Constrained minimum expenditure: the tangency point of the constraint and the isocost curve with the minimum value
Literature and further reading:

The Structure of Economics, 3rd ed., Eugene Silberberg
MATLAB Documentation, MathWorks


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