ESS: Fisherian sex ratio

Assumptions

- Mating is panmictic. In particular, the mating success of a male is independent of his mother’s sex-ratio genotype.
- Total reproductive expenditure by a female is independent of her sex-ratio genotype. That is, fecundity and sex ratio are independent apart from the effect of differential costs of producing sons and daughters.
- The population is large enough that the invader’s offspring are negligible fractions of their generation.

Notation

- \( p = \) proportion of reproductive resources invested in daughters (\( p' \) for the invader)
- \( f = \) cost of a daughter, \( m = \) cost of a son
- \( k = \) total reproductive resources per female; \( K = \) total grand-offspring per daughter
- \( N = \) total number of resident-type mothers in the population (assumed large)

Fitness

- Fitness is measured as numbers of grand-offspring

ESS analysis

Fitness of invader

\[
\begin{align*}
\# \text{ invader’s daughters} & = p' \cdot k / f \\
\# \text{ fitness through daughters} & = (p' \cdot k / f) \cdot K \\
\# \text{ invader’s sons} & = (1 - p') \cdot k / m \\
\# \text{ sons of resident-type mothers} & = N \cdot (1 - p) \cdot k / m \\
\text{invader’s sons’ share of grandoffspring} & = \frac{(1 - p') \cdot k / f}{[(1 - p') \cdot k / f] + [N \cdot (1 - p) \cdot k / f]} \\
& \approx \frac{(1 - p')}{N \cdot (1 - p)}
\end{align*}
\]
# grandoffspring in population
(ignoring offspring of invader’s daughters) \( \approx N \cdot (p \cdot k / f) \cdot K \)

# invader’s grandoffspring through sons \( \approx \frac{(1 - p')}{N \cdot (1 - p)} \cdot \frac{N \cdot p \cdot k \cdot K}{f} = \frac{(1 - p') \cdot pkK}{(1 - p)f} \)

invader’s fitness \( w(p', p) \)
\[ \approx \frac{p'kK}{f} + \frac{(1 - p') \cdot pkK}{(1 - p)f} = \frac{kK}{f} \left[ p' + \frac{(1 - p')p}{(1 - p)} \right] \]
\[ = \frac{kK}{f} \left[ p' \left( 1 - \frac{p}{1 - p} \right) + \left( \frac{p}{1 - p} \right) \right] \]

Maximizing invader’s fitness

\[ \frac{d}{dp} w(p', p) = \frac{kK}{f} \left[ 1 - \frac{p}{1 - p} \right] \]

The mutant’s fitness is maximized (the derivative equals 0) when the part in [ ] = 0,

\[ i.e. \quad 1 - \frac{p}{1 - p} = 0 \]

which occurs at

\[ p^* = \frac{1}{2}. \]

ESS Result

The ESS is \( p^* = 1/2, i.e. \) equal investment in sons and daughters.

The sex ratio (proportion female) at this ESS will be

\[ \frac{p^*}{f} + \left( \frac{1 - p^*}{m} \right) \]
\[ = \frac{1}{2f} + \frac{1}{2m} \]
\[ = \frac{m}{f + m} \]

so that if female offspring are more costly than male offspring \( (f > m) \), the proportion of females will be \(< 0.5, i.e. \) there will be a male-biased sex ratio, and vice versa if female offspring are less expensive than male offspring.