

Optimal group strategyproof cost sharing*

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Abstract

Units of a good are produced at some symmetric cost. We study mechanisms that allocate goods and their production cost to some agents based on their private valuation. In general, such mechanisms are susceptible to group manipulations. Herein, we introduce a novel mechanism that is resistant to group manipulations called the generalized average cost mechanism (GAC). Among group strategyproof mechanisms, GAC is the only Pareto selection that satisfied either of the three following properties: (1) treatment of equal agents equally, (2) population monotonicity in the utility profile, or (3) population monotonicity in the cost function. Even considering a budget-surplus, we found that no group strategyproof mechanism was efficient, including GAC. However, GAC minimizes the worst absolute surplus loss among group strategyproof mechanisms. Thus, GAC can be useful towards a wide variety of cost functions independent of the shape of their marginal cost.

Keywords: *Cost-sharing, worst absolute surplus loss, group strategyproof, average cost.*

JEL classification: C72, D44, D71, D82.

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1 Introduction

Traditional cost-sharing mechanisms that cover the production cost of serving some agents based on reports of their utility have been extensively studied. Many of these mechanisms are prone to manipulation by groups of agents and are thus suboptimal. By incorporating ‘group strategyproofness’ in these mechanisms, as we later describe, group manipulations are prevented. Here, we focus on production economies where the goods are indivisible and can be produced at some symmetric cost. Such production economies have a wide variety of applications that influence the shape of the cost function.

The most well-studied case of cost function is decreasing marginal cost, typically found in economies of scale or natural monopolies.¹ A canonical example is the network facility location problem with a single facility, where there is a homogeneous cost of connecting agents to the facility and a fixed cost of opening the facility. Applications include the production of cars, pharmaceutical goods and software.

In contrast, the case of increasing marginal cost has not received as much attention.² This case finds applications in the exploitation of natural resources (e.g., oil, natural gas or fisheries). Another interesting application is the scheduling of jobs, where the disutility of agents is the waiting time until being served (see Cres and Moulin[2001] and Juarez[2008] for discussions). A classic example of this cost occurs in the management of queues in networks, such as in the Internet.

For several applications, the marginal cost function might not be monotonically increasing nor decreasing. Again, as the Internet network as an example, imagine agents interested in connecting to capacity-constrained servers with fixed costs. If server 1 has cost c_1 and can serve up to 10 agents, then the average cost is decreasing if no more than 10 people request to be served. On the other hand, if 11 people request to be served, then a second server has to be opened at cost c_2 , which will increase the average cost.

In addition to complications arising from cost function in the design of the mechanism, there are issues of incentive compatibility and fairness. The former issue occurs in settings where the designer does not have enough information about the potential participants. For example, if dealing with agents in a large network such as the Internet, agents might intentionally be able to coordinate misreports to reduce their cost. To prevent such coordination from occurring between any group of agents, we considered mechanisms that are group strategyproof (GSP). The issue of fairness occurs when production economics are symmetric. For example, concerns will arise whenever only one indivisible good can be produced at a finite cost to be distributed among several agents with the same valuation. To address this, we

¹Moulin and Shenker[2001] evaluate the trade-off between efficiency and budget-balance in submodular cost functions. Masso et al.[2010] provides optimal mechanisms within the class of fixed cost functions, a very particular case of submodular cost functions. There is a large literature on the applications of cost-sharing mechanisms to computer science problems. See, for instance, Roughgarden et al.[2006a, 2006b] and Immorlica et al.[2005] for the use of submodular cross-monotonic mechanisms to approximate budget-balanced when the actual cost function is not submodular.

²Moulin[1999] is the first to consider budget-balanced cost-sharing mechanisms for supermodular cost functions that are immune to group manipulations. Juarez[2013] characterizes a class of sequential mechanisms which are appropriate for supermodular cost functions.

focus on mechanisms that treat equal agents equally (ETE). In our model, ETE implies that for any two agents with the same utility, the mechanism should either serve both of them at the same price or not serve any of them at zero price.³

We go further to address three additional concerns of fairness in our model: (1) population monotonicity in the utility profile (PMUP), (2) net-utility monotonicity in the utility profile (NUMUP), and (3) population monotonicity in the cost (CM). PMUP requires that the group of agents who get a units of the good does not shrink as the utility profile increases. NUMUP requires that the new utility of the agents does not decrease as the utility profile increases. Finally, CM requires that the group of agents served does not decrease as the cost decreases. This is a basic equity property that requires the collective share in the benefits when the technology to produce the good or service improves. In the next section, we introduce a mechanism that satisfies all aforementioned properties.

1.1 The generalized average cost mechanism

The most well-studied mechanism to share the cost equitably and in an incentive-compatible way is the average cost mechanism (AC). This mechanism serves the largest coalition of agents S such that everyone in coalition S has utility greater than or equal to the average cost of serving $|S|$ agents. Since the cost function has decreasing average cost, this maximum coalition exists and thus the mechanism is well-defined. However, AC is defined only for cost functions with decreasing average cost.

Alternatively, AC can be implemented in a more practical way by playing the demand game introduced by Moulin[1999]. The demand game starts by offering units of the good to the agents at a price equal to the average cost of serving all the agents. If all of them accept the offer, they get served at this price. On the other hand, if only a subset of agent accept, then they are re-offered units of the good at a price equal to the average cost of serving them. We continue similarly until all the agents being served accept the offer or no agent is served.

When the cost function has decreasing average cost, AC meets all five properties addressing incentive compatibility and fairness issues defined in the previous section. However, this mechanism is not well-defined for all other cost functions. To address this, we introduce the generalized average cost mechanism (GAC).

To describe GAC, consider an arbitrary cost function for n agents with average costs ac_1, ac_2, \dots, ac_n , where ac_i is the average cost of serving i agents and the average cost is not necessarily decreasing. Let $P_1^*, P_2^*, \dots, P_n^*$ be the smallest non-decreasing cost function bounded below by the average cost. That is, consider the set of cost functions: $\mathcal{P} = \{(P_1, P_2, \dots, P_n) \in \mathbb{R}_+^n | P_j \geq ac_j \text{ for all } j \text{ and } P_k \geq P_{k+1} \text{ for all } k = 1, \dots, n\}$ and let P^* be the smallest element in \mathcal{P} with the usual \geq relation. The *generalized average cost mechanism (GAC)* for the cost function with average cost ac_1, ac_2, \dots, ac_n , is the average cost mechanism applied to the cost function $P_1^*, P_2^*, \dots, P_n^*$ defined as above.

³Since the mechanisms studied by this paper allocate zero payments to the agents who do not get a unit of the good, ETE is equivalent to anonymity in welfare (see Ashlagi and Serizawa[2012]) which requires that agents with identical utilities get the same net utilities.

Importantly, notice that if the cost function has decreasing average cost, then GAC coincides with AC. However, if the cost function has increasing average cost, then GAC is the fixed price mechanism that offers units of the good to the agents at a price equal to the average cost of serving n agents. GAC satisfies GSP⁴ and all other fairness properties described above. Moreover, we demonstrate that even when another mechanism satisfy these properties, GAC will Pareto dominate it (Theorem 1), yet issues of allocative efficiency remain.

1.2 The trade-off of efficiency and GSP

Allocative efficiency of the mechanism requires that at every utility profile, the mechanism should serve the surplus-maximizing set of agents. Unfortunately, Theorem 2 shows that there is no mechanism that is efficient and is group strategyproof, even if it has a budget-surplus.⁵ Therefore, there is a need to identify a second-best measure to select mechanisms.

To this end, we used the worst absolute surplus loss measure (*wal*): the supremum of the difference between the efficient surplus and the surplus of the mechanism, where the supremum is taken over all utility profiles. This measure has been used recently in the literature as a second-best efficiency measure in similar cost-sharing models.⁶

Proposition 3 shows that GAC minimizes the worst absolute loss among the GSP mechanisms for any cost function. Moreover, if the cost function has decreasing average cost, GAC is the only mechanism that minimizes *wal* (Proposition 4). Unfortunately, when the cost function has increasing average cost, there exist mechanisms that are equally inefficient but Pareto dominate GAC. In this case, we propose the sequential average cost mechanism (SAC). Under SAC, the agents “sequentially” pay the same price. That is, given an arbitrary order of the agents, i_n, i_{n-1}, \dots, i_1 , SAC computes the agent $i_{\tilde{k}}$ in the smallest position whose utility is bigger than the average cost of producing \tilde{k} units (if there is no such agent, then none is served). In this case, every agent in $\{i_{\tilde{k}}, \dots, i_1\}$ is offered a unit of the good at this price.

Among the GSP mechanisms that share the cost equally, SAC guarantees the smallest cost-shares to the agents at which they might get a unit of the good (Proposition 5 *i*). The SAC mechanism minimizes the worst absolute loss among the GSP mechanisms (Proposition 5 *ii*). Moreover, when the cost function has increasing marginal cost, SAC is not budget-balanced but it reduces the worst absolute surplus loss by up to half the loss of the optimal budget-balanced mechanism (Proposition 5 *iv*).

⁴This is because GAC belongs to the class of cross-monotonic mechanisms as described by Moulin[1999] and Juarez[2013].

⁵This impossibility holds except in the trivial case of a linear cost function. This result contrasts with the classic impossibility of simultaneously meeting strategyproofness, budget-balance and efficiency (Vickrey[1961], Clarke[1971] and Groves[1973]) because it persists even when allowing a budget-surplus on the mechanism. Notice the VCG mechanisms are strategyproof and efficient but do not balance the budget.

⁶See Moulin and Shenker[2001] and Juarez[2008] for applications of the *wal*-measure. See Moulin[2008] for an application of the *best relative gain*, a similar worst-case measure.

2 The Model

For a vector $x \in \mathbb{R}^N$, we denote by x_S the sum of the S -coordinates of x , $x_S = \sum_{i \in S} x_i$. Let $\delta_i : 2^N \rightarrow \{0, 1\}$ be the classic indicator function, that is, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise. For the vectors $x, y \in \mathbb{R}^N$, we say that $x \geq y$ if $x_i \geq y_i$ for all $i \in N$.

A cost C is a vector in \mathbb{R}_{++}^n where C_i (or sometimes referred as $C(i)$) specifies the total cost of producing i units. The (symmetric) cost function $\tilde{C} : 2^N \rightarrow \mathbb{R}_{++}$ generated by the cost C is $\tilde{C}(S) = C_{|S|}$. When there is no confusion, C will be used to denote the cost C and its cost function. Given the cost C , the marginal cost vector (c_1, c_2, \dots, c_n) is such that $c_i = C_i - C_{i-1}$ is the marginal cost of producing good i (where $C_0 = 0$). Therefore, $C_i = c_1 + \dots + c_i$ is the total cost of producing the first i units. The cost C has decreasing (increasing) marginal cost whenever $c_1 \geq c_2 \geq \dots \geq c_n$ ($c_1 \leq c_2 \leq \dots \leq c_n$). The average cost of producing k units is $ac(k) = \frac{C(k)}{k}$. Clearly, if the marginal cost is decreasing (increasing), then the average cost is decreasing (increasing), but the converse is not true.

There is a finite number of agents $N = \{1, 2, \dots, n\}$. Every agent gets utility from (is willing to pay for) one unit of the good. Let u , where $u \in \mathbb{R}_+^N$, be the vector of these utilities. Therefore, if agent i gets a unit by paying x_i , his net utility is $u_i - x_i$. If he does not get a unit, his net utility is zero.

Definition 1 A mechanism $\xi = (S, \varphi)$ is a pair of functions $S : \mathbb{R}_{++}^n \times \mathbb{R}_+^N \rightarrow 2^N$ and $\varphi : \mathbb{R}_{++}^n \times \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ such that for each utility profile u :

- i. $\varphi_N(C, u) \geq C_{|S(C, u)|}$
- ii. $\varphi_i(C, u) = 0$ if $i \notin S(C, u)$
- iii. $u_i \geq \varphi_i(C, u) > 0$ if $i \in S(C, u)$.

When there is no confusion, if the cost C is fixed then the restriction of the mechanism $\xi = (S, \varphi)$ will just be denoted by $\xi(u) = (S(u), \varphi(u))$.

A mechanism assigns to every report of utilities, units of the good to some agents and cost shares to those agents. Condition (i) states that the mechanism covers the cost of the served agents. Condition (ii) requires that the agents who are not served pay nothing. Condition (iii) requires individual rationality; that is, the payment of the agents should never exceed their utility and should always be positive.

Given the cost C , the net utility of agent i in the mechanism $\xi = (S, \varphi)$, denoted by NU_i^ξ , is $NU_i^\xi(u) = \delta_i(S(u))(u_i - \varphi_i(u))$. Let $NU^\xi(u)$ be the vector of such net utilities.

Definition 2 (Group strategyproofness) We say the mechanism $\xi = (S, \varphi)$ is group strategyproof if for all $T \subseteq N$ and all utility profiles u and u' such that $u'_{N \setminus T} = u_{N \setminus T}$, if $\delta_i(S(u'))(u_i - \varphi_i(u')) > NU_i^\xi(u)$ for some $i \in T$, then there exists $j \in T$ such that $\delta_j(S(u'))(u_j - \varphi_j(u')) < NU_j^\xi(u)$.

Group strategyproofness (GSP) rules out coordinated misreports of any group of agents. That is, if a group of agents misreport, and the net utility of an agent in the group strictly increases, then the net utility of another agent in the group should strictly decrease. All the mechanisms studied in this paper satisfy GSP.

3 The generalized average cost mechanism

In order to define the generalized average cost mechanism, we first define the traditional average cost mechanism.

Definition 3 (Average cost mechanism) Consider the cost C with average cost $ac(s) = \frac{C_s}{s}$ for all $s = 1, \dots, n$. At the utility profile u such that $u_{i_1} \geq u_{i_2} \geq \dots \geq u_{i_n}$.⁷ Consider the largest k such that $u_{i_k} \geq ac(k)$. Then, the average cost mechanism (AC) serves the agents in $\{i_1, \dots, i_k\}$ at a price equal to $ac(k)$.

The average cost mechanism serves the largest group of agents who are willing to pay the average cost. The equilibrium of AC can be computed as the Nash equilibrium of the game where every agent decides to buy or not buy a unit of the good. If s agents decided to buy, each of them gets a unit of the good at a price equal to $ac(s) = \frac{C_s}{s}$.

The average cost mechanism is clearly budget-balanced for the cost C . In particular, it is a feasible mechanism for C . If the cost has decreasing average cost, the AC mechanism is also GSP (Moulin and Shenker[2001], Juarez[2013])

On the other hand, if the cost has increasing average cost, then a multiplicity of AC-equilibria that are not welfare equivalent is possible (see below). Moreover, AC is not strategyproof, therefore it is neither GSP. To see this, consider the profile $u = (ac(2) - \epsilon, ac(2) - \epsilon, 0, 0, \dots, 0)$ for $0 < \epsilon < \frac{C_2}{2}$. At equilibrium, only one agent can be served; without loss of generality we assume $S^{AC}(u) = \{1\}$ and $\varphi_1^{AC}(u) = C_1$. Consider $\tilde{u} = (ac(2) - \epsilon, ac(2), 0, 0, \dots, 0)$. The AC equilibrium only serves agent 2 at price C_1 , because $ac(2) > ac(2) - \epsilon$. Then, agent 2 can profit by misreporting \tilde{u} when the true profile is u .

Consider an arbitrary cost with average costs $(ac_1, ac_2, \dots, ac_n)$. Let $(P_1^*, P_2^*, \dots, P_n^*)$ be the smallest vector with non-increasing coordinates bounded below by the average cost. That is, consider the set of vectors:

$$\mathcal{P} = \{(P_1, P_2, \dots, P_n) \in \mathbb{R}_+^n \mid P_j \geq ac_j \text{ for all } j \text{ and } P_k \geq P_{k+1} \text{ for all } k = 1, \dots, n\}$$

Definition 4 (Generalized average cost mechanism) Given an arbitrary cost C , let P^* be the smallest element in \mathcal{P} under the \geq relation⁸. The **generalized average cost mechanism (GAC)** for the cost C with average cost ac_1, ac_2, \dots, ac_n , is the average cost mechanism applied to the cost $(P_1^*, P_2^*, \dots, P_n^*)$.

⁷Ties in the utility profile are broken arbitrarily.

⁸Notice this element exists because the elements in \mathcal{P} are decreasing and bounded below.

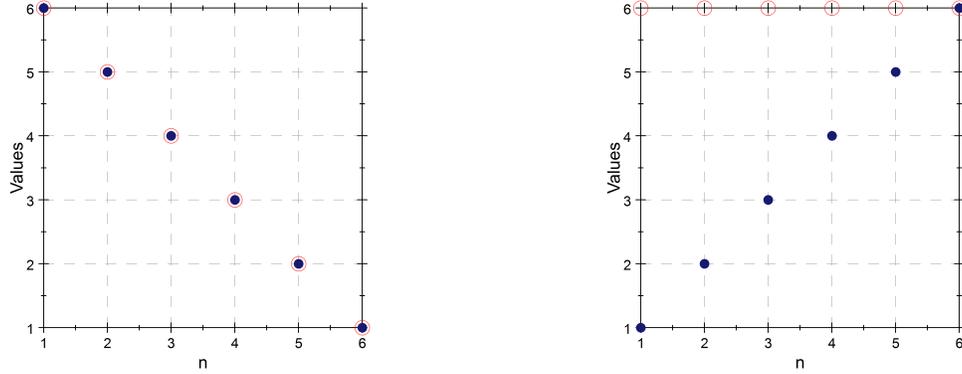


Figure 1: Computation of P^* for decreasing and increasing average cost. On left figure, P^* and ac coincide under decreasing average cost. On the right figure, P^* is constant for increasing average cost.

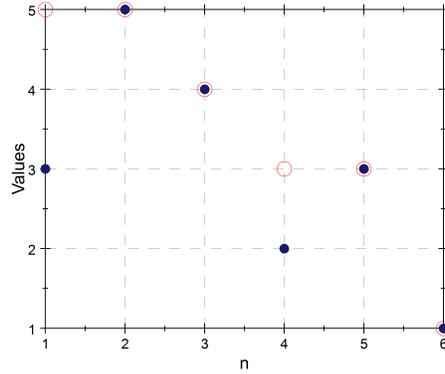


Figure 2: Computation of P^* for an arbitrary average cost. The blue filled circles represent ac . The red circles represent P^* .

4 Main result

We first discuss the set of properties needed to characterize the GAC mechanism. Given that the cost only depend on the size of the coalition that receives service, a natural requirement is to treat equal agents equally.

Definition 5 (Equal treatment of equals) *A mechanism treats equal agents equally (ETE) if $u_i = u_j$ implies that if $i \in S(C, u)$ then $j \in S(C, u)$ and $\varphi_i(C, u) = \varphi_j(C, u)$ for any cost C .*

When a mechanism meets ETE, then any pair of agents with the same utility should simultaneously get service at the same price or should not get service. The class of mecha-

nisms meeting GSP and ETE is a large class of cross-monotonic mechanisms discussed below (see Lemma 1).

Definition 6 (Equal share property) *A mechanism meets the equal share property (ESP) if $S(u) = S^*$, then $\varphi_i(u) = \varphi_j(u)$ for all $i, j \in S^*$.*

In the class of GSP mechanisms, ESP is weaker than ETE. The class of GSP and ESP mechanisms is also large. This class contains sequential mechanisms, cross-monotonic mechanisms and a combination of them (see, Juarez[2013]).

Next, we define two alternative properties related to the group of agents being served.

Definition 7 (Utility monotonicity) *i. The mechanism $\xi = (S, \varphi)$ is population monotonic in the utility profile (PMUP) if for any $u, \tilde{u} \in \mathbb{R}_+^N$ such that $u \leq \tilde{u}$, $S(C, u) \subseteq S(C, \tilde{u})$ for any cost C .*

ii. The mechanism $\xi = (S, \varphi)$ is net-utility monotonic in the utility profile (NUMUP) if for any $u, \tilde{u} \in \mathbb{R}_+^N$ such that $u \leq \tilde{u}$, $NU^\xi(u) \leq NU^\xi(\tilde{u})$ for any cost C .

A mechanism satisfies PMUP if the agents served does not decrease as the utility of the agents increase. Alternatively, a mechanism is NUMUP if the net utility of the agents increase as the utility increases. Within the class of GSP mechanisms, PMUP implies NUMUP, but the converse is not true (see the proof of Theorem 1).

The GAC mechanism meets PMUP because as the utility of the agents increase, the same group of agents who were originally served can afford to be served after their utility increase at their previous average cost or at potentially cheaper cost if more agents get service. The net-utility of the agents will not decrease, since the agents who were served before the increase in utility will continue receiving service at a price that is not larger than before.

Definition 8 (Cost monotonicity) *The mechanism $\xi = (S, \varphi)$ is cost monotonic (CM) if for any $u \in \mathbb{R}_+^N$ and any two arbitrary costs C and C' such that $C' \leq C$, then $S(C; u) \subseteq S(C'; u)$.*

A mechanism is cost monotonic if the group of agents served does not decrease as the technology to produce the goods becomes cheaper. This is a standard welfarist property requiring a share of the gains by all the agents in the society after the cost becomes cheaper.

The GAC mechanism satisfies cost-monotonicity because as the cost becomes cheaper, the average cost decreases, therefore the group of agents who were originally served can also afford the cheaper prices.

Definition 9 (Pareto domination) *• The mechanism ξ Pareto dominates the mechanism $\tilde{\xi}$ if for any cost C and any utility profile $u \in \mathbb{R}_+^N$: $NU^\xi(u) \geq NU^{\tilde{\xi}}(u)$.*

• A mechanism is a Pareto selection within a class of mechanisms if it is not Pareto dominated by another mechanism in this class.

Finally, we require that for every cost function, agents can guarantee a unit of the good if their utility is large enough, regardless of the utility of the other agents. This property was introduced by Moulin[1999].

Definition 10 (Consumer sovereignty) *The mechanism ξ satisfies consumer sovereignty (CS) if for any cost C , there exists a value \bar{x}^C such that $i \in S(C, (\bar{x}^C, u_{-i}))$ for any i and any $u_{-i} \in \mathbb{R}^{N \setminus i}$.*

All mechanisms studied here satisfy CS.

Theorem 1 *The GAC mechanism:*

- i. is the only Pareto selection among GSP and ETE mechanisms,*
- ii. is the only Pareto selection that meets the ESP among GSP, PMUP and CS mechanisms,*
- iii. is the only Pareto selection that meets the ESP among GSP, NUMUP and CS mechanisms,*
- iv. is the only Pareto selection that meets the ESP among GSP, CM and CS mechanisms.*

Part (i) single out GAC as the only optimal rule among the GSP mechanisms that meet ETE. GAC is also the only optimal rule among the GSP mechanisms that allocate payments equally and meet utility-monotonicity or cost monotonicity.

Refer to the appendix for the proof of this Theorem. The proof of part (i) uses a related result by Juarez[2013] characterizing the class of GSP and ETE mechanisms (Lemma 1). The proofs of part (ii) and (iii) use a new result that characterizes the necessary and sufficient conditions for a class of GSP mechanisms to be cross-monotonic (Lemma 2).

5 GSP and Efficiency

The **surplus** of the mechanism ξ at a utility profile u is the sum of the net utilities of all agents $\sigma^\xi(u) = \sum_{i \in N} NU_i^\xi(u)$. The **efficient surplus** at u is $eff(u) = \max_{S \subseteq N} u(S) - C_{|S|}$. A mechanism is **efficient** if it serves the group of agents that generate $eff(u)$ at any utility profile u .⁹ We say that the cost is linear if $C_i = iC_1$ for all i . Note that if the cost is linear then the marginal cost is constant.

Theorem 2 *If the cost is not linear, then there is no GSP mechanism that is efficient.*

If the cost is linear, then the fixed-cost mechanism where agent i is offered a unit of the good at price C_1 independent of the valuations of the other agents is efficient and is GSP.

Refer to the appendix for the proof of this result; we prove a more general result for cost functions that are not necessarily symmetric.

⁹The usual definition of efficiency requires that the mechanism give the efficient surplus at any utility profile. Our definition is more general, since we require only that the surplus maximizer set of agents be served.

5.1 The worst absolute loss

Given that efficiency and GSP are not compatible for non-trivial costs, we identify second-best optimal mechanisms that are GSP. These mechanisms have the smallest surplus loss relative to the efficient allocation. Two measures have been used recently to make this comparison. The first measure is the worst relative gain (Moulin[2008]), that is, the infimum of the ratio of the surplus of the mechanism and the efficient surplus, where the infimum is taken over all utility profiles. With this measure, any mechanism that is group strategyproof has zero worst relative gain. Hence, the measure is not informative.¹⁰

Definition 11 *Given the agents in N and the cost C , the worst absolute surplus loss of ξ is:*

$$wal(N, C, \xi) = \sup_{u \in \mathbb{R}_+^N} \text{eff}(u) - \sigma^\xi(u)$$

This measure is informative for mechanisms that satisfy consumer sovereignty; it is finite for any mechanism that allocates units to the agents with high utility independent of the profile of the other agents.¹¹

5.2 General cost

Not surprising, GAC is not efficient. However, it achieves the smallest worst absolute loss among all GSP mechanisms.

Proposition 3 *For any cost C , and any GSP and ETE mechanism ξ ,*

$$wal(N, C, GAC) \leq wal(N, C, \xi).$$

As we discuss in the next section, the inequality in this Proposition might or might not become strict depending on the shape of the cost C .

5.3 Decreasing average cost

When the cost function has decreasing average cost, the GAC mechanism coincides with AC. GAC is uniquely characterized in this class of cost functions.

Proposition 4 *If the cost C has decreasing average cost then:*

- i. $wal(N, C, GAC) < wal(N, C, \xi)$ for any GSP and ETE mechanism ξ .*

¹⁰To see this, consider a GSP mechanism and assume that the cost has increasing average cost (this is proved similarly for decreasing marginal cost and other non-linear cost vectors). By GSP, there is a finite payment for coalition N , denoted by x^N . Let the utility profile be the vector $u = x^N + \epsilon \cdot 1_N$, for some $\epsilon > 0$. At this profile, the surplus is $n\epsilon$. Notice the efficient surplus is positive at this profile because $x_i > C(i)$ for some i . Hence as ϵ goes to zero, the surplus of the mechanism goes to zero, but the surplus of the efficient mechanism remains bounded below by a positive number. Hence the worst relative gain is zero.

¹¹The proof of this claim can be found in Lemma 1 of Juarez[2008].

ii. $wal(N, C, GAC) = ac(1) + ac(2) + \dots + ac(n) - C(n)$.

When the cost has decreasing marginal cost, Moulin and Shenker[2001] prove a proposition similar to part (i) but they impose an additional budget-balanced restriction on the mechanism. This proposition shows that the same result holds even when we allow for a larger class of cost functions and mechanisms.

5.4 Increasing average cost

Definition 12 (Sequential average cost mechanism) *Given the cost C with increasing average cost, and an arbitrary order of the agents i_n, i_{n-1}, \dots, i_1 , the equilibrium of the sequential average cost mechanism ($SAC[i_n, i_{n-1}, \dots, i_1]$) is computed as follows: Let k^* be the largest index such that $u_{i_{k^*}} > ac(k^*)$, then every agent i_l such that $l \leq k^*$ and $u_{i_l} > ac(k^*)$ gets a unit of the good at price $ac(k^*)$.*

Given the order of the agents i_n, i_{n-1}, \dots, i_1 , the $SAC[i_n, i_{n-1}, \dots, i_1]$ mechanism offers a unit of the good to agents at price $ac(n)$ if $u_{i_n} > ac(n)$. If $u_{i_n} \leq ac(n)$ and $u_{i_{n-1}} > ac(n-1)$, then agents i_{n-1}, \dots, i_1 are offered a unit of good at price $ac(n-1)$. If $u_{i_n} \leq ac(n)$, $u_{i_{n-1}} \leq ac(n-1)$ and $u_{i_{n-2}} > ac(n-2)$, then agents i_{n-2}, \dots, i_1 are offered a unit of the good at price $ac(n-2)$, etc. When there is no confusion, $SAC[i_n, i_{n-1}, \dots, i_1]$ will just be denoted by SAC.

Clearly, SAC is feasible for the cost C . In contrast to AC, SAC is not budget-balanced. For instance, if u is such that $u_{i_n} > ac(n)$ and $u_{i_l} < ac(n)$ for $l < n$, then only agent i_n is served at price $ac(n)$, and $ac(n) > c_1$. The SAC mechanism does not treat equal agents equally, but it allocates equal cost-shares to the agents who get served.

Definition 13

Given the cost C and the mechanism $\xi = (S, \varphi)$, the lower bound on the payment of agent i is the smallest payment among all utility profiles where i gets service:

$$p_i^{min}(\xi) = \min_{u|i \in S(C,u)} \varphi(C, u).$$

If the mechanism has a finite worst absolute loss, then the lower bound in the payments always exist, since an agent is always guaranteed service if his utility is large enough.

The lower bound of the payment for agent i is the minimum level of utility at which agent i might get service. Any utility level below this threshold will not provide service to him. This threshold is particularly important for sequential mechanisms, since payments increase as the size of the coalition receiving service increases (see Juarez[2013]).

For SAC, the vector of lower bounds equals $(ac(n), ac(n-1), \dots, ac(1))$ (up to reordering the agents). We see below that this vector of lower bounds uniquely characterizes the SAC mechanism for increasing marginal cost.

Proposition 5 *If the cost C has increasing average cost, then:*

- i.* For any GSP mechanism ξ that satisfied the ESP, there exists an ordering of the agents i_n, \dots, i_1 such that $p^{\min}(\xi) > p^{\min}(SAC[i_n, \dots, i_1])$.
- ii.* $wal(N, C, SAC) \leq wal(N, C, \xi)$ for any GSP mechanism ξ ,
- iii.* $wal(N, C, GAC) = wal(N, C, SAC) = \max_k k[ac(n)] - C(k) = \max_k k[\frac{c_{k+1} + \dots + c_n}{n}] - (n - k)[\frac{c_1 + \dots + c_k}{n}]$,

If the cost C has increasing marginal cost, then:

- iv.* $\frac{wal(N, C, \xi)}{wal(N, C, SAC)} \leq 2$ for any GSP mechanism ξ that is budget-balanced.

From part (i), SAC gives the agents the lowest potential payments among all GSP mechanisms that meet the ESP.

Part (ii) shows the optimality of SAC among the GSP mechanisms. In contrast to the case of decreasing average cost, there is a large class of mechanisms that are optimal under the worse absolute loss measure when the cost function has increasing average cost.

Part (iv) shows that the loss of a GSP and budget-balanced mechanism can be reduced by up to half by allowing a budget-surplus. In the appendix we show that these bounds are tight when the marginal cost equals $c_i = i$.

When the marginal cost is increasing, it is not difficult to see that the worst absolute surplus loss of AC equals exactly $wal(N, C, SAC)$ (see lemma 2 in Juarez[2008] for details). Hence, by implementing SAC, we gain GSP and only lose budget-balance.

6 Conclusion

The design of GSP cost-sharing mechanisms was first discussed by Moulin[1999]. The paper characterizes cross-monotonic mechanisms by GSP, budget-balance, voluntary participation, non-negative transfers and strong consumer sovereignty. Juarez[2013] also characterizes GSP mechanisms under two orthogonal continuity conditions. Further, Moulin and Shenker[2001] evaluate the trade-off between efficiency and budget-balance. Therein, the Shapley value cross-monotonic mechanism (the analogue to the AC mechanism for non-symmetric cost functions) is characterized by GSP, budget-balance, voluntary participation, non-negative transfers and strong consumer sovereignty. Additionally, Masso et al.[2010] show that within the class of fixed cost functions, a very particular case of submodular cost functions, the average cost mechanism is also optimal among the class of strategy proof mechanisms using the worst absolute surplus loss measure. Therefore, Masso et al.[2010] cleverly extends Moulin and Shenker's[2001] result without assuming GSP. While important contributions to the field, these studies focus in production economies and mechanisms tailored for submodular cost functions.

We present a mechanism that addresses this situation by identifying optimal GSP mechanisms for any symmetric cost function. Our main result extends Moulin and Shenker[2001] in the case of symmetric cost functions. We impose no restriction on the shape of the cost

function; in particular, our cost function does not require decreasing marginal cost as in Moulin and Shenker[2001]. Moreover, we do not impose any budget-balanced restriction on the mechanism. Importantly, we show that our novel mechanism, GAC, is resistant to group manipulations. Among group strategyproof mechanisms, GAC is the only Pareto selection that satisfied either of the three following properties: (1) treatment of equal agents equally, (2) population monotonicity in the utility profile, or (3) population monotonicity in the cost function. Even considering a budget-surplus, we found that no group strategyproof mechanism was efficient, including GAC. However, GAC minimizes the worst absolute surplus loss among group strategyproof mechanisms. Thus, GAC can be useful towards a wide variety of cost functions independent of the shape of their marginal cost.

7 Appendix: Proofs

7.1 Preliminary Lemmas

7.1.1 Lemma 1

Definition 14 • A **cross-monotonic set of cost shares** (payments) $\chi^N = \{x^S \in \mathbb{R}_+^N \mid S \subseteq N\}$ is such that:

- i. $x_{N \setminus S}^S = 0$ for all $S \subseteq N$, and
 - ii. if $S \subseteq T$, then $x_i^S \geq x_i^T$ for all $i \in S$
- A mechanism (G, φ) is **cross-monotonic** if there exists a cross-monotonic set of cost shares χ^N such that for all $u \in \mathbb{R}_+^N : G(u) = \max_{\subseteq} \{S \in 2^N \mid x^S \leq u\}$ and $\varphi(u) = x^{G(u)}$.

Lemma 1 Any GSP and ETE mechanism is welfare equivalent to a cross-monotonic and ESP mechanism.¹²

Recall that $1_N = (1, \dots, 1) \in \mathbb{R}_+^N$. For a non-negative number x , let $x \cdot 1_N = (x, \dots, x) \in \mathbb{R}_+^N$.

By ETE, $G(x \cdot 1_N) = N$ or $G(x \cdot 1_N) = \emptyset$ for all $x \geq 0$, since all agents should either be served or not served at a symmetric utility profile.

Case 1. Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.

Proof.

Step 1.1. If $NU_k(u) = 0$ for all $u \in \mathbb{R}_+^N$ and $k \in N$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.

Proof. If $NU_k(u) = 0$ but $G(u) = S \neq \emptyset$ for some utility profile u , then $\varphi_i(u) = u_i$ for all $i \in S$. Thus, by SP, for $k \in S$ and $v_k > u_k : k \in G(v_k, u_{-k})$ and $\varphi_k(v_k, u_{-k}) = u_k$; thus, $NU_k(v_k, u_{-k}) > 0$.

Step 1.2. Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $NU(u) = 0$ for all $u \in \mathbb{R}_+^N$.

¹²This proof is similar to Juarez[2013] Proposition 2, only written as a reference.

Proof. Assume there is an agent k such that $NU_k(u) > 0$ at some utility profile u . Let $u^{max} = \max(u_1, \dots, u_n) \cdot 1_N$. Then, $G(u^{max}) = \emptyset$. Thus, when the true profile is u^{max} , agents in N help k by misreporting u : Agent k is strictly better off because he is getting a unit at a price below u_k , while any other agent j may or may not get a unit at a price less than or equal to u_j . This contradicts *GSP*.

Steps 1.1 and 1.2 combined prove Case 1.

Case 2. There exists $x^* \geq 0$ such that $G(x^* \cdot 1_N) = N$.

Proof.

By ETE, there exists $y^* \geq 0$ such that $\varphi_i(x^* \cdot 1_N) = y^*$ for all i .

Step 2.1. For all $u > y^* \cdot 1_N$, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.

Proof.

First assume that $x^* > y^*$. Let $v = x^* \cdot 1_N$. By *SP*, $1 \in G(u_1, v_{-1})$ and $\varphi_1(u_1, v_{-1}) = y^*$. Thus, by *GSP*, $G(u_1, v_{-1}) = N$ and $\varphi_i(u_1, v_{-1}) = y^*$ for all i . Changing the profiles one agent at a time $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$.

Now, assume that $y^* = x^*$. Consider \tilde{x} such that $\tilde{x} > x^*$. Let $\tilde{v} = \tilde{x} \cdot 1_N$. By ETE, $G(\tilde{v}) = \emptyset$ or $G(\tilde{v}) = N$. If $G(\tilde{v}) = \emptyset$, then when the true profile is \tilde{v} , coalition N can improve by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile. On the other hand, if $G(\tilde{v}) = N$, then $\varphi(\tilde{v}) = y^*$. Indeed, $\varphi(\tilde{v}) \geq y^*$ because $\tilde{x} > x^* = y^*$. If $\varphi(\tilde{v}) > y^*$, then when the true profile is \tilde{v} , all agents can improve by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile.

Therefore $G(\tilde{x} \cdot 1_N) = N$, $\varphi(\tilde{x} \cdot 1_N) = y^*$ and $\tilde{x} > y^*$. By the initial case, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.

We finish the proof of case 2 by induction in the number of agents. We assume that any *GSP* and *ETE* mechanism for less than n agents is welfare equivalent to a cross-monotonic mechanism that satisfies equal sharing (that is all, agents being served pay the same). We will prove it for a mechanism for n agents. We will divide the proof into steps 3 and 4 (and several multi-steps and cases).

Step 3. If an agent is served, then he will not pay less than y^* at any utility profile. That is, if $i \in G(u^*)$ for some $u^* \in \mathbb{R}_+^n$ then $\varphi_i(u^*) \geq y^*$.

Proof.

We will prove this step in cases 3.1 and 3.2.

Case 3.1 $G(u^*) \neq N$.

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent i .

Without loss of generality, also assume that $j \notin G(u^*)$ and $\varphi_i(u^*) < u_i^*$, so agent i gets a positive net utility at u^* .

Consider the profile $\tilde{u} = (0, u_{-j}^*)$. Then by *GSP*, $i \in G(\tilde{u})$ and $\varphi_i(\tilde{u}) = \varphi_i(u^*)$. Otherwise, j would help i at the profile that gives i higher utility.

Let $U^j = \{u \in \mathbb{R}^N | u_j = 0\}$ be the set of utility profiles where agent j has utility zero. By

induction, the restriction of the mechanism to U^j is welfare equivalent to a cross-monotonic mechanism for $N \setminus j$ agents that satisfies equal sharing.

Since $\tilde{u} \in U^j$ and $i \in G(\tilde{u})$ and the mechanism restricted to U^j is cross-monotonic with equal sharing, then we can find a utility profile $w \in U^j$ such that $w \geq \tilde{u}$ and $G(w) = N \setminus j$ and $\varphi_i(w) \leq \varphi_i(\tilde{u}) = \varphi_i(u^*) < y^*$.

Let $x^{N \setminus j} = \varphi(w)$. Clearly $x_k^{N \setminus j} = x_i^{N \setminus j} < y^*$ for all $k, i \in N \setminus j$, and $k \neq i$.

Let $\epsilon > 0$ be such that $y^* - \epsilon > x_i^{N \setminus j}$ and consider $u = ((y^* + \epsilon) \cdot 1_N)$. Then by step 2.1, $G(u) = N$ and $\varphi(u) = x^N$. By *SP*, $j \notin G(y^* - \epsilon, u_{-j})$. Thus by *GSP* $G(y^* - \epsilon, u_{-j}) = N \setminus j$ and $\varphi(y^* - \epsilon, u_{-j}) = x^{N \setminus j}$.

Since $u_k > x_k^{N \setminus j}$ for all $k \in N \setminus j$, then by *GSP* $G((y^* - \epsilon) \cdot 1_N) = N \setminus j$ and $\varphi((y^* - \epsilon) \cdot 1_N) = x^{N \setminus j}$. This contradicts *ETE*.

Case 3.2. $G(u^*) = N$

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent i .

Assume without loss of generality that $u_i^* > \varphi_i(u^*)$ (otherwise, at the new profile we can increase the utility of agent i and continue serving N or a proper subset of N (case 3.1 above)).

If agent $j \in G(u^*)$ is such that $\varphi_j(u^*) = u_j^*$, then at the profile $(0, u_{-j}^*)$ agent j is not served; that is, $j \notin G(0, u_{-j}^*)$. Moreover, $i \in G(0, u_{-j}^*)$ and $\varphi_i(0, u_{-j}^*) = \varphi_i(u^*)$ (otherwise, j helps i). Therefore, $\varphi_i(0, u_{-j}^*) = \varphi_i(u^*) < y^*$ and $G(0, u_{-j}^*) \neq N$; thus, we can apply the case 3.1 to the profile $(0, u_{-j}^*)$.

On the other hand, if $\varphi_k(u^*) < u_k^*$ for all $k \in N$, then consider the utility profile $v^* = \max\{u_1^*, u_2^*, \dots, u_n^*\} \cdot 1_N$. By *GSP* (replacing one agent at a time): $G(v^*) = N$ and $\varphi(v^*) = \varphi(u^*)$. By *ETE*, $G(v^*) = N$ and $\varphi_j(v^*) = \varphi_j(u^*) < y^*$ for all $i, j \in N$. Therefore, when the true profile is $x^* \cdot 1_N$ (recap $\varphi(x^* \cdot 1_N) = y^* \cdot 1_N$), coalition N can improve by reporting v^* . This contradicts *GSP*.

Step 4. The mechanism is welfare equivalent to a cross-monotonic and ESP mechanism.

Proof.

First, we construct the cost shares. For the agents in N , their cost shares will be $x^N = y^* \cdot 1_N$.

The cost shares of coalition S equal x^S , where x^S are the cost shares of coalition S at U^j for some $j \in N$, $S \subseteq N \setminus j$. These cost shares exist because the mechanism restricted to U^j is welfare equivalent to a cross-monotonic mechanism. Moreover, it is well defined because if $G(u) = S$ for some $u \in U^j$, and $G(\tilde{u}) = S$ for some $\tilde{u} \in U^k$, where $S \subseteq N \setminus \{j, k\}$, then $\varphi(u) = \varphi(\tilde{u})$. To see this, by the induction hypothesis, the mechanisms restricted to U^j and U^k are cross-monotonic with equal sharing; therefore, $\varphi_j(u) = \varphi_l(u)$ and $\varphi_j(\tilde{u}) = \varphi_l(\tilde{u})$ for $j, l \in S$. If $\varphi(u) < \varphi(\tilde{u})$, then when the true profile is \tilde{u} , agents in N can help S by reporting u . Similarly, if $\varphi(u) > \varphi(\tilde{u})$, then when the true profile is u , agents in N can help S by reporting \tilde{u} . Hence, $\varphi(u) = \varphi(\tilde{u})$.

The cost shares are cross-monotonic. Indeed, for every $j \in N$, these cost shares are cross-monotonic in U^j . Also, $x_i^N = y^* \leq x_i^S$ for every $i \in S \subset N \setminus j$ by step 3.

Next, we show that the mechanism coincides (welfare-wise) with the cross-monotonic mechanism generated by the cost shares above.

Let u be a utility profile.

Step 4.1. If $u_i \geq y^*$ for all $i \in N$, then the mechanism is welfare equivalent to the mechanism such that $G(u) = N$ and $\varphi_i(u) = y^*$.

If $u_i > y^*$ for all $i \in N$, then by step 2.1, $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$.

If $u_i > y^*$ for all $i \in S$ and $u_j = y^*$ for all $j \in N \setminus S$, then by step 3 no agent will pay less than y^* . If an agent $k \in S$ is paying more than y^* at u , then coalition N can help k by misreporting x^* , since all agents pay exactly y^* at that profile.

Step 4.2. If $u_i < y^*$ for some i , then $NU(u) = NU(0, u_{-i})$.

By step 3, $i \notin G(u)$. By *SP*, $i \notin G(0, u_{-i})$. Thus, $NU_i(u) = NU_i(0, u_{-i})$. Moreover, by *GSP* $NU_j(0, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(0, u_{-i}) > NU_j(u)$ then when the true profile is u , agent i helps j by misreporting 0. Similarly, if $NU_j(0, u_{-i}) < NU_j(u)$, then when the true profile is $(0, u_{-i})$, agent i helps j by misreporting u_i . Therefore, $NU_j(0, u_{-i}) = NU_j(u)$.

Note that the allocation at every profile u is welfare equivalent to serving the maximum reachable coalition at u using the above cross-monotonic set of cost shares.

If u satisfies the conditions of step 4.1, then the maximum reachable coalition given the cost shares is N . By step 4.1, the mechanism is welfare equivalent to serving N .

Now, assume u satisfies the conditions of step 4.2. Let S^* be the maximum reachable coalition for the cost shares above at the utility profile u . Clearly, $S^* \neq N$ because $u_i < y^* = x_i^N$ for some $i \in N$. Let $j \in N \setminus S^*$. Therefore, $x^{S^*} \in U^j$ and x^{S^*} is the cost share of coalition S^* in U^j . By the induction hypothesis, $(G(u), \varphi(u))$ is welfare equivalent to serving the maximum reachable coalition for the cost shares in U^j . Therefore, $(G(u), \varphi(u))$ is welfare equivalent to serving S^* at prices x^{S^*} .

7.1.2 Lemma 2

Consider a strategy-proof mechanism. Then, there exist arbitrary pricing functions $f_i : \mathbb{R}_+^{N \setminus i} \rightarrow [0, \infty]$ for $i = 1, \dots, n$, such that at the utility profile u , agent i is offered a unit of the good at price $f_i(u_{-i})$. That is, if $u_i > f_i(u_{-i})$, then i is served at price $f_i(u_{-i})$; if $u_i < f_i(u_{-i})$, then i is not served and pays nothing; and if $u_i = f_i(u_{-i})$, then i may or may not get a unit of the good at this price.

The pricing function f_i is non-increasing if $u_{-i}, v_{-i} \in \mathbb{R}_+^{N \setminus i}$ such that $u_{-i} \leq v_{-i}$, then $f_i(u_{-i}) \geq f_i(v_{-i})$. *This mean that the price to serve an agent does not increase if the other agents increase their utility profiles.*

Lemma 2 *Any GSP and CS mechanism with pricing functions that are non-increasing is welfare equivalent to a cross monotonic mechanism.*

Proof.

Consider the GSP mechanism (S, φ) generated by the pricing functions f_1, \dots, f_n .

Step 1. If $S(u) = S(v)$, then $\varphi(u) = \varphi(v)$.

Proof. Suppose $S(u) = S(v) = N$. Let $w_i = \max(u_i, v_i) + \epsilon$, for all $i \in N$. Then,

$$f_i(w_{-i}) \leq f_i(u_{-i}) \leq u_i < w_i \text{ for all } i \in N$$

Therefore, $S(w) = N$ and

$$\varphi(w) = (f_1(w_{-1}), \dots, f_n(w_{-n})) \leq (f_1(u_{-1}), \dots, f_n(u_{-n})) = \varphi(u).$$

If $\varphi(w) < \varphi(u)$, then coalition N can profit by reporting w when the true profile is u . Thus, $\varphi(u) = \varphi(w)$. Similarly, $\varphi(v) = \varphi(w)$. Hence, $\varphi(u) = \varphi(v)$.

Now, suppose that $S(u) = S(v) = T \neq N$. Without loss of generality, assume that $NU_i(u) > 0$ and $NU_i(v) > 0$ for $i \in T$. Then, by GSP $S(u_T, 0_{-T}) = T$ and $\varphi(u_T, 0_{-T}) = \varphi(u)$. Similarly, $S(v_T, 0_{-T}) = T$ and $\varphi(v_T, 0_{-T}) = \varphi(v)$. By restricting the mechanism to the hyperplane where the agents in $N \setminus T$ have zero utility, we get a GSP mechanism for the agents in T . Therefore, by the argument above, $\varphi(v_T, 0_{-T}) = \varphi(u_T, 0_{-T})$. Hence, $\varphi(v) = \varphi(u)$.

Step 2. Let $x_i^T = \lim_{x \rightarrow \infty} f_i(x_{T \setminus i}, 0_{-T})$ if $i \in T$; and $x_i^T = 0$ if $i \notin T$.

By the monotonicity of f_i , $x_i^T \geq 0$. Moreover, by consumer sovereignty and the definition of the mechanism, $x_i^T > 0$. By step 1, x_i^T is the payment of agent i when coalition T is served.

Note that the payments $\chi = \{x^Q | Q \subset N\}$ generate a cross-monotonic set of cost-shares. Indeed, consider the vectors $A^t = (t_T, 0_{-T})$ and $B^t = (t_N)$ where $t \in \mathbb{R}_+$ is large enough such that $S(A^t) = T$ and $S(B^t) = N$ (this value exists by consumer sovereignty). Since $A^t \leq B^t$, $f_i(A^t_{-i}) \geq f_i(B^t_{-i})$ for any $i \in T$. Hence, $x_i^T \geq x_i^N$ for any $i \in T$.

Step 3. The mechanism (S, φ) is welfare equivalent to the cross-monotonic mechanism generated by χ .

Suppose that $S(u) = T$ and $\varphi(u) = x^T \in \chi$. If T is not the maximal reachable coalition at u for the set of payments in χ , then there exists Q such that $T \subsetneq Q$, and $u \geq x^Q$. Let v such that $S(v) = Q$. Therefore, unless Q and T generate the same vector of net-utilities, the agents in N can profit by misreporting v when the true profile is u . This is a contradiction.

■

7.1.3 Lemma 3

Consider the GSP mechanism $\xi = (S, \varphi)$ and recall from Section 5.4 that $p_i^{min} = \min\{\varphi_i(u) | i \in S(u)\}$ is the minimum payment of agent i over all the utility profiles where he is served.

Let $y \in \mathbb{R}_+^N$ be a vector such that $y \leq p^{min}$ and consider the **composition mechanism** $\tilde{\xi} = (\tilde{S}, \tilde{\varphi})$ of ξ with y defined as:

if $u \geq y$ then $\tilde{S}(u) = N$ and $\tilde{\varphi}(u) = y$, and

if $u_S < y_S$ and $u_{-S} \geq y_{-S}$ then $\tilde{S}(u) = S(0_S, u_{-S})$ and $\tilde{\varphi}(u) = \varphi(0_S, u_{-S})$.

Lemma 3 *The composition mechanism $\tilde{\xi}$ is GSP.*

Proof. Consider a utility profile u . If $u \geq y$, then no coalition can manipulate, since all agents are receiving a non-negative net utility at their lowest possible cost-share.

Suppose that $u_S < y_S$ and $u_{-S} \geq y_{-S}$. Assume that coalition T can manipulate. Then, $T \cap S = \emptyset$, since the only way agents in S can affect their payment is by misreporting above y_S ; thus, they will be worse off.

Hence, the agents in T can manipulate $\tilde{\xi}$ at the profile $(0_S, u_{-S})$. This is a contradiction, since $\tilde{\xi}$ coincides with ξ at the profiles where agents in S have utility 0, and ξ is GSP. ■

7.2 Proof of Theorem 1.

7.2.1 Proof of Part i

Consider a mechanism ξ that is GSP and ETE. By Lemma 1, the mechanism is welfare equivalent to a cross-monotonic and ESP mechanism. Hence, by feasibility, the cross-monotonic set of payments are such that $x_i^S \geq ac(s)$ for all $i \in S$.

The cost-shares generated by GAC are clearly smaller than the cost-shares of ξ . Hence, GAC Pareto dominates ξ .

7.2.2 Proofs of Parts ii and iii

First, we note that utility monotonicity and net-utility monotonicity implies that the sharing functions are non-increasing in the utility profile of the other agents. To see this, suppose that there exists profiles such that $u_{-i} > \bar{u}_{-i}$ and $f_i(u_{-i}) > f_i(\bar{u}_{-i})$. Let w_i be such that $f_i(u_{-i}) > w_i > f_i(\bar{u}_{-i})$. Clearly, $i \in S(w_i, \bar{u}_{-i})$ but $i \notin S(w_i, u_{-i})$, this contradicts PMUP. This also contradicts NUMUP since $NU_i(w_i, \bar{u}_{-i}) > 0$ but $NU_i(w_i, u_{-i}) = 0$.

Therefore, by Lemma 2, the mechanism is cross-monotonic. Finally, any mechanism that meets ESP will be Pareto dominated by GAC, since feasibility is satisfied.

7.3 Proof of Theorem 2

Consider an arbitrary cost function $C : 2^N \rightarrow \mathbb{R}^N$. The cost function is additive if $C(S \cup T) = C(S) + C(T)$ for any $S, T \subset N$ such that $S \cap T = \emptyset$. The proof of Theorem 2 will be a trivial consequence of the following Lemma for the case of a symmetric cost function.

Lemma 4 *If the cost function is not additive, then there is no a GSP mechanism that is efficient.*

Proof.

Step 1. There is no GSP and efficient mechanism if $N = \{1, 2\}$ and $C(12) < C(1) + C(2)$.

Consider the profiles $u = (C(12) - C(2) + 2\epsilon, C(2) - \epsilon)$ and $\tilde{u} = (C(1) - \epsilon, C(12) - C(1) + 2\epsilon)$. For small ϵ , $S(u) = S(\tilde{u}) = \{1, 2\}$. To see this, $S(u) = 1$ or $S(u) = 2$ are not feasible because $u_1 = C(12) - C(2) + 2\epsilon < C(1)$ and $u_2 = C(2) - \epsilon < C(2)$. On the other hand, $S(u) = \emptyset$ is not efficient since $u_1 + u_2 - C(12) > 0$. Similarly $S(\tilde{u}) = \{1, 2\}$.

Let $x = \varphi(u)$ and $y = \varphi(\tilde{u})$. Notice $x \neq y$. To see this, assume the contrary. If $x = y$ then by voluntary participation $x_1 \leq C(12) - C(2) + 2\epsilon$ and $y_2 \leq C(12) - C(1) + 2\epsilon$, thus

$$x_1 + x_2 = x_1 + y_2 \leq 2C(12) - C(2) - C(1) + 4\epsilon < C(12)$$

for small ϵ . This contradicts the feasibility of φ .

By efficiency $S(C(1) - \epsilon, C(2) - \epsilon) = \{1, 2\}$ for small ϵ . By strategyproofness, $\varphi_1(C(1) - \epsilon, C(2) - \epsilon) = x_1$. To see this, if $\varphi_1(C(1) - \epsilon, C(2) - \epsilon) > x_1$ then agent 1 misreports u_1 when the true profile is $(C(1) - \epsilon, C(2) - \epsilon)$. On the other hand, if $\varphi_1(C(1) - \epsilon, C(2) - \epsilon) < x_1$ then agent 1 misreports $C(1) - \epsilon$ when the true profile is u .

Finally, GSP implies that $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) = x_2$. Indeed, if $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) < x_2$ then agent 1 helps 2 by misreporting $C(1) - \epsilon$ when the true profile is u . On the contrary, if $\varphi_2(C(1) - \epsilon, C(2) - \epsilon) > x_2$, then 1 helps 2 by misreporting u_1 when the true profile is $(C(1) - \epsilon, C(2) - \epsilon)$. This contradicts GSP. Hence $\varphi(C(1) - \epsilon, C(2) - \epsilon) = x$. Similarly, $\varphi(C(1) - \epsilon, C(2) - \epsilon) = y$. This is a contradiction because $x \neq y$.

Step 2. There is no GSP and efficient mechanism if $N = \{1, 2\}$ and $C(12) > C(1) + C(2)$.

By feasibility $S(C(1) + \epsilon, C(2) + \epsilon) \neq \{1, 2\}$ for small ϵ . Also by efficiency $S(C(1) + \epsilon, C(2) + \epsilon) \neq \emptyset$. Assume, w.l.g., that $S(C(1) + \epsilon, C(2) + \epsilon) = \{1\}$.

By efficiency, $S(C(1) + \epsilon, C(2) + 2\epsilon) = \{2\}$. We will prove that $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = C(2)$. By efficiency $S(0, C(2) + 2\epsilon) = \{2\}$ for all $\epsilon > 0$, then by strategyproofness and feasibility $\varphi_2(0, C(2) + 2\epsilon) = C(2)$. On the other hand, by GSP $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = \varphi_2(0, C(2) + 2\epsilon)$ (if one is smaller then agent 1 can help 2). Thus, $\varphi_2(C(1) + \epsilon, C(2) + 2\epsilon) = C(2)$. Hence by strategyproofness $2 \in S(C(1) + \epsilon, C(2) + \epsilon)$, otherwise 2 can improve by misreporting $C(2) + 2\epsilon$. This is a contradiction.

Step 3. Assume $n > 2$.

Because the cost function is not additive, there are $i, j \in N$, $S \subseteq N \setminus \{i, j\}$ such that

$$C(S \cup i) + C(S \cup j) \neq C(S \cup \{i, j\}) + C(S). \quad (1)$$

Let \bar{u} be a utility profile such that $\bar{u}_s > C(N)$ for all $s \in S$ and $\bar{u}_k = 0$ if $k \notin S \cup \{i, j\}$. By efficiency, the agents in S should be served and any agent not in $S \cup \{i, j\}$ should not be served. Agents i and j may or may not be served.

Consider the set of utility profiles $U = \{u \mid u_{[N \setminus \{i, j\}]} = \bar{u}_{[N \setminus \{i, j\}]}\}$.

Thus $S \subseteq S(u) \subseteq S \cup \{i, j\}$ for all $u \in U$.

By restricting the mechanism to U , we define a GSP mechanism for agents i and j . By equation 1, the cost function is not additive, hence by steps 1 and 2 the mechanism is not efficient at U .

7.4 Proof of Proposition 4.

7.4.1 Proof of Part i.

Step 1. Let (S, φ) be a GSP and ETE mechanism. Then, (S, φ) is cross-monotonic and the set of payments meets the equal share property.

Consider a utility profile u and assume $S(u) = S^*$. By feasibility $u_i \geq \varphi_i(u) \geq ac(S^*)$ for all $i \in S^*$.

Hence, at the average cost equilibrium, $S^* \subseteq S^{AC}(u)$. Since the average cost is decreasing, then $\varphi_i^{AC}(u) \leq \varphi_i(u)$ for all $i \in S^*$.

Step 2. Consider a mechanism $\xi = (S, \varphi)$ as in Step 1 above.

Because ξ is Pareto dominated by AC , $wal(N, C, AC) \leq wal(N, C, \xi)$. We now prove the strict inequality.

If ξ is not welfare equivalent to AC , then there is a utility profile u such that $\varphi_i(u) > AC(S^*)$ for some $i \in S^*$, $S^* = S(u)$.

Let $i \in S^*$, $\epsilon > 0$, and consider the utility profile $\tilde{u}(\epsilon)$ such that $\tilde{u}_{-S^*}(\epsilon) = (ac(|S^*| + 1) - \epsilon, \dots, ac(n) - \epsilon)$, and $\tilde{u}_{S^*}(\epsilon) = (ac(1) - \epsilon, ac(2) - \epsilon, \dots, ac(|S^*| - 1) - \epsilon, ac(|S^*|) + \delta)$ where $\delta = \frac{\varphi_i(u) - AC(S^*)}{2} > 0$.

First notice $S(\tilde{u}(\epsilon)) = \emptyset$. To see this, clearly $S(\tilde{u}(\epsilon)) \neq N$ because the payments of all agents should be at least $ac(n)$, so this is not feasible for agent n who has utility $ac(n) - \epsilon$. By cross-monotonicity, this agent is not served. Similarly, the agent with utility equal to $ac(|S^*| + 1) - \epsilon$ is not served. Continuing this way, $S(\tilde{u}(\epsilon)) \cap (N \setminus S^*) = \emptyset$.

Also notice S^* is not feasible at $\tilde{u}(\epsilon)$ because $ac(|S^*|) + \delta < \varphi_i(u)$. Hence this agent is not served. Continuing this way, $S(\tilde{u}(\epsilon)) = \emptyset$.

Hence, $wal(N, C, \xi) \geq eff(\tilde{u}(\epsilon))$. Clearly, $eff(\tilde{u}(\epsilon)) = ac(1) + \dots + ac(n) + \delta - (n-1)\epsilon - C(n)$ because $ac(i) \geq c_i$ for all i .

Hence, as ϵ goes to zero, $wal(N, C, \xi) \geq ac(1) + \dots + ac(n) + \delta - C(n)$. Since $\delta > 0$ then, $wal(N, C, \xi) > ac(1) + \dots + ac(n) - C(n)$. Thus, by part (ii) below $wal(N, C, \xi) > wal(N, C, AC)$.

7.4.2 Proof of Part ii.

Consider the profile $u(\epsilon) = (ac(1) - \epsilon, ac(2) - \epsilon, \dots, ac(n) - \epsilon)$ for $\epsilon > 0$. Clearly, $S^{AC}(u(\epsilon)) = \emptyset$. Thus, $wal(N, C, AC) \geq eff(\tilde{u}(\epsilon))$. Clearly, $eff(\tilde{u}(\epsilon)) = ac(1) + \dots + ac(n) - n\epsilon - C(n)$ because $ac(i) \geq c_i$ for all i . Hence, as ϵ goes to zero, $wal(N, C, AC) \geq ac(1) + \dots + ac(n) - C(n)$.

Consider a utility profile u such that $S(u) = \emptyset$. Up to reordering the vector, assume $u_1 \geq u_2 \geq \dots \geq u_n$. Then, clearly $u \leq (ac(1), ac(2), \dots, ac(n))$. Since the efficient surplus is increasing in the utility profile, then

$$eff(u) - \sigma^\xi(u) \leq ac(1) + \dots + ac(n) - C(n).$$

Next assume $S(u) = S^* \neq \emptyset$. Because the average cost is decreasing, $ac(i) \geq c_i$ for all i , and then the efficient surplus serves at least the agents in S^* . Thus we can reduce the utility of the agents in S^* up to $ac(S^*)$ without affecting the loss. That is:

$$eff(u) - \sigma^\xi(u) = eff(ac(S^*)1_{[S^*]}, u_{-S^*}) - \sigma^\xi(ac(S^*)1_{[S^*]}, u_{-S^*}) = eff(ac(S^*)1_{[S^*]}, u_{-S^*})$$

Up to renaming the agents, assume $u_{-S^*} = (u_{S^*+1}, u_{S^*+2}, \dots, u_n)$, $u_{S^*+1} \geq u_{S^*+2} \geq \dots \geq u_n$. Clearly, $u_{S^*+1} < ac(S^* + 1)$, $u_{S^*+2} < ac(S^* + 2)$, \dots $u_n < ac(n)$.

Hence, $eff(ac(S^*)1_{[S^*]}, u_{-S^*}) \leq eff(ac(1), ac(2), \dots, ac(n)) = ac(1) + \dots + ac(n) - C(n)$.
Hence, $wal(N, C, AC) = ac(1) + \dots + ac(n) - C(n)$.

7.5 Proof of Proposition 5.

7.5.1 Proof of Part i.

Consider the mechanism $\xi = (S(\cdot), \varphi(\cdot))$ that is feasible, meets the ESP and has finite wal . We start below with some preliminary results.

Step 1. If $S(u) = S(\tilde{u}) = S^*$, then $\varphi(u) = \varphi(\tilde{u})$.

Consider two profiles u and \tilde{u} such that $S(u) = S(\tilde{u}) = S^*$. If $\varphi_i(u) < \varphi_i(\tilde{u})$ for some agent $i \in S^*$, then by ESP $\varphi_j(u) < \varphi_j(\tilde{u})$ for all $j \in S^*$. Therefore it is a Pareto improvement to misreport u when the true profile is \tilde{u} . This contradicts GSP, hence $\varphi(u) = \varphi(\tilde{u})$.

Step 2. For any $S^* \subseteq N$ there is a profile u^{S^*} such that $S(u^{S^*}) = S^*$.

Consider the profile $\bar{u}^x = x \cdot 1_{S^*}$, for some $x > 0$. By the ESP and feasibility, the agents in $N \setminus S^*$ will never be served at \bar{u}^x since they are unable to afford the cost of being served.

If x is large then the efficient coalition serves S^* at \bar{u}^x . Therefore, $wal(N, C, \xi)$ is finite only if ξ serves S^* at the utility profile \bar{u}^x for some large x .

Note: Let x^T be the payment of coalition T in the mechanism $(S(\cdot), \varphi(\cdot))$. By steps 1 and 2 such payment exists and is well defined. Since the mechanism satisfies ESP, when there is no confusion we refer to x^T simultaneously as the vector of payments and also as the payment of an agent in T .

Step 3. If $u \gg x^N$ then $S(u) = N$.

By steps 1 and 2, there exist large \tilde{u} , $\tilde{u} \gg x^N$, such that $S(\tilde{u}) = N$ and $\varphi(\tilde{u}) = x^N$.

By strategyproofness, $1 \in S(u_1, \tilde{u}_{N \setminus 1})$ and $\varphi_1(u_1, \tilde{u}_{N \setminus 1}) = x_1^N$.

By group strategyproofness, $S(u_1, \tilde{u}_{N \setminus 1}) = N$ and $\varphi(u_1, \tilde{u}_{N \setminus 1}) = x^N$.

Similarly, by strategyproofness $2 \in S(u_{12}, \tilde{u}_{N \setminus 12})$ and $\varphi_2(u_{12}, \tilde{u}_{N \setminus 12}) = x_2^N$.

By group strategyproofness, $S(u_{12}, \tilde{u}_{N \setminus 12}) = N$ and $\varphi(u_{12}, \tilde{u}_{N \setminus 12}) = x^N$.

By repeating the same argument $n - 2$ additional times, $S(u) = N$ and $\varphi(u) = x^N$.

Step 4. There exists an agent i such that if $i \in S(u)$ then $\varphi_i(u) \geq x^N$.

We prove this step by induction on the number of agents. The proof is trivial for one agent, since the only strategyproof mechanisms are the posted price mechanisms.

We assume that for any GSP mechanism, of $n - 1$ agents or less, that meets the ESP and has a finite wal , there is an agent such that his minimal payment is achieved at the coalition of maximal size.

Consider the mechanism (S, φ) for n agents, and assume step 4 is not true. Thus, for every agent i there is a coalition S^i such that $i \in S^i$ and $x^{S^i} < x^N$.

Let $\bar{x} = \max_{i \in N} x^{S^i}$. Consider the profile $u^i(x) = (x \cdot 1_{[S^i]}, (\bar{x} + \epsilon) \cdot 1_{[N \setminus S^i]})$, where $\epsilon > 0$ is small.

Since wal is finite, then for x large $S^i \subset S(u^i(x))$.

Also, for x large and small ϵ , $\varphi_i(u^i(x)) \leq \bar{x}$. To show that, it is clear if $S(u^i(x)) = S^i$ since $\varphi_i(u^i(x)) = x^{S^i} \leq \bar{x}$. On the other hand, if $j \in S(u^i(x))$ for some $j \notin S^i$, then $u_j^i(x) = \bar{x} + \epsilon$. Thus by individual rationality, $\varphi_j(u^i(x)) \leq \bar{x}$ for ϵ small enough, because there is only a finite number of payment vectors. Then by ESP, $\varphi_j(u^i(x)) = \varphi_i(u^i(x)) \leq \bar{x}$.

Therefore by GSP $S^i \subset S(u^i(\bar{x} + \epsilon)) = S(\bar{x} + \epsilon, \bar{x} + \epsilon, \dots, \bar{x} + \epsilon)$ for small ϵ .

Since $i \in S^i$ for every $i \in N$, then $N \subseteq S(\bar{x} + \epsilon, \bar{x} + \epsilon, \dots, \bar{x} + \epsilon)$ for small ϵ . This is a contradiction to step 1 because $\bar{x} < x^N$.

Step 5.

Let i_n be the agent (found in Step 4) whose payments are greater than or equal to x^N . Let i_{n-1} be the agent whose payments are greater than or equal to $x^{N \setminus i_n}$. Let i_{n-2} be the agent whose payments are greater than or equal to $x^{N \setminus \{i_n, i_{n-1}\}}$, etc.

Since the mechanism (S, φ) is feasible and meets ESP, then $x^{N \setminus \{i_n, i_{n-1}, \dots, i_{t+1}\}} \geq ac(t)$. Therefore, at the mechanism ξ , agent i_t is always paying not less than $ac(t)$ at any utility profile because C has increasing average cost. Hence $SAC[i_1, i_2, \dots, i_n]$ Pareto dominates ξ .

The mechanism such that agent i has priority satisfies ESP, feasibility, GSP and Pareto dominates the other mechanism.

7.5.2 Proof of Part ii.

Step 1. $wal(N, C, SAC) = \max_{1 \leq k \leq n} k \cdot ac(n) - (c_1 + \dots + c_k)$

Let $x^{SAC} = (ac(n), \dots, ac(n))$, and consider the fixed-cost mechanism ξ that offers to every agent a unit of the good at price $ac(n)$, independent of the other agents report. We claim ξ has a worst absolute surplus loss equal to $\max_{1 \leq k \leq n} k \cdot ac(n) - (c_1 + \dots + c_k)$.

Indeed, let u be a utility profile. Assume $u_{[S]} \gg x_{[S]}^{SAC}$, $u_{[T]} \ll x_{[T]}^{SAC}$ and $u_{[N \setminus (S \cup T)]} = x_{[N \setminus (S \cup T)]}^{SAC}$.

Let $\epsilon > 0$ such that $\epsilon < |u_i - x_i^{SAC}|$ for all $i \in S \cup T$. Let

$$u^\epsilon = (x_{[S]}^{SAC} + \epsilon 1_{[S]}, x_{[T]}^{SAC} - \epsilon 1_{[T]}, u_{[N \setminus (S \cup T)]})$$

where $1_{[K]} = (1, \dots, 1) \in \mathbb{R}^K$

Then, $eff(u^\epsilon) - \sigma^\xi(u^\epsilon) \geq eff(u) - \sigma^\xi(u)$. To see this, let $i \in S$, then $x_i^{SAC} - \epsilon < u_i$. Since $i \in S(u)$, then by replacing u_i by $x_i^{SAC} - \epsilon$ the efficient surplus decreases at most by $u_i - (x_i^{SAC} - \epsilon)$ while $\sigma^\xi(u)$ decreases exactly by $u_i - (x_i^{SAC} - \epsilon)$. On the other hand, if $i \in T$, then $x_i^{SAC} - \epsilon > u_i$. Thus by replacing u_i by $x_i^{SAC} - \epsilon$ the efficient surplus increases at most by $x_i^{SAC} - \epsilon - u_i$ while $\sigma^\xi(u)$ does not increase. The agents outside $S \cup T$ do not affect the surplus.

Finally, by the same argument $eff(u^\epsilon) - \sigma^\xi(u^\epsilon)$ is not decreasing as ϵ approaches zero. Clearly, $\lim_{\epsilon \rightarrow 0} \sigma^\xi(u^\epsilon) = 0$ and $\lim_{\epsilon \rightarrow 0} eff(u^\epsilon) = \max_S |S|ac(n) - C(S)$.

Hence,

$$wal(N, C, \xi) = \max_{S \subseteq N} |S|ac(n) - C(S) = \max_k k \cdot ac(n) - C(k).$$

On the other hand, by choosing the utility profile $u = (ac(n) + \epsilon, \dots, ac(n) + \epsilon)$ and letting ϵ tend to zero, $wal(N, C, SAC) \geq \max_k k \cdot ac(n) - C(k)$.

Since SAC Pareto dominates ξ , then $wal(N, C, SAC) = wal(N, C, \xi)$.

Step 2. $wal(N, C, SAC) \leq wal(N, C, \xi)$ for any feasible mechanism ξ .

Consider a mechanism $\xi = (S, \varphi)$ with finite worst absolute surplus loss. Then for every agent i , there is x^i large such that i is served if $u_i > x^i$ independent of other agents reports (i.e., the mechanism meets consumer sovereignty). Indeed, if this does not occur, then we can find an agent i and collection of utility profiles u^1, u^2, \dots such that $u_i^k \rightarrow \infty$ and $i \notin S(u^k)$ for all k . Since $i \in eff(u^k)$ for all k such that $u_i^k > C(N)$, then $eff(u^k) - \sigma^\xi(u^k) \rightarrow \infty$ as $k \rightarrow \infty$. Thus $wal(N, C, \xi) = \infty$.

Given this, consider a utility profile \tilde{u} such that $\tilde{u}_i > x^i$ for all i . Then, $S(\tilde{u}) = N$. We can assume without loss of generality that all agents are getting positive net utility.

Assume $\varphi(\tilde{u}) = x^N$. Thus, $\tilde{u}_i > x_i^N$ for all $i \in N$. Let $\epsilon > 0$, by strategyproofness $i \in S(x_i^N + \epsilon, u_{-i})$ and $\varphi_i(x_i^N + \epsilon, u_{-i}) = x_i^N$.

On the other hand, by GSP $j \in S(x_i^N + \epsilon, u_{-i})$ and $\varphi_j(x_i^N + \epsilon, u_{-i}) = x_j^N$ for all $j \neq i$. To see this, if $j \notin S(x_i^N + \epsilon, u_{-i})$ or $\varphi_j(x_i^N + \epsilon, u_{-i}) > x_j^N$ then agent i helps j by misreporting u_i when the true profile is $(x_i^N + \epsilon, u_{-i})$. On the other hand, if $\varphi_j(x_i^N + \epsilon, u_{-i}) < x_j^N$ then i helps j by misreporting $x_i^N + \epsilon$ when the true profile is u .

Hence, by changing one agent at a time, $S(x^N + \epsilon 1_N) = N$ and $\varphi(x^N + \epsilon 1_N) = x^N$.

Notice $\sigma^\xi(x^N + \epsilon 1_N) = n\epsilon$. On the other hand, $eff(x^N + \epsilon 1_N) = \max_S x_S^N - C(S) + |S| \epsilon$. Thus,

$$\lim_{\epsilon \rightarrow 0} eff(x^N + \epsilon 1_N) - \sigma^\xi(x^N + \epsilon 1_N) = \max_S x_S^N - C(S).$$

Hence, $wal(N, C, \xi) \geq \max_S x_S^N - C(S)$.

Finally, since $x_N^N \geq C(N)$, $\max_S x_S^N - C(S) \geq \max_S |S| \frac{ac(n)}{n} - C(S)$.

Along with step 1, this proves step 2.

7.5.3 Proof of part iii.

This part follows immediately from step 1 in part (ii), and noticing:

$$\max_{1 \leq k \leq n} k \cdot ac(n) - (c_1 + \dots + c_k) = \max_k k \left[\frac{c_{k+1} + \dots + c_n}{n} \right] - (n - k) \left[\frac{c_1 + \dots + c_k}{n} \right] \quad (2)$$

7.5.4 Proof of Part iv.

The *incremental marginal cost mechanism (INC)* sequentially offers the agents a unit of the good at a price equal to marginal cost. That is, for an arbitrary order of the agents, say $1, \dots, n$ we offer agent 1 a unit of the good at price c_1 . Agent 2 is offered a unit of the good at price c_2 if 1 accepts, or at price c_1 if 1 did not accept. And similarly for the following agents. Only the incremental mechanisms are GSP, budget-balanced and consumer-sovereign when the marginal cost function is increasing (see Moulin[1999]).

Because the cost function has increasing marginal cost, the loss will be given at x^N . At this point, $x^N = (c_1, \dots, c_n)$. The surplus of the mechanism at $u = x^N + \epsilon \cdot 1_N$ is $n \cdot \epsilon$. Hence, as ϵ tends to zero:

$$\begin{aligned} wal(N, C, INC) &= Eff((c_1, \dots, c_n)) = \\ \max_S \sum_{i \in S} c_i - C(|S|) &= \left(c_{\lfloor \frac{n+1}{2} \rfloor} + \dots + c_n \right) - \left(c_1 + \dots + c_{\lfloor \frac{n-1}{2} \rfloor} \right). \end{aligned} \quad (3)$$

Hence, by substituting $k = \lfloor \frac{n}{2} \rfloor$ into equations 2 and 3:

$$\frac{wal(N, C, INC)}{wal(N, C, SAC)} \leq 2.$$

Finally, to see that this bound is tight, consider the marginal cost $c_i = i$.

$$\begin{aligned} \max_k k \left[\frac{c_{k+1} + \dots + c_n}{n} \right] - (n-k) \left[\frac{c_1 + \dots + c_k}{n} \right] &= \\ \frac{k \left(\frac{1}{2} (n+1)^2 - \frac{1}{2} n - \frac{1}{2} (k+1)^2 + \frac{1}{2} k \right)}{n} - \frac{(n-k) \left(\frac{1}{2} (k+1)^2 - \frac{1}{2} k - \frac{1}{2} \right)}{n} \end{aligned}$$

This has a maximum at $k = \frac{n}{2}$. By substituting, we get that the loss equals $\frac{n^2}{8}$.

On the other hand, it is easy to check, from part (iii), that $wal(N, C, INC) = \frac{n^2}{4}$.

References

- [1] Ashlagi, I., Serizawa, S., 2012. Characterizing Vickrey allocation rule by anonymity. *Social Choice and Welfare*, 38, 531-542.
- [2] Clarke E. H.: Multipart pricing of public goods. *Public Choice* 11 (1971)
- [3] Cres H., Moulin H.: Scheduling with opting out: improving upon random priority. *Operation Research* 49, 565-577 (2001)
- [4] Groves T.: Incentives in teams. *Econometrica* 41 (1973)
- [5] Immorlica N., Mahdian M., Mirrokni W.: Limitations of cross-monotonic cost sharing schemes. *SODA* (2005)
- [6] Juarez R.: The worst absolute surplus loss in the problem of commons: Random priority vs. Average cost. *Economic Theory* 34 (2008)
- [7] Juarez R.: Group strategyproof cost sharing: The role of indifferences. *Games and Economic Behavior* 82 (2013)
- [8] Juarez R.: Group strategyproof cost sharing: Mechanisms without indifferences. *Mimeo University of Hawaii*(2009)

- [9] Masso J., Nicolo A., Sen A.: Strategy-Proofness and Equal-Cost Sharing for Binary and Excludable Public Goods with Fixed Cost. Under Review (2010)
- [10] Mehta A., Roughgarden T., Sundararajan M.: Beyond Moulin Mechanisms, Games and Economic Behavior 67 (2009).
- [11] Moulin H.: Incremental Cost Sharing: characterization by coalitional strategy-proofness. Social Choice and Welfare **16**, 279-320 (1999)
- [12] Moulin H.: The price of anarchy of serial cost sharing and other methods. Economic Theory 36 (2008)
- [13] Moulin H., Shenker S.: Strategyproof sharing of submodular costs: budget balance versus efficiency. Economic Theory **18**, 511-533 (2001)
- [14] Roughgarden T. Sundararajan M: Approximately efficient cost-sharing mechanisms. CORR (2006a)
- [15] Roughgarden T., Sundararajan M.: New Trade-offs in Cost Sharing Mechanisms, STOC (2006b)
- [16] Roughgarden T., Dobzinski S., Mehta A., and Sundararajan M.: Is Shapley Cost Sharing Optimal? Symposium on Algorithmic Game Theory. (2008)
- [17] Vickrey W.: Counterspeculation, auctions, and competitive sealed tenders. Journal of Finance 16 (1961)