Solutions without dummy axiom for TU cooperative games

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Abstract

In this paper we study an expression for all additive, symmetric and efficient solutions, i.e., the set of axioms that traditionally are used to characterize the Shapley value except for the dummy axiom. Also, we obtain an expression for this kind of solutions by including the self duality axiom. These expressions allow us to give an alternative formula for the consensus value, the generalized consensus value and the solidarity solution. Furthermore, we introduce a new axiom called coalitional independence which replaces the symmetry axiom and use it to get similar results.

1 Introduction

Since the seminal axiomatic contribution of Shapley in 1953, the variations of his novel axioms have played a key role in the literature of transferable-utility cooperative games, mainly trying to understand different applications. Most variations substitute his original axioms either by weaker versions or by different axioms that fit certain application. However, very few papers consider the raw solutions that emerge by taking out one or two of the original axioms. This paper is one of them; it provides a closed form expression of semivalues without the dummy axiom. It also proposes a weakening of the symmetry axiom, and provides a closed form expression of these solutions.

Semivalues without the dummy axiom have applications in problems incompatible with a subsidy-free scenario. For instance, labor unions require employers to pay some minimum compensation even if an employee does not work: a waiter receives a minimum salary when the restaurant is empty; and, in certain countries, workers enjoy unemployment insurance from the government.

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Related literature starts with Dubey et al. (1981), which proposes a closed form expression of semivalues without efficiency. Other semivalues of this type can be found in Banzhaf (1965), Dubey et al. (1979), Roth (1977) and Weber (1979). On the other hand, symmetry has been primarily removed by introducing weights to the agents. This was done first by Shapley (1953), and complemented in different papers, for instance, Kalai et al. (1987), Banzhaf (1965), Nowak et al. (1995) and Hart et al. (1989).

This paper closes the gap in this literature by providing a systematic analysis of semivalues without the dummy axiom. Although, Nowak et al. (1997) introduced the solidarity value by exchanging the dummy axiom by the average marginal contribution axiom, we discuss this solution below.

This paper is divided in six parts. In part 2 we present preliminary notation and we recall Shapley's theorem. In part 3 we give an alternative proof for the general formula for semivalues without the dummy axiom. In part 4 we replace symmetry by a coalition independent axiom. In part 5 we relate the previous results with solutions in the literature. In part 6 we remark the extension of previous formulas to non-additive semivalues. All proofs are written in the appendix.

2 Preliminaries

By a game we mean a pair (N, v) where $N \subset \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a real function such that $v(\emptyset) = 0$. Let $G = G^N$ be the set of games with a fixed set of players N. We consider N fixed and $n = |N|$. Let (N, z) be the zero game, i.e., the game defined by $z(T) = 0$ for every $T \subseteq N$.

Definition 1 *By a solution in G we mean a continuous function $\varphi : G \to \mathbb{R}^N$. Let V be the set of solutions in G.*

A solution is a rule to divide the common gain or cost among the players in N. The requirement of continuity, in the definition of a solution, is necessary to obtain all of our results. Also, it is desirable to have a zero payoff for the zero game because in this case no coalition generates gain or cost.

It is easy to verify that G and V are vector spaces with the obvious operations.

Axiom 1 (Additivity) *The solution φ is additive if $\varphi(v + w) = \varphi(v) + \varphi(w)$ for every $v, w \in G$.*

The solution φ is said to be efficient if the amount that is distributed among the players is equal to the amount the grand coalition N obtains in every game.

Axiom 2 (Efficiency) *The solution φ is efficient if $\sum_{i \in N} \varphi_i(v) = v(N)$ for every $v \in G$.*

The player i is said to be a dummy player in the game (N, v) if $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N$. In other words, a dummy player does not modify the gain or cost of any coalition when he joins it. The dummy axiom requests that every dummy player in (N, v) gets a zero payoff.

Axiom 3 (Dummy) *If the player i is a dummy player in (N, v) then $\varphi_i(v) = 0$.*

Let us consider the set $S_N = \{\theta : N \to N | \theta \text{ is bijective}\}$. For every $\theta \in S_N$ and $v \in G$ we define another game $(N, \theta^* v)$ as follows

$$\theta^* v(S) = v(\theta^{-1}(S))$$

and for every $\theta \in S_N$ and $x \in \mathbb{R}^n$ we define the vector $\theta \cdot x \in \mathbb{R}^n$ whose $i - th$ coordinate is $x_{\theta(i)}$.

Axiom 4 (Symmetry) *The solution φ is said to be symmetric if and only if $\varphi(\theta^* v) = \theta \cdot \varphi(v)$ for every θ and $v \in G$.*

Theorem 1 (Shapley, 1953) *There exists a unique solution φ that satisfies additivity, symmetry, dummy and efficiency axioms. Furthermore it is given by*

$$\varphi_i(v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S)),$$

where $s = |S|$.

3 Solutions without a dummy axiom

3.1 General expression

In this section, we obtain an expression for all additive, symmetric and efficient solutions: that is, all the axioms that traditionally characterize the Shapley value except the dummy axiom. A similar, albeit different expression of linear, symmetric and efficient semivalues can be found in the appendix of Ruiz et al. (1998).

Proposition 1 *The solution φ satisfies additivity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{1}$$

for some $n - 1$ real numbers $\{\beta_s\}_{s=1}^{n-1}$.

Proof of this result, and the ones that follow, are given in the Appendix.

We denote by φ^β the solution φ with parameters $\{\beta_s\}_{s=1}^{n-1}$ when we need to refer to its parameters. Every additive, symmetric and efficient solution is of the form (1) and for every set of real numbers $\{\beta_s\}_{s=1}^{n-1}$ in (1) we get a solution that satisfies these three axioms. Furthermore, notice that for different sets of real numbers $\{\beta_s\}_{s=1}^{n-1}$ we get different solutions, therefore, the dimension of this (affine) subspace of solutions is $n-1$. Moreover, the Shapley value corresponds to the numbers $\beta_s = \frac{(s-1)!(n-s-1)!}{n!}$, $s = 1, \ldots, n-1$.

We may interpret (1) as follows. With respect to an efficient solution, a game v has the following information: $v(N)$ tells us the total amount to be shared and $v(S)$, $S \neq N$, the amount that coalition S claims for itself. So, we start with the egalitarian solution, i.e., we give $\frac{v(N)}{n}$ to each player. We keep going with one transference from $N \backslash S$ to S for each coalition $S \neq N, \emptyset$: Every player in $N \backslash S$ pays $s\beta_s v(S)$ and every player in S receives $(n-s)\beta_s v(S)$. Notice that we use the same factor β_s for coalitions with equal cardinality s. At the end, player i has an amount $\varphi_i(v)$ given by (1).

It is also possible to give an interpretation of (1) as a result of a random process: Let us take nonnegative real numbers β_s, $s = 1, \ldots, n-1$, such that $\sum_{s=1}^{n-1} \binom{n}{s} \beta_s < 1$ and assume that we choose the grand coalition with probability $1 - \sum_{s=1}^{n-1} \binom{n}{s} \beta_s$, whereas we select any other coalition $S \neq N$ with probability β_s, using this same number for every coalition with cardinality s. Now, let us assume that if the grand coalition is chosen every player receives $v(N)$, and if any other coalition $S \neq N$ is selected, every player in $N \backslash S$ pays $sv(S)$ and every player in S receives $(n-s)v(S)$. Then, it is easy to verify that the expected value that player i receives in this process is equal to the amount $\varphi_i(v)$ given by (1). However, this random process imposes additional conditions over the β_s, $s = 1, \ldots, n-1$, that we do not have with the axiomatic approach.

3.2 Self Duality property

Now, we study the Self Duality property. Several solutions have it: Shapley's value, Banzhaf's value and the egalitarian solution among them.

Definition 2 *Given a game $(N, v) \in G$, we define its dual game v^* as*

$$v^*(S) = v(N) - v(N \backslash S)$$

for every $S \subseteq N$.

Axiom 5 (Self duality) *We say that the solution φ is self dual if $\varphi(v) = \varphi(v^*)$ for every game $v \in G$.*

An efficient and self dual solution gives the same importance to the amounts that coalitions gain and those they miss to gain. The next corollary adds the self dual axiom to the proposition 1. This approximately halves the dimension of the space of solutions.

Corollary 1 *The solution φ satisfies additivity, symmetry, efficiency and self duality axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{2}$$

for some set of $[\frac{n+1}{2}]$ real numbers $\{\beta_s\}_{s=1}^{n-1}$ such that $\beta_s = \beta_{n-s-1}$.

3.3 Kernel of a solution

The kernel of a solution φ is the vector space of games v such that $\varphi(v) = 0$. In this subsection we give a basis for the kernel of a solution of the form (1). Let w_T be given by

$$w_T(S) = \begin{cases} 1 & \text{if } S = \{j\},\ j \notin T \\ \frac{\beta_1}{\beta_t} & \text{if } S = T \\ 0 & \text{otherwise} \end{cases}$$

for every $T \subset N$, $T \neq N$ and $|T| \geq 2$. Furthermore, let w_N be

$$w_N(S) = \begin{cases} 1 & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2 *Let φ be given by (1) such that $\beta_t \neq 0$ for $t = 1, \ldots, n-1$, then the set of games $\{w_T\}_{|T|\geq 2}$ form a basis for the kernel of φ.*

4 Solutions satisfying the coalitional independence axiom

In this section we replace the symmetry axiom in the previous propositions with a new one that we call coalitional independence axiom. This axiom looks like the fair ranking axiom of Chun (1989).

Definition 3 *We say that the two games (N, v) and (N, w) only differ in S if and only if $v(T) = w(T)$ for every coalition $T \neq S$.*

The coalitional independence axiom requests that the solution changes equally for any two players in S or any two players in $N \backslash S$ for every two games that only differ in S.

Axiom 6 (Coalitional independence) *We say that φ satisfies the coalitional independence axiom if*

$$\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w)$$

for every two games (N, v) and (N, w) that only differ in S and $i,\ j \in S$ or $i,\ j \in N \backslash S$.

Remark 2 *Notice that additivity and symmetry imply coalitional independence, but additivity and coalitional independence do not imply symmetry. Indeed, let φ be the solution given by,*

$$\varphi_i(v) = \sum_{j \in N} j v(\{j\}).$$

Clearly φ satisfies additivity and coalitional independence but not symmetry. Now, suppose that φ is an additive and symmetric solution. Let v,$w \in G$ be two games that only differ in S, and take either $i,\ j \in S$ or $i,\ j \in N \backslash S$. Let θ be the permutation of N that interchanges i and j. Then $\theta^*(v - w) = v - w$, so, $\varphi(v - w) = \varphi(\theta^*(v - w)) = \theta^* \varphi(v - w)$.

Therefore, $\varphi_i(v - w) = \varphi_j(v - w)$. Hence, φ satisfies coalitional independence.

In the rest of this section we mimic the previous section but replacing the symmetry axiom by the coalitional independence axiom.

Proposition 3 *The solution φ satisfies additivity, coalitional independence and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)[\beta_S v(S) - \beta_{N\setminus S} v(N\setminus S)] \tag{3}$$

for some set of $2^n - 2$ *real numbers* $\{\beta_S\}_{\emptyset\neq S\subsetneq N}$.

We get a similar expression as that of proposition 1, except that now we have one β_S for each non empty coalition $S \neq N$. Again, every solution that satisfies the additive, symmetry and efficiency axioms is of the form (3) and for every set of real numbers $\{\beta_S\}_{\emptyset\neq S\subsetneq N}$ we get a solution that satisfies these axioms. Now, the next corollary includes the self duality axiom as part of its hypotheses **3**. Again, including this axiom roughly halves the dimension of the space of solutions.

Corollary 2 *The solution φ satisfies additivity, coalitional independence, efficiency and self duality axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)[\beta_S v(S) - \beta_{N\setminus S} v(N\setminus S)] \tag{4}$$

for some set of $2^{n-1} - 1$ *real numbers* $\{\beta_S\}_{S\subseteq N}$ *such that* $\beta_S = \beta_{N\setminus S}$.

We conclude this section with a characterization of the Shapley value, we replace the symmetry axiom with the coalitional independence axiom.

Proposition 4 *The Shapley value is the unique solution that satisfies additivity, coalitional independence, dummy and efficiency axioms.*

5 Some special cases

In this section we briefly see some special solutions of the form (1) that do not satisfy the dummy axiom (i.e. different from Shapley's value). A first example is the Equal Surplus solution:

$$\varphi_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j\in N} v(\{j\})}{n}$$

which we get when we choose $\beta_1 = \frac{1}{n}$ and $\beta_s = 0$ for $s \neq 1$ in (1). Another self dual solution with a simple expression is

$$\varphi_i(v) = \sum_{S\ni i} \frac{v(S)}{s} - \sum_{S\not\ni i} \frac{v(S)}{n-s}$$

where $\beta_s = \frac{1}{s(n-s)}$ in (1).

5.1 The Consensus Value

The convex linear combination (rather than the linear combination) of two solutions of the form (1) gives another solution of the same form. Moreover, the parameters of the new solution are just the convex combination of those of the two original solutions. More precisely, for any two solutions φ^β and φ^γ, and a real number $\theta \in [0,1]$:

$$(1-\theta)\varphi^\beta + \theta\varphi^\gamma = \varphi^{(1-\theta)\beta+\theta\gamma}.$$

In this sense, Ju et al. (2004) prove that the consensus value is the middle point between the Equal Surplus solution and the Shapley value. Thus an expression for the consensus value is:

$$\frac{v(N)}{n}+\frac{1}{2}\left[v(\{i\})-\frac{\sum_{k\neq i} v(\{k\})}{n-1}\right]+\frac{1}{2}\sum_{S\ni i,|S|\neq n,n-1,1}(n-s)[\frac{(s-1)!(n-s-1)!}{n!}(v(S)-v(N\backslash S))].$$

In the same way, we could generate an expression for any generalized consensus value, i.e., we would only need to replace $\beta_1 = \frac{1-\theta}{n} + \frac{\theta}{n(n-1)}$ and $\beta_s = \frac{\theta(s-1)!(n-s-1)!}{n!}$ for $s = 2,..,n-1$, in (1).

5.2 Solidarity value

Nowak and Radzik (1997) introduce the solidarity value. They define, for any non-empty coalition T and any game $v \in G$,

$$A^v(T) = \frac{1}{t}\sum_{k\in T}[v(T) - v(T\backslash\{k\})]$$

Then, they define the solidarity value for player i as,

$$\psi_i(v) = \sum_{T\ni i}\frac{(n-t)!(t-1)!}{n!}A^v(T) \tag{5}$$

They characterized this value with the efficiency, additivity, symmetry and A-null player axioms, so the solidarity value must be a special case of (1). Indeed, if we expand (5) we get that the coefficient of $v(S)$, for a coalition T which does not contain i, is $\frac{(n-s-1)!s!}{n!}\frac{1}{s+1}$. Thus, this coefficient corresponds to $s\beta_s$ in (1), and therefore

$$\beta_s = \frac{(n-s-1)!(s-1)!}{(s+1)n!}$$

which gives us an alternative expression for (5):

$$\psi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N}\frac{(n-s)!(s-1)!}{n!}\left[\frac{v(S)}{s+1} - \frac{v(N\backslash S)}{n-s+1}\right].$$

Observe that the solidarity value is not self dual since $\beta_s \neq \beta_{n-s}$.

5.3 Least Square Prenucleolus

Lastly, Ruiz et al. (1998) introduce the Least Square Prenucleolus solution,

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[\sum_{S\ni i}(n-s)v(S) - \sum_{S\not\ni i} sv(S)\right]$$

This solution is also of the form (1). The corresponding parameters are $\beta_s = \frac{1}{n2^{n-2}}$.

Remark 3 *Current literature does not say much about non-additive semivalues. This paper does not explore this topic; however, the non-additive and non-dummy semivalues can be easily characterized by modifying (1). Indeed, replace the constant β_i by any function $\beta_i^* : G^N \to \mathbb{R}$ that is symmetric in G. This function is clearly efficient and symmetric. It will be additive when β_i^* is constant, and by construction the dummy axiom does not hold. Similar formulas can be given by replacing symmetry by the coalition independence axiom.*

6 Appendix

Lemma 1 *A solution satisfies the additivity axiom if and only if it is linear.*

Before we continue with the proofs, we need to define a game χ_S for every $S \subseteq N$,

$$\chi_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2 *The solution φ satisfies the additivity and symmetry axioms if and only if there exist real numbers $\{\beta_s\}_{s=1}^{n} \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$ such that*

$$\varphi_i(v) = \sum_{S\ni i}\beta_s v(S) + \sum_{S\not\ni i}\widetilde{\beta}_s v(S) \text{ for every } i \in N.$$

Proof. Let φ be an additive and symmetric solution, from Lemma 1, φ is linear. Clearly $\{\chi_S\}_{S\subset N}$ is a basis for G, and for every game $v \in G$, $v = \sum_{S\subset N} v(S)\chi_S$. Set $\beta_S^i = \phi_i(\chi_S)$, then

$$\varphi_i(v) = \sum_{S\subseteq N} v(S)\beta_S^i$$

for every $i \in N$. On the other hand, let $U, V \subseteq N$ be such that $|U| = |V|$, $k \in U$, $l \in V$ and θ a permutation of N such that $\theta(U) = V$ and $\theta(k) = l$. Since $\theta(\chi_U) = \chi_V$ then

$$\varphi_k(\chi_U) = \varphi_l(\chi_V)$$

by symmetry. Therefore, $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \in U$, $l \in V$. Similarly, we can conclude that $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \notin U$, $l \notin V$. Thus

$$\varphi_i(v) = \sum_{S\ni i}\beta_s v(S) + \sum_{S\not\ni i}\widetilde{\beta}_s v(S)$$

for some constants $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. The proof in the other direction is straightforward $\square$

Proof of Proposition 1. By Lemma 2,

$$\varphi_i(v) = \sum_{S\ni i} \beta_s v(S) + \sum_{S\not\ni i} \widetilde{\beta}_s v(S)$$

for some numbers $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. Since φ is efficient, we have that

$$\sum_{i\in N} \varphi_i(\chi_S) = s\beta_s + (n-s)\widetilde{\beta}_s = 0$$

for every $S \subsetneq N$, and

$$\sum_{i\in N} \varphi_i(\chi_N) = n\beta_n = 1.$$

Therefore, $\widetilde{\beta}_s = -\frac{s}{n-s}\beta_s$ for $s = 1, 2, ..., n-1$ and $\beta_n = \frac{1}{n}$. Thus

$$\begin{aligned} \varphi_i(v) &= \textstyle\sum_{S\ni i} \beta_s v(S) - \sum_{S\not\ni i} \frac{s}{n-s}\beta_s v(S) \\ &= \textstyle\sum_{S\ni i} \beta_s v(S) - \sum_{S\ni i} \frac{n-s}{s}\beta_{n-s} v(N\backslash S) \end{aligned}$$

for some numbers $\{\beta_s\}_{s=1}^{n-1}$. Now, if we replace $q_s = \frac{\beta_s}{n-s}$ we get

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)(q_s v(S) - q_{n-s} v(N\backslash S)).$$

$\square$

Proof of Corollary 1 Let us take an arbitrary player $i \in N$, then by Proposition 1 there exists a set of numbers $\{\beta_s\}_{s=1}^n$ such that

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N\backslash S)].$$

Let $S \subsetneq N$ be any coalition different to the grand coalition and let ξ_S be given by

$$\xi_S(T) = \begin{cases} 0 & \text{si } T \neq S \text{ and } T \neq N\backslash S \\ 1 & T = S \text{ or } T = N\backslash S \end{cases}.$$

Then

$$(n-s)[\beta_{n-s} - \beta_s] = \varphi(\xi_S^*) = \varphi(\xi_S) = (n-s)[\beta_s - \beta_{n-s}],$$

therefore $\beta_s = \beta_{n-s}$. Thus

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)\beta_S[v(S) - v(N\backslash S)].$$

It is straightforward to prove that solutions of this kind satisfy the self duality axiom. □

Proof of Proposition 2 First, notice that we can rewrite (1) as,

$$\varphi(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} (n-s)\beta_s v(S) - \sum_{S \not\ni i} s\beta_s v(S) \tag{6}$$

Now,

a) $w_T \in Ker(\varphi)$: We will prove first that $\varphi(w_T) = 0$, for $|T| \geq 2, T \neq N$ using (6). Fix $i \in T$. Recall that $w_T(S)$ is non zero only if $S = T$ or if $S = \{j\}$, with $j \notin T$. Thus,

$$\varphi_i(w_T) = (n-t)\beta_t \frac{\beta_1}{\beta_t} - \sum_{j \notin T} \beta_1 = 0.$$

Similarly, for $i \notin T$:

$$\varphi_i(w_T) = (n-1)\beta_1 - \sum_{j \notin T, j \neq i} \beta_1 - t\beta_t \frac{\beta_1}{\beta_t}$$

$$= (n-1)\beta_1 - (n-t-1)\beta_1 - t\beta_1 = 0.$$

Lastly, $\varphi(w_N) = 0$ by the symmetry and efficiency axioms.

b) Now, we prove that the set of games $\{w_T\}_{|T| \geq 2}$ is linearly independent. Assume

$$\sum_{|T| \geq 2} \alpha_T w_T = 0,$$

then evaluating in $S \neq N$, $|S| \geq 2$:

$$\alpha_S \frac{\beta_1}{\beta_s} = \sum_{|T| \geq 2} \alpha_T w_T(S) = 0,$$

so $\alpha_S = 0$ for $|S| \geq 2$, $S \neq N$. Hence,

$$\alpha_N w_N = 0$$

and therefore $\alpha_N = 0$ also.

c) φ has full rank. Given $x \in R^n$, let v be given by

$$v(S) = \begin{cases} \frac{x_i}{n\beta_1} & \text{if } S = \{i\} \\ \sum x_i & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}$$

it is easy to verify that $\varphi(v) = x$.

Thus, the kernel of φ has dimension $\dim G - \dim \mathbb{R}^n = 2^n - 1 - n$ which is the dimension of the span of the $\{w_T\}\square$

Lemma 3 *The solution φ satisfies additivity and coalitional independence axioms if and only if there exist numbers $\{\beta_S\}_{S\subseteq N} \cup \{\widetilde{\beta}_S\}_{S\subsetneq N}$ such that*

$$\varphi_i(v) = \sum_{S\ni i} \beta_S v(S) + \sum_{S\not\ni i} \widetilde{\beta}_S v(S) \textit{ for every } i \in N.$$

Proof. Recall that we have already defined a basis $\{\chi_S\}$ for G where

$$\chi_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise} \end{cases}.$$

Note that, for every S, χ_S and the zero game differ only on S, thus if φ is any linear and coalitional independent solution

$$\varphi_i(\chi_S) = \varphi_j(\chi_S), \text{if } i,j \in S \text{ or if } i,j \in N \setminus S.$$

For every S define $\beta_S := \varphi_i(S)$ for any $i \in S$; similarly, for every $S \neq N$, define $\widetilde{\beta_S} := \varphi_j(S)$ for any $j \notin S$. Then, for every game v

$$\varphi_i(v) = \varphi_i(\sum_S v(S)\chi_S) = \sum_{S\in i} \beta_S v(S) + \sum_{S\not\ni i} \widetilde{\beta_S} v(S).$$

It is straightforward to check the converse $\square$

Proof of Proposition 3. Keeping the same notation as above, we know that if φ is in addition an efficient solution, then, for every $S \neq N$

$$0 = \chi_S(N) = \sum_i \varphi_i(\chi_S) = s\beta_S + (n-s)\widetilde{\beta_S};$$

thus $\widetilde{\beta_S} = -\frac{s}{n-s}\beta_S$, $S \neq N$. Similarly, $\beta_N = \frac{1}{n}$. Hence,

$$\begin{aligned} \varphi_i(v) &= \textstyle\sum_{S\ni i} \beta_S v(S) - \sum_{S\not\ni i} \frac{s}{n-s}\beta_S v(S) \\ &= \textstyle\sum_{S\ni i} \beta_S v(S) - \sum_{S\ni i} \frac{n-s}{s}\beta_{N\setminus S} v(N\setminus S) \end{aligned}$$

Let us take $q_S = \frac{\beta_S}{n-s}$ then

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} (n-s)\left[q_S v(S) - q_{N\setminus S} v(N\setminus S)\right]$$

for some numbers $\{q_S\}_{S\subsetneq N}$. $\square$

Proof of Corollary 2. Assume $i \in N$, then by Proposition **3** there exists a set of numbers $\{\beta_S\}_{S \subsetneq N}$ such that

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N \setminus S} v(N \setminus S)].$$

For every $S \neq N$ we define a game ξ_S as follows

$$\xi_S(T) = \begin{cases} 0 & \text{si } T \neq S \text{ and } T \neq N \setminus S \\ 1 & T = S \text{ or } T = N \setminus S \end{cases}.$$

Therefore

$$(n-s)[\beta_{N \setminus S} - \beta_S] = \varphi(\xi_S^*) = \varphi(\xi_S) = (n-s)[\beta_S - \beta_{N \setminus S}],$$

so, $\beta_S = \beta_{N \setminus S}$. Thus

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)\beta_S[v(S) - v(N \setminus S)].$$

It is easy to show that solutions of this kind satisfy the self duality axiom. □

Proof of proposition 4. We leave to the reader to verify that Shapley's value satisfies the coalitional independence axiom.

Now, let φ be a solution that satisfies additivity, coalitional independence, dummy and efficiency axioms. Fix a player i and consider coalitions $T \subset N \setminus \{i\}$ but $T \neq N \setminus \{i\}$. First of all, notice that i is a dummy player for $\chi_{T \cup \{i\}} + \chi_T$, therefore -using the expression for coalitional independent solutions from Proposition 3-

$$0 = \varphi_i(\chi_{T \cup \{i\}} + \chi_T) = (n-t-1) \cdot \beta_{T \cup \{i\}} - t \cdot \beta_T.$$

Thus, $\beta_T = \frac{n-t-1}{t} \beta_{T \cup \{i\}}$ for each $T \subsetneq N \setminus \{i\}$.

From this, it follows that the solution is determined by the single number $\beta_{N \setminus \{1\}}$. Moreover, from

$$0 = \varphi_1(\chi_N + \chi_{N \setminus \{1\}}) = \frac{1}{n} - (n-1)\beta_{N \setminus \{1\}},$$

we conclude that the solution is unique. Since the Shapley value satisfies these axioms, $\varphi = Sh$. □

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