Supplementary Material for: Free Intermediation in Resource Transmission

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1 Remarks and Examples

Remark 1
Either of the following conditions is sufficient for the monotonicity of the problem (u, F):

- *u is strictly monotonic on the total price paid by the planner. That is, $u(x, t) > u(x, \tilde{t})$ for any $t < \tilde{t}$ and $x \in A$.*

- *The outcome possibility function F is strongly monotonic in prices.*

Proof. Suppose that $u(x, t)$ is monotonic in t. Let p, p' and S defined as above. By the monotonicity of the OPF, $F(S, p) \subseteq F(S, p')$. Thus, any allocation of resource $x \in F(S, p)$, is also feasible for prices p', that is $x \in F(S, p')$. Let $y = x$, $u(x, p'(S)) > u(x, p(S))$, the problem (u, F) is monotonic.

On the other hand, consider a price p and a utility-maximizing group S at prices p. Consider intermediary $n \in S$ and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. Let x be a utility-maximizing allocation in $F(S, p)$ and assume that $u(x, p(S)) > u((0, \ldots, 0), 0)$. Since F is strongly monotonic in p, for $v(p) = u(x(p), p(S)$ and $x(p) \in F(S, p)$, there is $\epsilon > 0$, s.t. $B_\epsilon(x(p)) \subset F(S, p')$. Since the preferences represented by $u(x, t)$ are monotonic in A, then there exists $y \in B_\epsilon(x(p))$, s.t. $u(y, p(S)) > u(x(p), p(S) \geq u(x, p(S))$. Therefore, $u(y, p'(S)) \geq u(y, p(S)) > u(x, p(S))$. ■

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Remark 2

Either of the following conditions on the intermediation problem guarantees that the problem is cross-monotonic:

a. *u is product separable: there exists functions $\alpha : \mathbb{R}_+^M \mapsto \mathbb{R}$ and $\beta : \mathbb{R}_+ \mapsto \mathbb{R}$ such that $u(x,t) = \alpha(x)\beta(t)$ for any x and t. F is independent of prices: $F(S,p) = F(S)$ for any S and p.*

b. *F is product separable: there exists functions $\gamma : 2^{\mathcal{N}} \mapsto 2^A$ and $\delta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $F(S,p) = \gamma(S)\delta(p(S))$ for any S and p. The utility function is independent of prices and homothetic: $u(x,t) = \tilde{u}(x)$ and $\tilde{u}(\lambda x) = \lambda\tilde{u}(x)$ for any x, $t \geq 0$ and $\lambda > 0$.*

Proof. a. $F(S,p) = F(S)$, $u^*(S,p) = \max_{x\in F(S)} u(x,p(S)) = \max_{x\in F(S)} \alpha(x)\beta(p(S))$. $u^*(S,\mathbf{0}) \leq u^*(T,\mathbf{0})$, which is $\max_{x\in F(S)} \alpha(x)\beta(0) \leq \max_{x\in F(T)} \alpha(x)\beta(0)$, equivalent with $\max_{x\in F(S)} \alpha(x) \leq \max_{x\in F(T)} \alpha(x)$. Then, $\max_{x\in F(S)} \alpha(x)\beta(p(S)) \leq \max_{x\in F(T)} \alpha(x)\beta(p(T))$ for $p(S) = p(T)$. Therefore, $u^*(S,p) \leq u^*(T,p)$ whenever $p(S) = p(T)$.

b. $u(x,p) = u(x)$, $u^*(S,p) = \max_{x\in F(S,p)} u(x,p(S)) = \max_{x\in\gamma(S)\delta(p(S))} u(x)$. $u^*(S,\mathbf{0}) \leq u^*(T,\mathbf{0})$, means $\max_{x\in\gamma(S)\delta(0)} u(x) \leq \max_{x\in\gamma(T)\delta(0)} u(x)$. Assume x^1 and x^2 solves $\max_{x\in\gamma(S)\delta(0)} u(x)$ and $\max_{x\in\gamma(T)\delta(0)} u(x)$ respectively, there is $u(x^1) \leq u(x^2)$. Since the preferences are homothetic, $u(\delta(t)x^1) \leq u(\delta(t)x^2)$, and $\delta(t)x^1$, $\delta(t)x^2$ solves the problem $\max_{x\in\gamma(S)\delta(t)} u(x)$ and $\max_{x\in\gamma(T)\delta(t)} u(x)$, thus $u^*(S,p) \leq u^*(T,p)$, with $t = p(S) = p(T)$. ■

The following two examples show that the result in Theorem 2 does not hold when either monotonicity or cross-monotonicity is removed.

Example 1

We show a monotonic problem that is not cross-monotonic where Theorem 2 does not hold.

Consider the planner's utility function $u(x,t) = \tilde{u}(x)$ that is independent of the prices paid to the intermediaries. Also, consider the OPF F that is strongly monotonic in prices such that for intermediaries 1 and 2, $F(\{1\},\mathbf{0}) = F(\{2\},\mathbf{0}) = F(\mathcal{N},\mathbf{0})$. Moreover, for some price vector $p = (p_1,p_2,0,\ldots,0)$ where $p_1,p_2 > 0$, we have that $F(S,p) \subsetneq F(\{1,2\},p) = F(\{1\},p) = F(\{2\},p)$ for any $S \subseteq \mathcal{N}$. First note that problem (u,F) is monotonic because F is strongly monotonic in prices. However, (u,F) is not cross-monotonic. To see this, assume that (u,F) is cross-monotonic. Then, $u^*(\{1\},\mathbf{0}) = u^*(\{1,2\},\mathbf{0})$ implies that $u^*(\{1\},p) = u^*(\{1,2\},(p_1,\mathbf{0}_{-1}))$. Furthermore, $u^*(\{1,2\},(p_1,\mathbf{0}_{-1})) > u^*(\{1,2\},p)$ by strong monotonicity in prices of F. Hence, $u^*(\{1\},p) > u^*(\{1,2\},p)$, which contradicts $F(\{1,2\},p) = F(\{1\},p)$. Finally, note that $\mathbf{0}$ and p are prices that are robust SPNE in the problem (u,F), since group $\{1,2\}$ is a utility-maximizing group at both prices, hence when *such group is chosen by the planner no intermediary has the incentive to deviate by strong monotonicity in prices of F.*

Example 2

We show a cross-monotonic problem that is not monotonic where Theorem 2 does not hold.

Consider the planner's utility function $u(x,t) = \tilde{u}(x)$ that is independent of the prices paid to the intermediaries. Also, consider an OPF F that satisfies the following conditions: (1) $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$, $\forall n$; (2) $F(\mathcal{N}, \mathbf{0}) = F(S, p)$ for some prices $p = (p_1, \mathbf{0}_{-1})$ such that $p_1 > 0$ and a group S such that $1 \in S$; (3) $\forall n \neq 1$, for prices p^n with $p^n = (p_1, 0, \dots, p_n, 0, \dots, 0)$, $p_n > 0$, $x^*(\mathcal{N}, \mathbf{0}) \cap F(T, p^n) = \emptyset$ $\forall T$ with $n \in T$, and $F(\mathcal{N} \setminus \{n\}, p^n) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$; finally, (4) for any price vector $p' = (p'_1, \mathbf{0}_{-1})$, $p'_1 > p_1$, $x^*(\mathcal{N}, \mathbf{0}) \cap F(T, p') = \emptyset$, $\forall T$ with $1 \in T$. The problem (u, F) meeting these conditions is not monotonic in prices since u does not depend on t and F is not strongly monotonic in prices by condition (2).

We now see that the problem (u, F) has multiple robust SPNE. Indeed, first notice that since condition (1) is satisfied, $\mathbf{0}$ is a FIE by Theorem 1, and thus it is a robust SPNE. We now show that $p = (p_1, \mathbf{0}_{-1})$ is also a robust SPNE. Indeed, by condition (2), S is a utility-maximizing group at prices $\mathbf{0}$. Assume that the planner chooses S and pays p_1 to intermediary 1, and the maximal utility that the planner could achieve at prices $\mathbf{0}$ is $\bar{u}$. Then, intermediary 1 has no incentive to decrease its price. On the other hand, if intermediary 1 increases its price to p'_1, from condition (4), the planner cannot get any utility-maximizing allocation when choosing a group that contains intermediary 1. However, at prices p' the planner can get utility $\bar{u}$ by choosing group $\mathcal{N} \setminus \{1\}$, since by condition (1), $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{1\}, \mathbf{0}) = F(\mathcal{N} \setminus \{1\}, p')$. Thus, the planner will not choose intermediary 1 if his price increases to p'_1. Alternatively, consider the case where intermediary $n \neq 1$ deviates to a price p_n. By condition (3), the planner will have utility less than $\bar{u}$ using any group with intermediary n and get $\bar{u}$ with $\mathcal{N} \setminus \{n\}$. Thus, if intermediary n charges a positive price he will not be used by planner. So no intermediary has incentive to deviate from $p = (p_1, 0, \dots, 0)$, and hence p is a robust SPNE.

We now turn our attention to outcome possibility functions that guarantee a FIE and unique robust SPNE. Consider the situation where every intermediary has an exact duplicate at prices $\mathbf{0}$. For instance, we can imagine a situation where an economy is replicated by doubling the intermediaries along with their abilities. The following definition formalizes this situation.

Definition 1 (Duplicated OPF)

*An outcome possibility function F is **duplicated** if it is defined for $N = 2k$ intermediaries* *and for any* $S \subset \{1,\ldots,k\}$ *and* $T \subset \{k+1,\ldots,2k\}$*, we have that* $F(S \cup T, \mathbf{0}) = F(S \cup T(-k), \mathbf{0})$*, where* $T(-k) = \{n-k | n \in T\}$.

Under a minimally competitive OPF no intermediary is unique. That is, for any intermediary n, there is an intermediary n' that brings exactly the same outcome as n. In particular, this happens when the OPF is additive and any intermediary has an exact replica.

Definition 2 (Minimally Competitive OPF)
An outcome possibility function F *is* ***minimally competitive*** *if for any intermediary* n*, there exists* $n' \neq n$*, such that* $F(S \cup \{n\}, \mathbf{0}) = F(S \cup \{n'\}, \mathbf{0})$ *for any group* S.

Corollary 1 (Sufficient Conditions that Guarantee FIE)
Suppose that the problem (u, F) *is monotonic in prices and cross-monotonic. Any of the following conditions is sufficient to guarantee the existence of a free intermediation equilibrium and a unique robust SPNE:*

a. *The problem has a minimally competitive OPF.*

b. *The problem has a duplicated OPF.*

c. *There exists a group of intermediaries* S *such that* $F(S, \mathbf{0}) = F(\mathcal{N} \setminus S, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$.

Proof. a. We show that $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \emptyset$ is satisfied. For any intermediary n, let $S_n = \mathcal{N} \setminus \{n\}$. From the condition of minimally competitive outcome, $F(\mathcal{N}, \mathbf{0}) = F(S_n, \mathbf{0})$, for any n. $\max_{x \in F(S_n, \mathbf{0})} u(x, 0) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0)$, thus $S_n = \mathcal{N} \setminus \{n\}$ is a utility-maximizing group. Then $\bigcap_n S_n = \emptyset$, so the intersection of utility-maximizing group at prices $\mathbf{0}$ is $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \emptyset$. From Theorems 1 and 2, there exists FIE and unique robust SPNE.

b. From the definition of duplicate OPF, $F(\mathcal{N}, \mathbf{0}) = F(\{1,\ldots,k\}, \mathbf{0})$. Hence, for any intermediary i, $F(\mathcal{N} \setminus \{i\}, \mathbf{0}) = F(\{1,\ldots,k\}, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Similar with part (a), from Theorem 1 and 2, there exists FIE and unique robust SPNE.

c. There exists a group of intermediaries S s.t. $F(S, \mathbf{0}) = F(\mathcal{N} \setminus S, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Let $S_1 = S$ and $S_2 = \mathcal{N} \setminus S$. There is $\max_{x \in F(S_1, \mathbf{0})} u(x, 0) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0) = \max_{x \in F(S_2, \mathbf{0})} u(x, 0)$. So S_1 and S_2 are utility-maximizing group at prices $\mathbf{0}$. $S_1 \cap S_2 = \emptyset$, from Theorem 1 and 2, there exists FIE and unique robust SPNE. ∎

This result implies that either by replicating the existing intermediaries and their OPFs or by finding a group of intermediaries that have the same abilities as their complement, will result in FIE and unique robust SPNE. Part (c) also illustrate comparative statics with respect to the addition of intermediaries: if a new group of intermediaries arrive and have exactly the same abilities as the original intermediaries, then a FIE and a unique robust

SPNE will be created. The intuition behind this corollary is similar to Theorem 1: perfect competition among the intermediaries occurs when every intermediary can be substituted by another group of intermediaries that achieve an equal level of utility.

2 Applications

The generality of the paper provides a unified framework for the study of different literatures that seem disconnected, ranging from resource allocation problems in networks to minimal cost spanning tree models. Herein, we discuss these applications and technical details.

Resource Transmission in a Network under Fixed Proportional Constraints: Consider the case where a planner is interested in transmitting a divisible resource to agents (such as money). The planner has preferences over the different allocations of the resource to the agents. The planner can reach the agents via a group of intermediaries that may differ in the *types* of agents they can reach as well as the *quality* in which they can reach the agents. The types of agents that intermediaries reach are represented by a network. The quality in which intermediaries reach the agents can be interpreted as the effective transmission of the resource from the intermediaries to the agents. This is represented by the total amount of the resource that an intermediary sends to the agents per unit of resource received, as well as by the proportions in which every agent receives a resource relative to another from a given intermediary.

This model can be applied to the transmission of advertising money in companies. A company looking to promote their product can use different media (the intermediaries) to reach the advertising target of their product; such intermediaries include TV channels, radio stations, Internet websites, newspapers, etc. The quality of the connections is important because, within the media, there are different channels that target to specific demographics of agents and may influence the planner's objective differently. Alternatively, this model can incorporate the allocation of government's money to people in need via charities. The government may decide to send the money via charities that will charge an indirect cost for the use of their services. The connections of the charities as well as their quality are exogenous information that the planner cannot control, and they are typically taken into account when making a decision on how to allocate the resources.

Resource Transmission in Networks under Unit-Capacities: Consider a planner interested in distributing a fix amount of a divisible resource to agents via a set of links owned by intermediaries. Multiple layers of intermediation are possible, and thus the planner might need to contract more than one intermediary to reach an agent. We assume that links have unit capacities, which decrease the amount transmitted to the agents by the product of the capacity of the links used.

A particular case of this problem occurs when intermediaries are directly connected to agents and have 'waste-constraints' where intermediaries are directly connected to a subset of agents but only transmit a portion of the amount sent through them. Such is the case of universities or charities, where an overhead cost is charged for every dollar sent to them, and the planner can choose where every charity spends the resources —unlike in the case of proportional constraints, where the charities have exogenous priorities. The problem can also be applied to more complex layers of intermediation arising in network flow problems. For instance, when there is ground water that must be distributed to agents via private canals (intermediaries) that have an evaporation loss or other conveyance losses[1] that are proportional to the amount of water transmitted and might be different across canals. The owners of the canals may charge the planner for the use of their canals, and therefore the planner should consider the trade-offs between allocating resources to cheap canals with high conveyance losses as opposed to more efficient but relatively more expensive canals.

Resource Transmission under Capacity Constraints: Consider a planner interesting in distributing a fix amount of good to agents via a network of intermediaries. Intermediaries, who own the links, are constrained by the capacity of every link. Multiple layers of intermediation are possible, and thus the planner might need to contract more than one intermediary to reach an agent.

This model can be applied to the distribution of resources when natural disasters occur. For instance, an organization interested in transmitting the resources to regions in need may be faced with transportation capacities (such as cargo in ships and planes). Multiple layers of intermediation might be required as goods sent to remote regions might require more than one mode of transportation. This model also has applications to the transmission of data in the internet. Data transmitted in networks often goes through intermediaries which charge for the use of their links. These links are often capacity constrained and might require the user of the link to pay in order for the goods to flow in the network.[2]

Minimal Cost Trees and Related Models: The generality of our model also encompasses problems of network building that might not be explicitly used for the transmission of a

[1]Conveyance losses are typical in these models, and typically depend as a proportion of the length of the canal and structure.

[2]This can be seen in the recent dispute between Time Warner Cable Company (TWC) vs Netflix and other streaming devices, where TWC was interested in controlling the quality of streaming movies due to capacity constrains on its network. A recent agreement on the payment by Netflix to TWC has been reached.

divisible good. Such is the case for a planner seeking to build a minimum cost spanning tree that connect agents (nodes) using links in a network owned by intermediaries. When intermediaries post prices for the use of their links, the planner can choose any set that connects the agents at the minimal cost. Applications of this model include the construction of electricity and water networks.

2.1 Resource Transmission in Networks under Proportional Constraints

Consider the case where there are fixed links between the intermediaries $\mathcal{N} = \{1, \ldots, N\}$ and the agents $\mathcal{M} = \{1, \ldots, M\}$. Every intermediary is connected to a group of agents and can transmit resources to the agents that it is connected with some fixed quality (ability), this is denoted by the *sharing-rate.* Let q_{nm} be the sharing-rate of intermediary n connected to agent m, where $q_{nm} \geq 0$ for each intermediary n. The matrix of sharing-rates is $Q = (q_{11}, \ldots, q_{NM})_{N\times M}$, and $Q_n = (q_{n1}, \ldots, q_{nM})$ is the ability of intermediary n to transmit the resource to the agents. We assume that if there is no link between intermediary n and agent m, then $q_{nm} = 0$. The sharing-rate distinguishes the way in which intermediaries transmit resources to agents per unit of money given.[3] Two intermediaries connected to the same group of agents might have different impacts on the agents, and thus one might be better aligned than the other to the planner's preferences.

The planner has a utility function $u(x, p) = u(x)$ that is independent of the price paid to the intermediaries. That is, the planner cares only about the final resource transmitted to the agents in $\mathcal{M}$. Assume that the total resource available for the planner to transmit is I. Given the matrix of sharing rates Q, the outcome possibility function is

$$F(S, p) = \{\sum_{n\in S} Q_n y_n | \sum_{n\in S} y_n \leq I - \sum_{n\in S} p_n \text{ and } y_n \geq 0\} \text{ if } \sum_{n\in S} p_n \leq I$$

$$F(S, p) = \{(0, \ldots, 0)\} \text{ if } \sum_{n\in S} p_n > I$$

That is, the possibility set of a group S when posted prices are p is the transmission of not more than $I - \sum_{n\in S} p_n$ units of the resource using the abilities given by Q of the

[3] One application of this model includes in the allocation of resources to charities who have a pre-determined set of priorities among agents. When $\sum_{m=1}^{M} q_{nm} < 1$, we can interpret the intermediary (charity) as being inefficient. Such inefficiencies happen often in charities (and universities) where every dollar spent is often decreased due to indirect cost which serves to pay for the administration. The case of $\sum_{m=1}^{M} q_{nm} > 1$ implies that a dollar transmitted using that intermediary increases, for instance when charities or universities offer matching funds from donors. Previous results in the transmission of resource in networks do not distinguish in the quality of the links or assume that the sharing-rate is equal across intermediaries.

intermediaries in S.

Example 3 (Perfect Substitute Utility Function)
Consider a planner with utility function $u(x) = \sum_{m=1}^{M} \alpha_m x_m$, where α_m is the weight of the final resource allocated to agent m. Given the sharing-rates $\{q_{nm}\}_{\{n\in\mathcal{N}, m\in\mathcal{M}\}}$, the marginal utility of resource allocated to intermediary n is constant and given by $MU_n = \sum_{m=1}^{M} \alpha_m q_{nm}$. Without loss of generality we rename the intermediaries based on a non-increasing order of their marginal utility, that is $MU_1 \geq MU_2 \geq \cdots \geq MU_N$.

When $MU_1 = \cdots = MU_k > MU_{k+1}$ and $k \geq 2$, the planner is indifferent between allocating the resources to any of the intermediaries from $\{1, \ldots, k\}$ when their prices are zero. If only one intermediary from $\{1, \ldots, k\}$ has a price zero, then he can raise the price to slightly below the second lowest price posted by a different intermediary. Alternatively, if no intermediary from $\{1, \ldots, k\}$ has zero price, then at most one of them will be chosen, and the ones who are not chosen have the incentive to decrease their price. Therefore, a SPNE requires that at least two intermediaries from $\{1, \ldots, k\}$ have price zero. It is easy to verify that every price allocation such that $p_i = p_{i'} = 0$, for some $i, i' \in \{1, \ldots, k\}$ and $p_n \geq 0$, $\forall n \neq i, i'$ is a SPNE. Thus, in this example there are multiple FIEs.

When $MU_1 > MU_2$, the intermediary 1 has some market power to price above zero and continue being chosen. In a SPNE, $p_2 = 0$ and $p_1 = I(1 - \frac{MU_2}{MU_1})$, $p_n \geq 0$, $\forall n \geq 3$ and intermediary 1 is chosen to transmit $I - p_1$ units of resource. The planner's utility would be $I \cdot MU_2$, which is welfare equivalent to the utility given by allocating all resources to the intermediary with the second highest marginal utility when he prices at 0. In particular, there is no FIE.

An alternative way to prove the existence of a robust SPNE is by computing the utility-maximizing groups at $\mathbf{0}$ and applying Theorem 2 (since monotonicity and homotheticity of the preferences are clearly satisfied). Indeed, if $1, \ldots, k$ are the intermediaries with marginal utility $MU_n = MU_1$, $\forall 1 \leq n \leq k$, then each of $\{1\}, \ldots, \{k\}$ is a utility-maximizing group at $\mathbf{0}$. Therefore, if $k > 1$ then $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \emptyset$, hence a unique robust SPNE exists. However, if $k = 1$, then $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \{1\}$, thus no FIE exists.

Example 4 (Symmetric Network)
Assume that the planner with utility function $u(x) = min\{x_1, x_2, x_3\}$ cares about the agent who is allocated the least resources. The network in Figure 1 represents the connections from intermediaries to agents given by the matrix of sharing-rates $Q = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Every intermediary is connected to two agents and would always send the resource equally

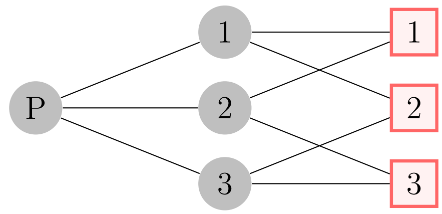

Figure 1: Network with three symmetric intermediaries

to the agents connected. Note that if the planner only uses two intermediaries, the optimal allocation is to transmit half the resources through each intermediary. Thus, the agent connected to both intermediaries would get half of the resource and each of the other two agents would get one quarter of the resource transmitted. In this case, the resource cannot be allocated equally to three agents and results in a waste of resources and an inefficiency for the planner. Thus, the planner-optimal allocation can only be achieved by using the three intermediaries in conjunction. Hence, every intermediary has some market power to post a positive price in equilibrium.

There is a symmetric equilibrium where every intermediary posts price $\frac{I}{6}$, the planner would use all the intermediaries $b(p) = \{1, 2, 3\}$, and the allocation of resource to agents is $x(p) = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$.

There is another equilibrium price allocation which results when every intermediary posts price equal to total resource I, that is $p = (I, I, I)$, and the planner pays one of the intermediaries (say, intermediary 1, $b(p) = \{1\}$) all the resource without transmitting anything, which means $x = (0, 0, 0)$. In this equilibrium, there is no incentive for intermediary 1 to deviate since it gets all the resource. For intermediary 2 or 3, even if one decreases his price, the planner cannot get positive utility because one intermediary is not connected to all the agents and at least one agent would receive 0 resource. Thus, paying all resource to intermediary 1 is still a best strategy for planner. The SPNE with planner's utility equal to 0 exists because intermediaries 2 and 3 cannot cooperate by lowering their prices simultaneously.

There is an easier way to verify that no FIE exists in this case. Indeed, note that the only utility-maximizing group at the vector of prices $\mathbf{0}$ is $\{1, 2, 3\}$. Hence, the necessary conditions to guarantee a FIE in Theorem 1 do not hold.

Let $conv(Q) = \{\sum_{n=1}^{N} \lambda_n Q_n | \sum_{n=1}^{N} \lambda_n = I, \lambda_n \geq 0, \forall n\}$ be the convex hull of the sharing rates $Q_1, \ldots, Q_N$ of intermediaries. The points in $conv(Q)$ are the feasible allocations of the resource to agents subject to the constraints Q given by the intermediaries. Let Q_{-n} be the matrix where the row Q_n is removed from Q. Let $conv(\mathbf{0}, Q)$ be the convex hull of Q and the vector of zeros. Let $x^{\star}(Q, u) = \{x \in conv(Q) | u(x) \geq u(x'), \forall x' \in conv(Q)\}$ be the set of allocations to the agents that maximize the planner's utility. Note that, when the planner's preferences are convex the set $x^\star(Q,u)$ is a convex set. Moreover, when the planner's preferences are strictly convex the set $x^\star(Q,u)$ contains a unique point.

The next result follows from the two main Theorems in the paper. We need to recognize that, due to the restrictions of the model, the assumptions in Theorem 2 regarding monotonicity and cross-monotonicity of a problem can be simply implied by the strong monotonicity and homotheticity of the planner's preferences, respectively.

Corollary 2

a. *Given the sharing rates of intermediaries $Q_1,\dots,Q_N$, there exists a FIE (or $\mathbf{0}$ is the unique robust SPNE) for any strongly monotonic*[4] *and homothetic preferences of the planner if and only if for every intermediary n, $Q_n \in conv(\mathbf{0}, Q_{-n})$.*

b. *Suppose that preferences of the planner are homothetic, strongly monotonic and strictly convex. A FIE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if the utility-maximizing allocation $x^\star(Q,u)$ belongs to the intersection of $\bigcap_{n\in\mathcal{N}} conv(Q_{-n})$.*

Part (a) provides conditions for the existence of a FIE for any strongly monotonic and homothetic preferences of the planner. Such conditions imply that the ability Q_n to transmit the resource by intermediary n can be replicated by a subset of other intermediaries. On the other hand, part (b) focuses on a specific utility function u of the planner that is monotonic and strictly convex. It requires that the utility-maximizing allocation belongs to $conv(Q_{-n})$ for any n. Thus, no intermediary is unique, as his ability can be replicated by the ability of others.

2.2 Resource Transmission in Networks under Unit-Capacities

We consider the problem of intermediation with unit-capacity constraints. A finite directed network $G=(V,E)$ without cycles that connects a single source P and sinks $\mathcal{M}=\{1,\dots,M\}\subset V$ is interpreted as connecting the planner with agents $\mathcal{M}$. The link $e\in E$ has a unit-capacity constraint c_e, which means that every unit of resource transmitted using link e would receive at most c_e units. Consider the case where the planner is endowed with I units of resource to distribute to the agents. Thus, for instance, if I units of good are transmitted in the sequence of links with unit capacities $c_1,\dots,c_l$, then $c_1\cdots c_l I$ is the maximal amount of resource that reaches its destination.

[4]The preferences represented by a utility function u are strongly monotonic if for any x and x' such that $x\ge x'$ and $x\ne x'$, $u(x)>u(x')$. While we use strong monotonicity in Corollaries 2 and 3, the same results apply for some non-monotonic preferences such as those represented by a perfect complements utility function $u(x)=\min_{i\in\mathcal{N}} x_i$.

Assume the intermediaries in the set $\mathcal{N} = \{1, \ldots, N\}$ own the links in the network. Let $E = \{E_1, \ldots, E_N\}$ be a partition of the links E, where E_n represents the links owned by intermediary n.[5]

The planner has preferences over allocations in $\mathbb{R}^M_+$ denoted by a utility function $u : \mathbb{R}^M_+ \to \mathbb{R}$ that is independent of the prices p. Thus, for instance, if the planner only cares about the total allocation to the agents, then $u(x) = \sum_m x_m$, but in general the planner might care about the worst individual $u(x) = \min_{m \in \mathcal{M}} x_m$ or some other utility function.

Assume the intermediaries post prices $p = (p_1, \ldots, p_N)$ for the use of their links.[6] Given the prices, the planner decides on the group of intermediaries to contract by paying the prices posted, and distributes the rest of the resources. Thus, for instance, if the planner is selecting group S, then he pays a total price of $\sum_{n \in S} p_n$ for the use of links in S, and $I - \sum_{n \in S} p_n$ units of resource are left for transmission to the agents.

For intermediaries $S \subset \mathcal{N}$ and agent $m \in \mathcal{M}$, let $PG(S, m)$ be the paths in G connecting the planner with agent m in the network where the capacities of the intermediaries in $\mathcal{N} \setminus S$ are zero. For a given path w with unit capacities $(c_1 \ldots c_l)$ on the links, let $c(w) = c_1 \cdots c_l$ be the unit capacity of the path. Given an agent m and intermediaries $S \subset \mathcal{N}$, let $\bar{c}^m(S) = \max_{w \in PG(S,m)} c(w)$ be the maximum unit capacity of the paths that connect agent m with the planner in the network. Note that since there is a finite number of paths, $\bar{c}^m(S)$ is easily computable. Given the group of intermediaries S, the maximal unit capacity is $\bar{c}^m(S)$ for agent m. Let $x^{m,S} = (0, \ldots, 0, \bar{c}^m(S), 0, \ldots, 0) \in \mathbb{R}^M_+$ be the vector representing the maximal transmission to the agent m using intermediaries in S. The OPF for group S and vector of prices p is

$$F(S,p) = \{x \in \mathbb{R}^M_+ \mid x \leq \sum_{m=1}^{M} \lambda^m x^{m,S}(I - p(S)), \sum_{m=1}^{M} \lambda^m = 1, \lambda^m \geq 0, \forall m\} \text{ if } \sum_{n \in S} p_n \leq I$$

$$F(S,p) = \{(0, \ldots, 0)\} \text{ if } \sum_{n \in S} p_n > I$$

Definition 3 (Non-zero Corners Utility Function)
The utility function has non-zero corners if for any $x \in \mathbb{R}^M_+$ such that $x_m = 0$ for some m, then $u(x) = 0$; and if $x > (0, \ldots, 0)$, then $u(x) > 0$. The preferences of the planner

[5] The canonical case of this model occurs when every intermediary owns one link. Another traditional case occurs when intermediaries own the span of links emanating from nodes. Moreover, the model where there is a single agent and every link has capacity 1 is discussed in Choi, Galeotti and Goyal[5]. Their results from Theorem 1 can be easily obtained from our Corollary 3 below.

[6] We focus on the case where each intermediary posts a single price for the use of all his links. We do not study the case of multiple pricing, but it is also an interesting case.

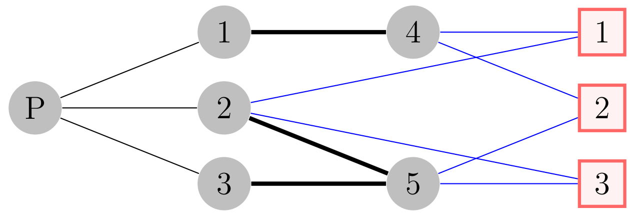

Figure 2: Network with Multiple Layers of Intermediation

are non-zero corners if there exists a non-zero corners utility function that represents such preferences.

The perfect complements utility function $u(x) = min\{x_1, \ldots, x_M\}$ and the Cobb-Douglas utility function $u(x) = \prod_{m=1}^{M} x_m^{\alpha_m}$ satisfy non-zero corners. Given that the preferences of planner are homothetic, the problem (u, F) is monotonic if the preferences are strongly monotonic or the utility function has non-zero corners (we prove this in the proof of Corollary 3 below).

Example 5

Consider the network in Figure 2. The intermediaries $\mathcal{N} = \{1, \ldots, 5\}$ are represented by the middle nodes in the network. Each of them own the links that originate from their node. The agents $\mathcal{M} = \{1, 2, 3\}$ are in the final layer of network. The black (thick) links have a unit capacity of 1, while the blue links have a unit capacity $c_j = 0.5$. The planner has a perfect complement utility function $u(x) = \min\{x_1, x_2, x_3\}$ over the final allocation of the resource to the agents in $\mathcal{M}$.

In this example no intermediary is fundamental. That is, the unit capacity of resource transmission to agent m with all intermediaries except n is $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N}) = 0.5$, $\forall m, n$. There is a FIE and unique robust SPNE, $p = \mathbf{0}$, $b(p) = \{1, 2, 4\}$, $x(p) = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$.

Consequences of Theorems 1 and 2 in the problem of resource transmission under unit-capacities are described below.

Corollary 3

a. *Suppose that for any agent $m \in \mathcal{M}$ and intermediary $n \in \mathcal{N}$ we have that $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$. Then, for any homothetic preferences of the planner, the price vector $p = \mathbf{0}$ is a FIE and unique robust SPNE. Conversely, if for any strongly monotonic utility function of the planner there exists a FIE (or $\mathbf{0}$ is the unique robust SPNE) then $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ for any agent m and intermediary n.*

b. *Suppose the planner's utility function is homothetic and has non-zero corners. A FIE* *exists (or* $\mathbf{0}$ *is the unique robust SPNE) if and only if* $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ *for any* $m \in \mathcal{M}$ *and* $n \in \mathcal{N}$*.*

This corollary establishes the sufficient conditions for the existence of a FIE and for the prices $\mathbf{0}$ to be the unique robust SPNE. These conditions require that the maximal unit capacity that can be transmitted to an agent in the network should not change when any intermediary is removed. Part (a) shows that this property is necessary if we want the existence for any monotonic utility function of the planner. On the other hand, part (b) shows that the same condition is necessary when we restrict to a single set of preferences of the planner that satisfy non-zero corners.

2.3 Resource Transmission in Networks under Total-Capacities

We consider the case of intermediation with total capacity constraints on the links. A finite directed network $G = (V, E)$ without cycles that connects a single source P and sinks $\mathcal{M} = \{1, \ldots, M\}$ is interpreted as connecting the planner with agents $\mathcal{M}$. Every link $l \in E$ in the network has a capacity constraint c_l, which is the maximal capacity that can be transmitted in that link.[7] Assume the intermediaries in the set $\mathcal{N} = \{1, \ldots, N\}$ own the links in the network. Let $E = (E^1, \ldots, E^N)$ be a partition of G, where E^n represents the links owned by intermediary n.The planner is endowed with I units of the resource and has preferences over the final allocations of the agents, denoted by a utility function $u(x) : \mathbb{R}^M_+ \to \mathbb{R}$. Unlike in the case of unit-capacities discussed above, the links have total capacities, therefore if I units of good are transmitted in the sequence of links with total capacities $(c_1, \ldots, c_l)$, then $\min\{c_1, \ldots, c_l, I\}$ reach their destination. The allocation of resource follows the same posting-price mechanism as in the case of unit-capacities.

Given an agent m and intermediaries $S \subset \mathcal{N}$, let $\bar{c}^m(S, I)$[8] be the maximal amount of resource that can be transmitted to agent m using the links owned by intermediaries in S[9] when I units are available for transmission. Notice $\bar{c}^m(S, I)$ is easily computable in the network, for instance the simple Ford-Fulkerson algorithm ([7]) computes the max-flow in a network. Let $x^{m,S} = (0, \ldots, 0, \bar{c}^m(S, I), 0, \ldots, 0) \in \mathbb{R}^M_+$ be the vector representing the maximal transmission to the agent m using intermediaries in S. The OPF for group S and vector of prices p is

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[7] Similar models with capacity constraints in links have been studied in the literature, Bochet, Ilkilic, Moulin and Sethuraman[4] discuss the transmission of a divisible resources from suppliers to demanders in a network with similar capacity constraints over the links.

[8] Unlike in the previous section, the results under total capacity depend on the total resource I, see below.

[9] Alternatively, we can re-interpret this as saying that the capacities of all the links owned by the intermediaries in $\mathcal{N} \setminus S$ are changed to zero.

$$\tilde{F}(S,p) = \{x \in \mathbb{R}_+^M \mid x \leq \sum_{m=1}^{M} \lambda^m x^{m,S}(I - p(S)), \sum_{m=1}^{M} \lambda^m = 1, \lambda^m \geq 0, \forall m\} \text{ if } \sum_{n \in S} p_n \leq I$$

$$\tilde{F}(S,p) = \{(0,\ldots,0)\} \text{ if } \sum_{n \in S} p_n > I$$

Unlike the previous two applications, the OPF in this example is not additive. This can be readily seen in an example of two links l_1, l_2 owned by different intermediaries, where (l_1, l_2) is the only path connecting the planner to a single agent. Each link has capacity 1. If the planner selects l_1 or l_2, then he cannot transmit anything to the agent. However, if the planner selects l_1 and l_2, then he can transmit 1 unit.

Furthermore, unlike in the previous two applications, the OPF $\tilde{F}$ is not homothetic in the resource I. Thus, an increase in the amount of the resource I may change the multiplicity of equilibria and welfare of the planner at equilibrium, as illustrated in the following example.

Example 6
Consider a graph with three parallel links directly connecting the planner with a single agent. The links are owned by different intermediaries and have capacities 10, 10 *and* 11, *respectively. The planner cares about transmitting the maximal amount of the resource to the agent (i.e.,* $u(x) = x$*). If the planner has* $I = 18$ *units of resource, then every pair of links can transmit the full resource (thus every pair of intermediaries would maximize the utility). The prices* $p = (0,0,0)$, $b(p) = \{1,2\}$ *and* $x(p) = 18$ *is a FIE and a unique robust SPNE. At the same time, prices* $p = (8,8,8)$, $b(p) = \{3\}$ *and* $x(p) = 10$ *is also SPNE, since intermediaries* 1 *and* 2 *cannot coordinate to lower the prices and get higher utility.*

If the planner has $I = 40$ *units of resource, there is a SPNE with* $p = (0,0,20)$ *and intermediaries 1 and 3 (or 2 and 3) being used. Note that in this equilibrium, intermediary* 3 *has a link with a larger capacity constraint than intermediaries* 1 *and* 2, *but he posts a positive price and gets a larger benefit than intermediaries 1 and 2. There are multiple SPNE, for example* $p = (30,30,30)$ *and only intermediary 3 being used. However, there is a unique robust SPNE.*

This example also shows that when resource I increases, the planner's utility at the equilibrium may not increase and the increase resource is paid to intermediaries.

This example also illustrates that the problem $(u, \tilde{F})$ is not monotonic, hence the results in Theorem 2 might not apply. Indeed, once the full capacity of the network has been reached, a strict increase in one of the prices may not strictly decrease the OPF. Therefore, the results in Theorem 2 may not apply. Consequences of Theorem 1 in the problem of resource transmission under total-capacities are described below.

Corollary 4

a. *For any monotonic utility function u there exists a FIE if and only if the full transmission of the resource to any agent m without using the links of intermediary n is possible, that is $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ for any agent m and intermediary n.*

b. *Suppose that the planner's utility function has non-zero corners. In the problem without capacities, i.e., capacities are infinity for every link, a FIE exists (or* **0** *is the unique robust SPNE) if and only if there is no intermediary who owns link(s) on every path from the planner to some agent.*

We use a simple argument of the max-flow min-cut Theorem to prove part a. A particular case of part b is discussed in Choi, Galeotti, Goyal[5], which proves the case that connects sellers and buyers, and they generate a surplus of 1 if they connect, and a surplus of 0 if they do not connect.

2.4 Separable Utility: Minimum Cost Spanning Trees and Related Models

In this section we restrict our attention to intermediation problems (u, F) with a separable utility function, $u(x, p_S) = u(x) - p(S)$, and an outcome possibility function that is independent of the price p, $F(S, p) = F(S)$. Intermediation problems with such structure capture more stylistic settings previously discussed in the literature, as shown below.

Example 7 (MCST and Related Models, Moulin and Velez[29)
] Let $\mathcal{B} = \{B_1, \ldots, B_c\} \subset 2^{\mathcal{N}}$ be a collection of acceptable subsets of intermediaries such that if $B_i \in \mathcal{B}$ and $B_i \subset D$ then $D \in \mathcal{B}$. Consider the outcome space $A = \mathbb{R}$, the utility of the planner $\bar{u}(x, p(S)) = x - p(S)$ and OPF equal to $F(S, p) = [0, 1]$ if $S \in \mathcal{B}$ and $F(S, p) = \{0\}$ if $S \notin \mathcal{B}$. Thus, the planner has a quasilinear utility function with numeraire good equal to the total price paid. The OPF has a positive element only when it is part of an acceptable set.

For instance, if $\mathcal{B}$ contains at least two individual intermediaries, say $\{i\}$ and $\{j\}$ are acceptable, then at a SPNE, the planner gets utility 1 and pays no money for the intermediaries. This is similar to a Bertrand competition model, where intermediaries lower their prices to zero in hopes to be chosen by the planner.

One particular case of this setting occurs in the minimal cost spanning tree (MCST) discussed in Moulin and Velez[29], where the links E in a network connecting a set of *nodes $\mathcal{M}$ are owned by the group of intermediaries $\mathcal{N}$. Let $(E_1, \ldots, E_N)$ be a partition of the set of links E, where E_n represents the links owned by intermediary n. The set of acceptable intermediaries $\mathcal{B} \subset 2^{\mathcal{N}}$ contain the groups of intermediaries whose links connect to all nodes in $\mathcal{M}$. Note this might not necessarily be a spanning tree. In the case where every intermediary owns exactly one link, the set $\mathcal{B}$ contains all spanning trees.*

Other related models of interconnection in trees can be similarly encompassed by this analysis, including the Steiner tree problem where the shortest interconnect for a given set of objects is found.

Let $u_S = \max_{x \in F(S)} u(x)$ be the maximal utility achieved when using the intermediaries in S and $\bar{u} = u_{\mathcal{N}} = \max_{x \in F(\mathcal{N})} u(x)$ be the maximal utility achieved when using all the intermediaries. The straightforward consequence of Theorems 1 and 2 are discussed below.

Corollary 5

a. *Consider an intermediation problem (u, F) with a separable utility function, $u(x, p(S)) = u(x) - p(S)$, and an outcome possibility function that is independent of the prices p, $F(S, p) = F(S)$. A FIE exists (or $\mathbf{0}$ is the unique robust SPNE) exists if and only if the group of intermediaries who achieve the maximal utility, $\mathcal{S} = \{S_i \subseteq \mathcal{N} | u_{S_i} = \bar{u}\}$, satisfy $\bigcap_{S_i \in \mathcal{S}} S_i = \emptyset$.*

b. *For the model in Example 7, a FIE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if the intersection of the acceptable sets is empty, that is $\bigcap_{B_i \in \mathcal{B}} B_i = \emptyset$. Furthermore, in the MCST problem a FIE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if for every node $m \in \mathcal{M}$ there are at least two intermediaries with links to node m.*

Proofs of Results in Applications

Proof of Corollary 2

Proof. We prove that the problem (u, F) is monotonic and cross-monotonic.

Recall that the preferences of the planner are independent of price $u(x, p) = u(x)$, strongly monotonic and homothetic in x. Consider a price p and group S such that $\sum_{n \in S} p_n \leq I$. Then, the OPF equals $F(S, p) = \{\sum_{n \in S} Q_n y_n | \sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n \text{ and } y_n \geq 0\}$. Consider prices p', s.t. $p'_n < p_n$, $p'_{-n} = p_{-n}$ for $n \in S$. For any $x \in F(S, p)$, assume $x = \sum_{n \in S} Q_n y_n$, $\sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n < I - \sum_{n \in S} p'_n$. Thus, for any y', s.t. $\sum_{n \in S} y'_n \leq I - \sum_{n \in S} p'_n$, $x' = \sum_{n \in S} Q_n y'_n \in F(S, p')$, so there exists $x' \in F(S, p')$ such that $x' > x$. By monotonicity of u, $u(x') > u(x)$. Thus the problem (u, F) is monotonic in prices p.

In order to show that the problem (u, F) is cross-monotonic, note that

$$F(S,p) = \{\sum_{n\in S} Q_n y_n | \sum_{n\in S} y_n \leq I - \sum_{n\in S} p_n \text{ and } y_n \geq 0\}$$

$$= (I - \sum_{n\in S} p_n)\{\sum_{n\in S} Q_n y_n | \sum_{n\in S} y_n \leq 1 \text{ and } y_n \geq 0\}.$$

Hence, remark 2 (b) is satisfied.

a. First, if $\forall n$, $Q_n \in conv(\mathbf{0}, Q_{-n})$, then the OPF $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$. Thus, for each n there exists a utility-maximizing group S_n at prices $\mathbf{0}$, s.t. $n \notin S_n$. Therefore, $\bigcap_{n\in\mathcal{N}} S_n = \emptyset$. So, from Theorem 1 and 2, there exists FIE and unique robust SPNE.

Second, if there exists FIE for any monotonic and homothetic preferences. Suppose there is intermediary n, s.t. $Q_n \notin conv(\mathbf{0}, Q_{-n})$. Consider the utility function $u(x) = \min\{\frac{x_1}{q_{n1}}, \ldots, \frac{x_M}{q_{nM}}\}$, then the indirect utility function satisfies $v(\mathbf{0}) > v_{-n}(\mathbf{0}_{-n})$. Hence, $p = \mathbf{0}$ is not equilibrium price allocation. Given prices $\mathbf{0}_{-n}$, intermediary n has incentive to deviate and post positive price $p'_n > 0$ with $p' = (p'_n, \mathbf{0}_{-n})$ and $v(p') > v_{-n}(\mathbf{0}_{-n})$. Thus, intermediary n would a higher payoff, which is a contradiction. Hence, $Q_n \in conv(\mathbf{0}, Q_{-n})$, $\forall n$.

b. When the planner has strictly convex preferences, there is a unique point $x \in F(\mathcal{N}, \mathbf{0})$ that maximizes the utility at prices $\mathbf{0}$. Let $x = x^\star(Q, u)$. Note that $x \in \bigcap_{n\in\mathcal{N}} conv(Q_{-n})$ is equivalent for the group $S_n = \mathcal{N} \setminus \{n\}$ to be a utility-maximizing group at prices $\mathbf{0}$. Hence, $\bigcap_{n\in\mathcal{N}} S_n = \emptyset$. Thus, from Theorem 1 and 2, there exists FIE and unique robust SPNE.

For the converse, if a FIE exists, then by part a, $Q_n \in conv(0, Q_{-n})$ for every intermediary n. Thus, $F(N, \mathbf{0}) = conv(0, Q) = conv(0, Q_{-n})$. Therefore, $x^\star(Q, u) \in conv(0, Q_{-n})$ for all n. Hence, $x^\star(Q, u) \in \bigcap_{n\in\mathcal{N}} conv(0, Q_{-n})$. By monotonicity of u, $x^\star(Q, u)$ is in the boundary of $conv(0, Q_{-n})$, then $x^\star(Q, u) \in \bigcap_{n\in\mathcal{N}} conv(Q_{-n})$ ■

Proof of Corollary 3

Proof. a. For any agent m and intermediary n we have that $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$, which implies that $F(\mathcal{N} \setminus \{n\}, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Thus, group of intermediaries $S_n = \mathcal{N} \setminus \{n\}$ is utility-maximizing group at prices $\mathbf{0}$, there exists $S_j(\mathbf{0}) = S_n$, so $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \emptyset$.

We now show that the problem (u, F) is monotonic. Indeed, since preferences are strongly monotonic and homothetic, for price p and p', with $p'_n < p_n$, $p'_{-n} = p_{-n}$, $n \in S$, $p(S) \leq I$, then $p'(S) < I$. Thus, there exists $y \in F(S, p')$, s.t. $y \geq x$ and $y \neq x$. Since the preferences are strongly monotonic, $u(y) > u(x)$.

In order to show that (u, F) is cross-monotonic, note that the preferences are homothetic and $F(S, p) = \gamma(S)\delta(p(S))$ for a set $\gamma(S) \subset \mathbb{R}^M$. Thus, it satisfies cross-monotonicity from Remark 2 (b). From Theorem 1 and 2, there exists FIE and unique robust SPNE.

Conversely, for any monotonic utility function, there exists a FIE (or unique robust SPNE). Suppose $\bar{c}^m(\mathcal{N} \setminus \{n\}) < \bar{c}^m(\mathcal{N})$, then if utility function $u(x) = x_m$, the planner only cares about the resource allocated to agent m, and deleting intermediary n would decrease the maximal unit capacity allocated to agent m. Thus intermediary n could post price $p_n > 0$ and get positive benefit. So there is no FIE.

b. Similar to part (a), the problem (u, F) is cross-monotonic due to the homotheticity of preferences. We now show that the problem (u, F) is monotonic when preferences are homothetic and the utility function is non-zero corners. Indeed, we simply prove that the outcome possibility function F is strongly monotonic in prices (which implies the monotonicity of (u, F) by Lemma 3). For price p and p', with $p'_n < p_n$, $p'_{-n} = p_{-n}$, $n \in S$, for $x \in F(S, p)$, $u(x) > u((0, \dots, 0))$, thus $x_m > 0$, $\forall m$, there are links to all agents with group S. $I - p(S) < I - p'(S)$, there exists $y \in F(S, p')$ and $y > x$.

Since preferences are homothetic and the utility function has non-zero corners, $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$, then we have that a FIE or a unique robust SPNE exists.

Conversely, suppose that x is a FIE (or a unique robust SPNE exists). By Theorem 2, $\bigcap_{j=1}^{J} S_j(\mathbf{0}) = \emptyset$, since the problem (u, F) is monotonic and cross-monotonic. Suppose that $x = (x_1, \dots, x_M) \in F(S, \mathbf{0})$. $x_m > 0$, for any m, so $x = \lambda^m x^{m,S} I$.

Assume $\exists m, n$ such that $\bar{c}^m(\mathcal{N} \setminus \{n\}) < \bar{c}^m(\mathcal{N})$. To prove $v(\mathbf{0}) > v_{-n}(\mathbf{0}_{-n})$, there is $x \in F(\mathcal{N} \setminus \{n\}, \mathbf{0})$ s.t. $u(x) = v_{-n}(\mathbf{0}_{-n})$. Here to prove $\exists x' \in F(\mathcal{N}, \mathbf{0})$, s.t. $x' > x$ which means $x'_i > x_i$, $\forall i$. $x^i = (0, \dots, 0, \bar{c}^i(\mathcal{N}), 0, \dots, 0) \in \mathbb{R}^M$, assume $x^i_{-n} = (0, \dots, 0, \bar{c}^i(\mathcal{N} \setminus \{n\}), 0, \dots, 0) \in \mathbb{R}^M$. Since the preferences are non-zero corners, $x_i > 0$, $\forall i$. Thus there is $\lambda^i > 0$ with $\sum_i \lambda^i = 1$, s.t. $x = \sum_{i=1}^{M} \lambda^i x^i_{-n} I$. $\exists \epsilon > 0$ small enough, with $\lambda'^m = \lambda^m - \epsilon \geq 0$, s.t. $\lambda'^m \bar{c}^m(\mathcal{N}) > \lambda^m \bar{c}^m(\mathcal{N} \setminus \{n\})$. Let $\lambda'^i = \lambda^i + \frac{\epsilon}{M-1}$, $\forall i \neq m$, $\sum_{i=1}^{M} \lambda'^i = 1$, there is $\lambda'^i \bar{c}^i(\mathcal{N}) > \lambda'^i \bar{c}^i(\mathcal{N} \setminus \{n\})$. Then $x' = \sum_{i=1}^{M} \lambda'^i x^i_{-n} I > x$ and $x' \in F(\mathcal{N}, \mathbf{0})$. The preferences are monotonic and $x' > x$, then $v(\mathbf{0}) \geq u(x', 0) > u(x, 0) = v_{-n}(\mathbf{0}_{-n})$. At prices $p_{-n} = \mathbf{0}_{-n}$, from Lemma 2, intermediary n has incentive to deviate and charge positive price $p_n > 0$. So there is no FIE. ■

Proof of Corollary 4

Proof.

a. First, we prove $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I)$ for every intermediary n and agent m if and only if $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ for every intermediary n and agent m. To see that, if $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$, then $\bar{c}^m(\mathcal{N}, I) \geq \bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ and $\bar{c}^m(\mathcal{N}, I) \leq I$, so $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I)$. To prove the converse, consider the problem of maximal flow from the planner to agent m, then $\bar{c}^m(\mathcal{N}, I)$ is the maximal flow. Thus, there exists intermediary

n who owns a link in the minimal cut such that after deleting his link, the maximal flow decreases. Hence, $\bar{c}^m(\mathcal{N}, I) > \bar{c}^m(\mathcal{N} \setminus \{n\}, I)$, which is a contradiction.

Note that $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I)$ $\forall m, n$ if and only if coalition $N \setminus n$ is a utility-maximizing coalition at prices $\mathbf{0}$ for all n. Hence, by Theorem 1, a FIE exists.

Conversely, for any monotonic utility function, there exists a FIE (or unique robust SPNE). Suppose $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) < \bar{c}^m(\mathcal{N}, I)$ and planner's utility function is $u(x) = x_m$, then if $p_{-n} = \mathbf{0}_{-n}$, since $u^*(\mathcal{N} \setminus \{n\}, \mathbf{0}) = \bar{c}^m(\mathcal{N} \setminus \{n\}, I) < \bar{c}^m(\mathcal{N}, I) = u^*(\mathcal{N}, \mathbf{0})$, intermediary n has incentive to post positive price $p_n > 0$, s.t. $p' = (p_n, \mathbf{0}_{-n})$ and $u^*(\mathcal{N} \setminus \{n\}, \mathbf{0}) < u^*(\mathcal{N}, p')$, intermediary n gets p_n. Thus, $\mathbf{0}$ is not an equilibrium.

b. Given the non-zero corner utility function, the planner needs to use path to each agent. If there exists intermediary n that owns link on every path to agent m, then $c^m(\mathcal{N} \setminus \{n\}, I) = 0 < c^m(\mathcal{N}, I) = I$, so posting a zero price for intermediary n is not optimal for him, hence there is no FIE. Conversely, if there is no intermediary n that owns link on every path connected to agent m, then $c^m(\mathcal{N} \setminus \{n\}, I) = c^m(\mathcal{N}, I) = I$ (due to the infinite capacities). From Theorem 1, there exists a FIE. ■

Proof of Corollary 5

Proof.

a. Since F is independent of the price, any group $S_j \subseteq \mathcal{N}$ such that $u_{S_j} = \bar{u}$ is a utility-maximizing group $S_j(\mathbf{0})$ at prices $\mathbf{0}$. Then $\bigcap_j S_j = \emptyset$ is equivalent with $\bigcap_j S_j(\mathbf{0}) = \emptyset$. Furthermore, the utility function $u(x, t) = u(x) - t$ is strictly decreasing in total price paid t. Hence, from Lemma 3, the problem (u, F) is monotonic. $u^*(S, \mathbf{0}) = u_S \leq u^*(T, \mathbf{0}) = u_T$ if and only if $u^*(S, p) = u_S - p(S) \leq u^*(T, p) = u_T - p(T)$ with $p(S) = p(T)$; from Definition of Cross-Monotonic, the problem is cross-monotonic. Therefore, $\bigcap_{S_j \in \mathcal{S}} S_j = \emptyset$ if and only if there exists FIE and unique robust SPNE.

b. The monotonic and cross-monotonicity of the problem (u, F) is trivial. Note that for any group of intermediaries $B_k \in \mathcal{B}$, $u^*(B_k, \mathbf{0}) = 1 = u^*(\mathcal{N}, \mathbf{0})$. So B_k is a utility-maximizing group at prices $\mathbf{0}$. From Theorem 1 and 2, there exists FIE and unique robust SPNE if and only if $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$.

We now show that in the MCST, $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$ is equivalent to have every node linked via at least two intermediaries. If every node is linked to at least two intermediaries, then $\mathcal{N} \setminus \{n\}$ is an acceptable set, $\mathcal{N} \setminus \{n\} \in \mathcal{B}$. Hence $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$. On the other hand, if m is uniquely linked to intermediary n, then $\mathcal{N} \setminus \{n\}$ is not acceptable, thus $u_{\mathcal{N}} = 1 > 0 = u_{\mathcal{N} \setminus \{n\}}$, intermediary n could charge positive price at equilibrium. Thus, there is no FIE. ■