

Intermediation Free Equilibrium in Resource Transmission Games*

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February 14, 2017

Abstract

We provide a unified framework for the study of the allocation of a divisible resource from a planner to agents via intermediaries. Our framework incorporates several models ranging from the transmission of resources to agents connected in a network to the allocation of costs in spanning trees.

Intermediaries are able to transmit the resource to a group of agents depending on the intermediaries selected as well as the fees posted by them for the use of their abilities. The planner selects a group of intermediaries to transmit the resource to agents, and chooses a feasible allocation that maximizes his utility over the resource allocated to the agents as well as the fees paid to the intermediaries selected.

We provide necessary and sufficient conditions for the existence of a perfectly competitive equilibrium, where the intermediaries used by the planner earn zero profit (*intermediation free equilibrium*). Multiplicity of equilibria may occur. We provide the necessary and sufficient conditions for the uniqueness of a *robust equilibrium*, whereby the intermediaries who are not used by the planner charge zero price.

Keywords: Resource-Sharing, Intermediation, Bertrand Competition, Networks

JEL classification: C70, D85

*We are grateful for comments received by Francis Bloch, Mihai Manea, Herve Moulin and participants at the Southwest Economic Theory Conference in Riverside, Coalition Theory Networks Conference in Moscow, Game Theory Festival in Stony Brook and the World Congress of the Game Theory Society in Maastricht.

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1 Introduction

There are many markets for which intermediaries play an essential role. The most common markets for which intermediaries are critical include the transmission of goods and resources to agents. For instance, the allocation of government resources to agents often require the use of private for-profit companies, called intermediaries, that are more closely connected to the agents than the government agency. Intermediaries therefore enable the government agency to more effectively target their agents. This top-down structure provides opportunity for competition between intermediaries with the potential for added benefits. Such benefits, however, are largely dependent on the way in which the intermediaries transmit the resources to the agents and the type of resource (i.e., divisible vs indivisible) that is to be distributed.

Although much attention has been paid to the case of intermediation for indivisible goods, few studies focus on intermediation for divisible goods and resources. Herein, we introduce a general model of intermediation where a planner is interested in transmitting a divisible resource to agents (such as money). Although the planner is not directly linked to the agents, it can do so via a group of intermediaries. Different groups of intermediaries have the ability to transmit different allocations of the goods to the agents. Thus, groups of intermediaries differ not only on the agents they can reach, but also the quality of their intermediation. For instance, two groups of intermediaries who can reach the same agents may be very different from the planner's perspective, since they may transmit different amounts to the agents.

We focus on the case where intermediaries are private and independent entities that can charge the planner for access to their agents. While intermediaries care about maximizing the amount paid by the planner, the planner has preferences over the different allocations of the resource to the agents as well as the total amount paid to the intermediaries employed.

We study the case of perfect information where the planner and intermediaries are aware of the preferences of the planner as well as intermediaries' abilities to transmit the resource. The planner solicits bids from intermediaries to access their abilities and select a group of intermediaries to contract for the transmission of the resource. In order to model the behavior of the planner and intermediaries, we used a game theoretical approach.

In the first stage of our game, intermediaries independently and simultaneously report their fees for providing the planner with access to their agents. The fees might affect the transmission of the resource. In the second stage, the planner selects the intermediaries and feasible amounts of the resource allocated to each of them for transmission to agents. The intermediaries who are not selected do not get paid. The ultimate goal of the intermediary is to be contracted and maximize the price paid by the planner. The goal of the planner is to distribute as much resource to the agents in a way that maximizes his preferences. We use a subgame perfect Nash equilibrium (SPNE) to describe the result of the strategic behavior between the planner and the intermediaries. The equilibrium price of intermediaries depends on the utility function of the planner and the abilities of the intermediaries to transmit the resource to the agents.

An important challenge is to identify the necessary conditions for existence of a perfectly competitive equilibrium, where the intermediaries used by the planner earn zero profit. In particular, our first equilibrium concept, the *intermediation free equilibrium (IFE)* is a SPNE where the intermediaries used by the planner charge zero price. This equilibrium does not

preclude the intermediaries who are not used by the planner to post a positive price. However, at an IFE, all intermediaries regardless of whether they are used by the planner earn zero profit. Thus, an IFE resembles a competitive equilibrium where the planner is directly transmitting the resource to the agents as if there are no intermediaries.

Even when an IFE exists, other SPNEs may also exist. This multiplicity of equilibrium is undesirable as it decreases the predictive power of equilibrium. We introduce a second refinement of the SPNE, the *robust SPNE*, where the group of intermediaries who are not selected by the planner price at zero. In particular, a robust SPNE is a refinement of a collusion-proof Nash equilibrium for the intermediaries who are not selected by the planner. One can imagine that if intermediaries are not selected, then they have the incentive to undercut their prices (individually or in groups) trying to get selected. Thus, a robust SPNE prevents group manipulation by the intermediaries who are not selected.¹ Note that there is a unique IFE that is robust; it requires all intermediaries to price at zero regardless of whether they are used by the planner.

The main contributions of the paper are two-fold. First, we provide the necessary and sufficient conditions on the utility function of the planner and the abilities of the intermediaries that guarantee the existence of an IFE. Second, we provide the necessary and sufficient conditions for the vector of zero prices to be the unique robust SPNE. Our work is the first paper in the literature that works for a wide variety of planner's preferences and is able to encompass a large class of intermediation settings.

1.1 Overview of the Results

We describe the main results of our paper using a simple yet illustrative example. Consider a planner who is endowed with I units of a resource and seeks to transfer as much resource to two agents. His preferences over the allocation of the resource (x_1, x_2) are given by a perfect complement utility function $u(x_1, x_2) = \min\{x_1, x_2\}$. While the planner cannot directly connect to the agents, he can do so using a group of intermediaries. The intermediaries vary on their ability to transmit the resource to the agents. These differences come from the group of intermediaries selected as well as the price paid to them. This variation is captured by an *outcome possibility function (OPF)* F that assigns to every group of intermediaries and prices a set of potential outcomes available for the planner to select from. For this example, assume there are three intermediaries, and given the vector of prices $p = (p_1, p_2, p_3)$ of the intermediaries the OPF F is given by

$$F(\{1\}, p) = \{(x, 0) | 0 \leq x \leq \frac{6}{5}(I - p_1)\},$$

$$F(\{2\}, p) = \{(0, x) | 0 \leq x \leq \frac{6}{7}(I - p_2)\},$$

$$F(\{3\}, p) = \{(x, x) | 0 \leq x \leq \frac{I - p_3}{2}\},$$

¹This does not prevent group manipulations by individuals who are selected by the planner. In fact, it is easy to see that a full coalition-proof Nash equilibrium does not exist for almost any intermediation problem.

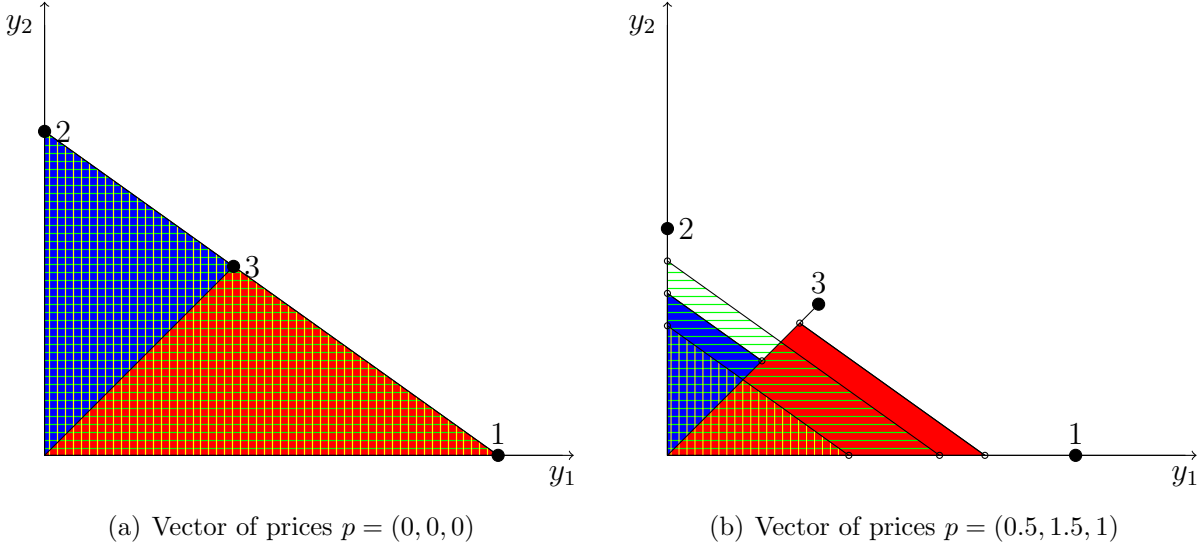


Figure 1: (a) and (b) illustrate the outcome possibility function F when $I = 5$ and prices are $p = (0, 0, 0)$ and $p = (0.5, 1.5, 1)$, respectively. The sets $F(\{1\}, p)$, $F(\{2\}, p)$ and $F(\{3\}, p)$ correspond to the line connecting the origin with the points 1, 2 and 3, respectively. The sets $F(\{2, 3\}, p)$, $F(\{1, 3\}, p)$, $F(\{1, 2\}, p)$ and $F(\{1, 2, 3\}, p)$ correspond to the shaded areas in blue, red, green/horizontal-lines and yellow/vertical-lines, respectively.

$$F(S, p) = \text{conv}(\cup_{n \in S} F(\{n\}, p), (\sum_{i \in S} p_i e^n)) \text{ for } S \subset \{1, 2, 3\},$$

where conv is the convex hull of the sets and $e^n \in \mathbb{R}^3$ is the vector equal to 1 on the n -th coordinate and zero otherwise.

Thus, the outcome possibility function is such that intermediary 1 can only transmit the resource to agent 1 and intermediary 2 can only transmit the resource to agent 2. On the other hand, intermediary 3 can transmit the resource to agents 1 and 2, but it can only do so in equal proportions. Moreover, every unit of money sent to intermediary 1 is increased by 20%, whereas every unit sent to intermediary 2 is decreased by $\frac{1}{7}$. The ability of groups of intermediaries to transfer the resource is just the convex combination of the abilities of individual intermediaries at the prices of the entire group.

When the intermediaries price at zero, the planner can transmit the resource and achieve his maximal utility by using three potential groups. The planner can use intermediaries 1 and 2, and transmit $\frac{5I}{12}$ and $\frac{7I}{12}$ via intermediaries 1 and 2, respectively. The final allocation to the agents is $(\frac{I}{2}, \frac{I}{2})$. Alternatively, the planner can allocate all the resource to intermediary 3, and the agents will also receive the same allocation $(\frac{I}{2}, \frac{I}{2})$. Moreover, the planner can also use intermediaries 1, 2 and 3 to transmit the resource with maximal utility by selecting any convex combination of the above.

Now, assume that the intermediaries post prices for the use of their OPF, then the planner chooses a group of intermediaries and transmit the resource to the agents. Let (p_1, p_2, p_3) be the vector of prices, where p_n is the price that intermediary n reports. In this two stage dynamic game, there are two types of SPNEs. The first equilibrium prices $(0, 0, 0)$ is the

intermediation free equilibrium (IFE), where the planner is able to transmit the resource as if there are no intermediaries. That is, at the IFE, the planner fully transmits the resource to the agents without any amount paid to the intermediaries. This is an equilibrium because if an intermediary who is used by the planner increases its price above zero, then the planner will not select it, as it can use another group of intermediaries to transmit the resource more efficiently.

The second type of SPNE is a “planner-inefficient” equilibrium, (p_1, p_2, p_3) , where $p_1 \geq I$, $p_2 \geq I$ and $p_3 = I$. In this equilibrium, the planner pays intermediary 3 an amount equal to I and transmits no resource to the agents. This is an equilibrium because neither intermediary 1 or 2 can decrease its price to undercut intermediary 3. Intermediary 3 has no incentive to decrease its price because it is being selected.

Two results of our study relate to the existence of an IFE, where the intermediaries used charge zero price at equilibrium. Theorem 1 shows that an IFE exists if and only if the intersection of the utility maximizing groups at prices $(0, 0, 0)$ (in our example above $\{1, 2\}$, $\{3\}$ and $\{1, 2, 3\}$) is empty.

For the second result, we introduce the *robust SPNE*, where the group of intermediaries who are not selected price at zero. In our example, $(0, 0, 0)$ is the unique robust SPNE, since in the second type of SPNE intermediaries 1 and 2 charge positive prices. Theorem 2 shows that if the problem (u, F) is monotonic and cross-monotonic,² then $(0, 0, 0)$ is the unique robust SPNE if and only if the intersection of the utility maximizing groups at prices $(0, 0, 0)$ is empty.

The paper also discusses specific classes of outcome possibility functions that guarantee the existence of IFE and uniqueness of a robust SPNE. In particular, Corollary 1 shows that either by replicating the existing intermediaries and their production functions, or by finding groups of intermediaries who perfectly complement in the OPF, will result in the existence of an IFE and unique robust SPNE.

The generality of this model provides a framework for the study of different literatures that seemed disconnected. In particular, in Appendix B we demonstrate the applicability of our results, especially Theorems 1 and 2, to a variety of new and old problems. Our work is the first paper in the literature that works for a wide variety of planner’s preferences and is able to encompass a large class of intermediation settings where the abilities of groups of intermediaries to transmit the resource to agents can be represented by an OPF.

1.2 Related Literature

The allocation of divisible resources has been prolific, especially in the network literature (see, Jackson[20] for the most comprehensive survey in networks). This includes Hougaard, et al.[15, 16, 17, 19, 18], Moulin[31, 30], Moulin et al.[32], Bochet et al.[7] and Juarez et al.[25, 26, 24, 22, 23]. However, we study the problem of transmitting a divisible good in networks with intermediaries, which not surprisingly creates substantial differences in the equilibria, strategies and difficulty of the model. A closely related paper is Moulin and Velez[33], which study the price of imperfect competition for the problem of spanning tree.

²Monotonicity occurs, for instance, when the planner is strictly worse-off as all prices increase. Cross-monotonic preferences includes the case of homothetic preferences in prices, as well as a variety of other weaker conditions.

Related results to the spanning tree model are specifically covered in Section B.4, but our equilibrium results have broader applicability, mainly due to the generality on the abilities of the intermediaries (such as connections in the networks as well as quality of the connections) and utility function of the planner.

There is also a large and growing literature in the transmission of indivisible goods and services with intermediaries. Condorelli and Galeotti [13] survey strategic models of intermediation in network. Manea[28] study dynamic game on bilateral bargaining in network with intermediation, Siedlarek[35] study a stochastic model of multilateral bargaining in a market with competition on different routes through the network. Kotowski and Leister[27] study intermediary traders in network with an auction mechanisms to set prices and analyze the welfare implications of stable and equilibrium networks. Blume et al.[6] study the effects of intermediation in markets with posted prices. Gale and Kariv[12] study a market with intermediaries and discover that the pricing behavior converges to the competitive equilibrium in an experiment. Choi, Galeotti and Goyal[10] study, theoretically and experimentally, pricing in complex structures of intermediation. In particular, their theoretical result can be obtained as a particular case of the results in Sections B.2, B.3 or B.4.

Competition and pricing in networks has also been studied. For instance, Bloch[3] surveys targeting and pricing in social networks. Bloch and Querou[5] study the monopoly pricing in social networks with consumer externalities. Campbell[8] studies monopoly targeting and pricing with communication in the network of consumers. Chawla and Roughgarden[9] study the price of anarchy and price of stability in network pricing game, in which the sellers of links have price competition facing the demand of consumers. Goyal, Heidari and Kearns[14] study the competition between firms seeking the adoption of products by consumers in social network and find bounded price of anarchy under the property of decreasing returns to local adoption. Our paper is related to this literature as we study the competition behavior among the intermediaries in a targeting problem, including network settings. However, in our resource transmission problem, the intermediaries may have different quality of transmission of the resource, which generates substantially more difficulties in both the existence and computability of equilibrium.

Our model generalizes the classical Bertrand[1] price competition model and equilibrium in two dimensions. First, groups of intermediaries might have different abilities, which can be used to differentiate from other intermediaries. Second, the planner not only cares about the price paid to the intermediaries, but also about the quality of transmission of the resource. These differences generate substantial challenges regarding the existence of equilibrium, especially since our game is discontinuous on the strategy of the intermediaries. Simon and Zame[36] prove the existence of (mixed-strategy) Nash equilibrium in discontinuous games, including the Bertrand competition game, when the sharing rule is endogenous. Reny[34] proves the existence of a pure strategy Nash equilibrium in compact, quasi-concave and better-reply secure games. More recently, Bich and Laraki[2] extend Reny's work to obtain tighter conditions for the existence of approximate equilibria. They also show that many sharing rules, especially related to competition models like this paper, generate pure and mixed strategy equilibria. Reny's result is used to prove the existence of equilibria in our setting.

1.3 Roadmap

Section 2 introduces the intermediation problem and the resource transmission game. Section 3 studies the sufficient and necessary conditions for the existence of an intermediation free equilibrium and uniqueness of a robust subgame perfect Nash equilibrium. Section 4 studies conditions on the OPFs that guarantee the existence of an intermediation free equilibrium. Appendix A contains the proofs of the main results and Appendix B discusses four applications.

2 The Model

Let $A = \mathbb{R}_+^M$ be the set of feasible outcomes. The planner is interested in choosing one of these outcomes but cannot directly select it. Instead, a group of $\mathcal{N} = \{1, \dots, N\}$ intermediaries are able to access subsets of the outcomes and set fixed prices $p = (p_1, \dots, p_N)$ for the use of their ability. Given a group of intermediaries $S \subset \mathcal{N}$, the aggregate price of group S is denoted by $p_S = \sum_{n \in S} p_n$, and the projection of the vector of prices p over $\mathbb{R}_+^{|S|}$ is denoted by $p_{[S]} \in \mathbb{R}_+^{|S|}$. For simplicity, we denote $p_{-n} = p_{[\mathcal{N} \setminus \{n\}]}$. Given prices p and p' , we say that $p' < p$ if $p'_i < p_i$ for all $i \in \mathcal{N}$. One important price vector is the vector of zero prices $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}_+^N$. In order to avoid confusion, we reserve the vector $(0, \dots, 0) \in \mathbb{R}_+^M$ to represent the allocation of the agents, whereas $\mathbf{0}$ represents the vector of zero prices.

Definition 1 (Intermediation Problem)

An intermediation problem is a pair of functions (u, F) such that:

- $u : A \times \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the **planner's preferences** over the chosen outcome, as well as the aggregate price paid to the intermediaries chosen. We assume that u is continuous, monotonic in A and non-increasing in \mathbb{R}_+ ³
- $F : 2^{\mathcal{N}} \times \mathbb{R}_+^N \rightarrow 2^A$ is an **outcome possibility function (OPF)** that assigns to every group of intermediaries and vector of prices a set of potential outcomes. We assume that F satisfies the following conditions:
 - a. $F(S, p)$ is a compact set for any group $S \in 2^{\mathcal{N}}$ and vector of prices $p \in \mathbb{R}_+^N$. Furthermore, $F(\emptyset, \mathbf{0}) = \{(0, \dots, 0)\}$
 - b. $F(S, p)$ is continuous in the vector of prices p for any group $S \in 2^{\mathcal{N}}$
 - c. F is non-decreasing in the group of intermediaries at price $\mathbf{0}$. That is, if $S \subseteq T$ then $F(S, \mathbf{0}) \subseteq F(T, \mathbf{0})$

³The utility function u is monotonic if for any $x > y$ and t we have that $u(x, t) > u(y, t)$. For the sake of brevity, we omit the other standard definitions for utility functions, but we follow standard definitions from Mas-Colell et al.[29], Chapter 3.

⁴For a given $x \in A$ and $\epsilon > 0$ the ball with center x and radius ϵ is denoted by $B_\epsilon(x) = \{x' \in A \mid |x - x'| < \epsilon\}$. In order to formally define continuity, for a given sequence of sets $\{\mathcal{M}^i\}_i$, the closure $cl(\{\mathcal{M}^i\}_i)$ is defined as $x \in cl(\{\mathcal{M}^i\}_i)$ if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that $B_\epsilon(x) \cap \mathcal{M}^i \neq \emptyset$ for any $i > \delta$. $F(S, p)$ is continuous in p whenever for any sequence in prices $\{p^i\}_i$ that converges to p , that is $\lim_{i \rightarrow \infty} p^i = p$, we have that $F(S, p) = cl(\{F(S, p^i)\}_i)$.

- d. F only depends on the prices of the chosen group. That is, $F(S, q) = F(S, p)$ for $q_{[S]} = p_{[S]}$ and for any $S \in 2^N$
- e. F is non-increasing in prices. That is, if $p \leq q$ then $F(S, q) \subseteq F(S, p)$ for any $S \in 2^N$

An intermediation problem is composed of two functions u and F . First, the function u represents the planner's preferences over the chosen outcome as well as the aggregate price paid to the intermediaries who are contracted. We assume the planner's utility does not decrease as more resources are allocated to the agents and the planner's utility is non-increasing on the total amount paid to the intermediaries.

Second, intermediaries vary on their ability to transmit the resource to the agents. These differences come from the group of intermediaries selected as well as the prices paid to them. This variation is formally described by an outcome possibility function F that assigns a set of potential outcomes to every group of intermediaries and price vector. We interpret $F(S, p)$ as the outcomes available for the planner to use after he has contracted group S and paid prices $p_{[S]}$. We have five assumptions regarding F . The first two assumptions are technical assumptions needed to guarantee the existence of equilibrium. In particular, $F(\emptyset, \mathbf{0}) = \{(0, \dots, 0)\}$ gives the planner the possibility of inaction. Selecting more intermediaries when prices are $\mathbf{0}$ should lead to no fewer feasible outcomes, which is the spirit of the third assumption. The fourth assumption guarantees that the ability of a group of intermediaries should only depend on themselves, and not on the prices posted by intermediaries outside the group. The last assumption, which relates to monotonicity, represents the fact that higher prices paid to intermediaries lead to no more resources available to transmit by the planner.

We study a two-stage perfect information game where at the first stage intermediaries choose simultaneously and independently a price p for having access to their outcome set. In the second stage, after observing the price vector p charged by the intermediaries, the planner chooses a group of intermediaries $b(p) \subset N$ and a feasible outcome $x(p) \in F(b(p), p)$.

Definition 2 (Resource Transmission Game)

Given an intermediation problem (u, F) , the **resource transmission game** is a sequential game of perfect information such that:

- The strategy space of intermediary n is $[0, P_n]$, where $0 \leq P_n \leq +\infty$. The **strategy of intermediary n** is to set a fixed price $p_n \in [0, P_n]$ that the planner has to pay for the use of his ability. P_n is the maximum price that intermediary n is allowed to post, when $P_n = +\infty$ there is no upper bound on the price of intermediary n . Let $p = (p_1, \dots, p_N)$ be the vector of strategies by the intermediaries.
- The **strategy of the planner** is a pair of functions $b : \mathbb{R}_+^N \rightarrow 2^N$ and $x : \mathbb{R}_+^N \rightarrow A$ such that $x(p) \in F(b(p), p)$.
- The objective of each intermediary is to maximize the price paid by the planner. The **utility $V^n(p, b, x)$ of intermediary n** is $V^n(p, b, x) = p_n$ if $n \in b(p)$, and $V^n(p, b, x) = 0$ if $n \notin b(p)$. That is, only the intermediaries selected might get positive utility equal to their proposed price.

- We focus on the case where the planner only pays for the intermediaries used. Therefore, the **utility of the planner** equals $u(x(p), \sum_{n \in b(p)} p_n)$.

In most of the paper, we impose no restriction on whether P_n is finite or infinite. We do impose a finite maximal price P_n , for every intermediary n , in Lemma 1.

Given prices p and a group of intermediaries S , the set of *utility maximizing allocations* at (S, p) is the set $x^*(S, p) \subset F(S, p)$ such that $x \in x^*(S, p)$ if and only if $u(x, p_S) \geq u(x', p_S)$ for any $x' \in F(S, p)$. Since $F(S, p)$ is compact and u is continuous, the set $x^*(S, p)$ is non-empty. The *maximal utility* $u^*(S, p)$ given prices p for group S equals $u(x, p_S)$ for $x \in x^*(S, p)$.⁵ Given prices p , the group $S(p) \subset \mathcal{N}$ is a *utility maximizing group* at p if $u^*(S(p), p) \geq u^*(T, p)$ for any $T \in 2^{\mathcal{N}}$.

Given the perfect information and sequentiality of the resource transmission game, we use a subgame perfect Nash equilibrium as a predictor of the behavior of the planner and intermediaries.

Definition 3 (Subgame Perfect Nash Equilibrium)

The strategies from intermediaries $p \in \mathbb{R}_+^{\mathcal{N}}$ and planner (b, x) are a **subgame perfect Nash equilibrium (SPNE)** if

- $V^n(p, b, x) \geq V^n(\tilde{p}_n, p_{-n}, b, x)$ for any $n \in \mathcal{N}$ and $\tilde{p}_n \in \mathbb{R}_+$.
- For any prices p , the selected group $b(p)$ is a utility maximizing group at p and $x(p) \in x^*(b(p), p)$.

When there is no confusion, a SPNE (p, b, x) will simply be referred to as the vector of prices p .

The ideal for the planner is to finding conditions under which there is no waste of resources used to pay the intermediaries. We capture this in the definition of an intermediation free equilibrium where the final allocation implemented as a SPNE is welfare-equivalent for the planner as if all the intermediaries price at zero.

Definition 4 (Intermediation Free Equilibrium)

An **intermediation free equilibrium (IFE)** (p, b, x) is a vector of strategies such that (p, b, x) is a SPNE and $u(x(p), p_{b(p)}) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, \mathbf{0})$.

Note that an IFE requires that the allocation to agents $x(p)$ and prices paid to intermediaries selected $p_{[b(p)]}$ are *planner-optimal*, that is, they achieve the maximal utility $\max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, \mathbf{0})$. However, at an IFE not all intermediaries need to be pricing at zero.

Definition 5 (Indirect Utility Function)

- The **indirect utility function** $v(p) = \max_{x \in F(S, p), S \in 2^{\mathcal{N}}} u(x, p_S)$ is the maximal utility that the planner can achieve given the prices p .
- The **indirect utility function without intermediary n** is denoted by $v_{-n}(p_{-n}) = v_{\mathcal{N} \setminus \{n\}}(p_{-n}) = \max_{x \in F(S, p), S \in 2^{\mathcal{N} \setminus \{n\}}} u(x, p_S)$.

⁵The maximal utility $u^*(S, p) = \max_{x \in F(S, p)} u(x, p_S)$ depends on prices p rather than p_S because $F(S, p)$ depends on p .

Note that since the OPF F and utility function u are non-increasing in prices, then the indirect utility function v is non-increasing in prices as well. Continuity of the indirect utility function is guaranteed, mainly due to the continuity of the utility function u and OPF F . This is proven in Lemma 2 and used in the main results.

3 Intermediation Free Equilibria

We formalize below a situation where a strict decrease in the prices of the intermediaries leads to a strict increase in the utility of the planner.

Definition 6 (Monotonicity)

The problem (u, F) is **monotonic in prices** if for any $S \subseteq \mathcal{N}$, intermediary $n \in S$, and prices p and p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$ we have that for any $x \in F(S, p)$ such that $u(x, p_S) > u((0, \dots, 0), 0)$ there exists $y \in F(S, p')$ such that $u(y, p'_S) > u(x, p_S)$.

When there is no confusion, we refer to the problem (u, F) as *monotonic* instead of monotonic in prices. Monotonicity in prices is a weak property that occurs in a large class of applications, including the four applications discussed in Appendix B. Our definition of monotonicity of the intermediation problem (u, F) implies that the indirect utility function of the planner is monotonic in prices. This is proven in Lemma 3 and widely used in the proofs of the main results.

The monotonicity of the intermediation problem (u, F) may be coming from two forces. On one hand, it might be that the planner has a utility function u that is strictly monotonic in prices. On the other hand, the OPF F may be strongly monotonic in prices.

Remark 1

Either of the following conditions is sufficient for the monotonicity of the intermediation problem (u, F) :

- a. u is strictly monotonic on the total price paid by the planner. That is, $u(x, t) > u(x, \tilde{t})$ for any $t < \tilde{t}$ and $x \in A$.
- b. The outcome possibility function F is strongly monotonic in prices.⁶

Lemma 3 also proves the claims in this remark. Conditions a and b provide simpler avenues to verify monotonicity of an intermediation problem. These conditions have important implications for the planner. They require that as price decreases, the planner should be strictly better off either due to paying less for intermediation (condition a) or having strictly better off options available for the planner to choose from (condition b). The example discussed in Section B.4 satisfies condition a, whereas a large class of networks discussed in sections B.1, B.2 and B.3 satisfy condition b.

The next property relates to the utility of the planner at the limit of the strategy space of every intermediary in the resource transmission game. It requires that for every intermediary

⁶ We say that F is strongly monotonic in prices if for any $S \subseteq \mathcal{N}$, intermediary $n \in S$, and prices p and p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$ we have that for any $x \in F(S, p)$ such that $u(x, p_S) > u((0, \dots, 0), 0)$ there exists $y \in F(S, p')$ such that $y > x$.

there exists a large enough price such that the planner is (weakly) better-off not selecting any group containing that intermediary.

Definition 7 (Price-Satiated)

The resource transmission game generated by (u, F) is **price-satiated** if the strategy space of every intermediary is bounded and for any group of intermediaries $S \subseteq \mathcal{N}$ and intermediary $n \in S$ with maximum price P_n , $u(x, P_n) \leq u((0, \dots, 0), 0)$ for any $x \in F(S, (P_n, \mathbf{0}_{-n}))$.

Price-satiation guarantees that the planner has the choice of inaction, selecting no groups of intermediaries, when they price is high enough. A particular case of price-satiation occurs when at the maximum price of every intermediary there is no remaining resource to transmit, that is $F(S, (P_n, \mathbf{0}_{-n})) = \{(0, \dots, 0)\}$ for every $S \subset N$ and intermediary $n \in S$. This condition is satisfied in the first three applications discussed in Appendix B.

In general, the existence of a SPNE in a resource transmission game is not guaranteed. This can be easily seen in a Bertrand competition game with producers who have different marginal cost and where the planner splits the resource equally in case of ties. However, under the appropriate tie-breaking rule chosen by the planner, an equilibrium exists (Reny[34]). The following results provide conditions for the existence of equilibrium in a large class of resource transmission games.

Lemma 1 (Existence of SPNE)

Every price-satiated resource transmission game generated by a monotonic problem (u, F) has a SPNE.

The proof of this result is based on Reny[34], who provides the existence of pure-strategy equilibrium in games that are better-reply secure. Contrary to the rest of the paper, the existence of equilibria requires a finite upper bound on the price that every intermediary can charge (implied by price-satiation), as Reny’s result requires a compact strategy space for every player.

One important utility maximizing group of intermediaries occurs when the prices posted are zero. The structure of such groups, especially with regard to their intersection, is important to understand the existence of an intermediation free equilibrium.

Theorem 1 (Existence of IFE)

Assume the utility maximizing groups at zero prices are $S_1(\mathbf{0}), S_2(\mathbf{0}), \dots, S_J(\mathbf{0})$. There is an intermediation free equilibrium if and only if $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$.

The existence of an IFE implies that there is no group of intermediaries who belong to all the utility maximizing groups at zero prices. The intuition is that if a group of intermediaries belong to this intersection, then these intermediaries will have sufficient market power to price above zero, thus creating an equilibrium that is not planner-optimal. The extreme case occurs in the traditional Bertrand competition model where symmetric producers with zero marginal cost of production compete for a price and the unique SPNE leads to an equilibrium price equal to zero. However, when producers have different marginal cost of production, a SPNE where producers price above zero is possible.

Theorem 1 provides testable conditions for the existence of an IFE. For most of the applications shown in Appendix B, these conditions are simple to compute, and are typically

not more difficult than computing 2^N utility maximization problems. Thus, for instance, if the only utility maximizing group is the grand coalition, then every intermediary has sufficient market power to price above zero, hence there is no IFE.

3.1 Robust SPNE

Multiplicity of equilibria often occurs, as will be seen in Example 1 and other examples in Appendix B. However, we can argue that some of the equilibria might not be as likely to occur because there are groups of intermediaries who may gain by offering their abilities at a lower price.⁷ In particular, intermediaries who are not chosen by the planner always have the incentive to undercut their prices in hopes of being chosen. In this section we look at a robustness of SPNE, where intermediaries who are not used by the planner cannot jointly decrease their prices and affect the equilibrium. Formally, a SPNE is a robust SPNE when the intermediaries who are not used by the planner charge prices equal to zero.

Definition 8 (Robust SPNE)

The subgame perfect Nash equilibrium (p, b, x) is **robust** if intermediaries who are not used by the planner post zero prices. That is, the SPNE (p, b, x) is robust whenever $n \notin b(p)$ implies $p_n = 0$.

A robust SPNE is an equilibrium refinement weaker than a collusion-proof Nash equilibrium, since intermediaries who are not selected by the planner cannot collude and gain by lowering their prices. Note that at a robust SPNE it is possible for intermediaries who are used by the planner to charge positive prices, thus a robust SPNE might not be an IFE. Furthermore, it is possible for a robust SPNE not to exist or for multiple robust SPNE to exist. Our analysis in this section will focus on finding the conditions on the intermediation problem for the existence and uniqueness of the robust SPNE.

The intermediation problem is cross-monotonic whenever the ranking of groups at the maximal utility is maintained as prices change.

Definition 9 (Cross-Monotonic)

The problem (u, F) is **cross-monotonic** if $\max_{x \in F(S, 0)} u(x, 0) \leq \max_{x \in F(T, 0)} u(x, 0)$, then $\max_{x \in F(S, p)} u(x, p_S) \leq \max_{x \in F(T, p)} u(x, p_T)$ for any p with $p_S = p_T$.

Cross-monotonicity is satisfied by a variety of intermediation problems, including the cases of homothetic⁸ preferences and product separable OPF.

Remark 2

Either of the following conditions on the intermediation problem guarantees that the problem is cross-monotonic:

⁷In the example in Section 1.1, assuming intermediaries are playing the equilibrium prices $p = (I, I, I)$ where intermediary 3 is chosen by the planner, we can see that intermediaries 1 and 2 can gain by lowering their prices simultaneously to a vector of prices p such that $p_1 + p_2 < I$.

⁸Preferences are *homothetic* if and only if there exists a utility function such that $u(\lambda x) = \lambda u(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}_+^M$.

- a. u is product separable: there exists functions $\alpha : \mathbb{R}_+^M \mapsto \mathbb{R}$ and $\beta : \mathbb{R}_+ \mapsto \mathbb{R}$ such that $u(x, t) = \alpha(x)\beta(t)$ for any x and t . Moreover, F is independent of prices: $F(S, p) = F(S)$ for any S and p .
- b. F is product separable: there exists functions $\gamma : 2^{\mathcal{N}} \mapsto 2^A$ and $\delta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $F(S, p) = \gamma(S)\delta(p_S)$ for any S and p . Moreover, the utility function is independent of prices and homothetic: $u(x, t) = \tilde{u}(x)$ and $\tilde{u}(\lambda x) = \lambda \tilde{u}(x)$ for any $x, t \geq 0$ and $\lambda > 0$.

The proof of this Remark is in Appendix A.

Theorem 2 (Uniqueness of Robust SPNE)

Assume that the problem (u, F) is monotonic in prices and cross-monotonic. $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$ if and only if the price vector $\mathbf{0}$ is the unique robust SPNE.

This Theorem complements the existence results of Theorem 1. Under the conditions of monotonicity in prices and cross-monotonicity of the intermediation problem, and when no group of intermediaries belongs to all the utility maximizing groups, there exists a unique robust SPNE.

The proof of the converse of this Theorem is readily seen. Indeed, if the prices $\mathbf{0}$ is a (robust) SPNE, then $\mathbf{0}$ is also an IFE. Therefore, by Theorem 1, $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$. The other side of the Theorem is substantially more difficult than the proof of Theorem 1. Its proof requires a variety of intermediate results related to the continuity and monotonicity of the indirect utility function (Lemmas 2 and 3) as well as a result that provides conditions that a SPNE satisfies even when it is not an IFE (Lemma 4).

Remark 3

Cross-monotonicity and Monotonicity in prices are necessary for Theorem 2 to hold.

The proof of this Remark is in Appendix A.

4 OPFs and IFE

We now turn our attention to outcome possibility functions that guarantee an IFE and unique robust SPNE. Consider the situation where every intermediary has an exact duplicate at prices $\mathbf{0}$. For instance, we can imagine a situation where an economy is replicated by doubling the intermediaries along with their abilities. The following definition formalizes this situation.

Definition 10 (Duplicated OPF)

An outcome possibility function F is **duplicated** if it is defined for $N = 2k$ intermediaries and for any $S \subset \{1, \dots, k\}$ and $T \subset \{k+1, \dots, 2k\}$, we have that $F(S \cup T, \mathbf{0}) = F(S \cup T(-k), \mathbf{0})$, where $T(-k) = \{n - k | n \in T\}$.

Under a minimally competitive OPF no intermediary is unique. That is, for any intermediary n , there is an intermediary n' that brings exactly the same outcome as n . In particular, this happens when the OPF is additive⁹ and any intermediary has an exact replica.

⁹ An OPF F is additive if $F(S, \mathbf{0}) = \text{conv}(\cup_{n \in S} F(\{n\}, \mathbf{0}))$ for any $S \subset \mathcal{N}$ and prices $\mathbf{0}$. Such is the case of the example discussed in Section 1.1.

Definition 11 (Minimally Competitive OPF)

An outcome possibility function F is **minimally competitive** if for any intermediary n , there exists $n' \neq n$, such that $F(S \cup \{n\}, \mathbf{0}) = F(S \cup \{n'\}, \mathbf{0})$ for any group S .

Corollary 1 (Sufficient Conditions that Guarantee IFE)

Suppose that the problem (u, F) is monotonic in prices and cross-monotonic. Any of the following conditions is sufficient to guarantee the existence of an intermediation free equilibrium and a unique robust SPNE:

- a. The problem has a minimally competitive OPF.
- b. The problem has a duplicated OPF.
- c. There exists a group of intermediaries S such that $F(S, \mathbf{0}) = F(\mathcal{N} \setminus S, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$.

This result implies that either by replicating the existing intermediaries and their OPFs or by finding a group of intermediaries that have the same abilities as their complement, will result in IFE and unique robust SPNE. Part (c) also illustrate comparative statics with respect to the addition of intermediaries: if a new group of intermediaries arrive and have exactly the same abilities as the original intermediaries, then an IFE and a unique robust SPNE will be created. The intuition behind this corollary is similar to Theorem 1: perfect competition among the intermediaries occurs when every intermediary can be substituted by another group of intermediaries that achieve an equal level of utility.

5 Conclusion

This paper investigates how intermediation affects the resource transmission between a planner and agents. We build a game theory model to study the market power of intermediaries to charge the planner a price for the use of their abilities to transmit the resource. We discover and describe the necessary and sufficient conditions for the existence of IFEs and uniqueness of a robust SPNE. We demonstrate how properties in the OPFs can achieve an IFE, including one that replicates the economy.

The generality of our model allows for the application of the results to a wide variety of new and old intermediation problems, some of them described in Appendix B. This paper is a start to the analysis of the transmission of a divisible resource from a planner to agents via intermediaries. Future work could include studies involving the case of competition between multiple planners, variable-pricing instead of fixed-pricing by the intermediaries, and imperfect information about the OPFs.

A Appendix: Proofs of Main Results

In order to prove Lemma 1 and Theorem 2, we introduce three intermediate results related to the continuity and monotonicity of the indirect utility function as well as a result that provides necessary and sufficient conditions for a SPNE.

Lemma 2 (Continuity of the Indirect Utility Function)

For any problem (u, F) , the indirect utility function $v(p)$ is continuous in p . Furthermore, the maximal utility $u^*(S, p)$ given group S is continuous in prices p .

Proof. We first show that the indirect utility function is continuous. We prove this in five steps, breaking the problem into converging sequences bounded from above and below the limit price.

Consider a decreasing sequence $\{p^i\}$, s.t. $\lim_{i \rightarrow \infty} p^i = p$, $p^i \geq p$. The indirect utility function $v(p)$ is non-increasing in p , since u is non-increasing in the aggregate prices and F is non-increasing in prices. Thus, $v(p^i) \leq v(p)$ for any i . Note that $\lim_{i \rightarrow \infty} v(p^i)$ exists by the monotonicity of $\{v(p^i)\}_i$. Therefore, $\lim_{i \rightarrow \infty} v(p^i) \leq v(p)$. From Definition 1, for any S , $F(S, p^i) \subseteq F(S, p)$. Let $S(p)$ and $x(p)$ are a utility maximizing group and optimal allocation at prices p , respectively. Let $S = S(p)$, then $v(p) = u(x(p), p_S)$. Since $F(S, p)$ is continuous in p , there exists a sequence of allocations $\{x^i\}_i$, s.t. $x^i \in F(S, p^i)$ for all i , $\lim_{i \rightarrow \infty} x^i = x(p)$. Since the utility function $u(x, t)$ is continuous, and $\lim_{i \rightarrow \infty} p^i_S = p_S$, then $\lim_{i \rightarrow \infty} u(x^i, p^i_S) = u(x(p), p_S)$. Since $v(p^i) \geq u(x^i, p^i_S)$, then $\lim_{i \rightarrow \infty} v(p^i) \geq \lim_{i \rightarrow \infty} u(x^i, p^i_S) = u(x(p), p_S) = v(p)$. This inequality together with the inequality above imply that $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

For any sequence $\{p^i\}$ with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \geq p$. We can find a decreasing sequence $p^{i'}$, s.t. $p^{i'} \geq p^i$ and $\lim_{i \rightarrow \infty} p^{i'} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{i'}) \leq \lim_{i \rightarrow \infty} v(p^i) \leq v(p)$, and since $\lim_{i \rightarrow \infty} v(p^{i'}) = v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

Consider a increasing sequence $\{p^i\}$, with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \leq p$. There exists limit for the monotonic decreasing sequence $v(p^i)$ and $v(p^i) \geq v(p)$, so $\lim_{i \rightarrow \infty} v(p^i) \geq v(p)$. Assume $v(p^i) = u^*(S^i, p^i)$, the group of intermediary $S^i \in 2^N$, there exists group S , s.t. $\{p^k\}$ which is a subsequence of $\{p^i\}$, $S^k = S$ and $\lim_{k \rightarrow \infty} u^*(S, p^k) = \lim_{i \rightarrow \infty} v(p^i)$. Assume x^k is the utility maximizing allocation in $F(S, p^k)$, $u^*(S, p^k) = u(x^k, p^k_S)$, $x^k \in F(S, p^k) \subset F(\mathcal{N}, \mathbf{0}) \subset \mathbb{R}_+^M$. $F(\mathcal{N}, \mathbf{0})$ is compact, so $F(\mathcal{N}, \mathbf{0})$ is sequentially compact, there exists a convergent subsequence of x^k , name as x^h , s.t. $\lim_{h \rightarrow \infty} x^h = x$, $x^h \in F(S, p^h)$. OPF is continuous in prices p , so $x \in F(S, p)$. Since $\lim_{h \rightarrow \infty} x^h = x$ and $\lim_{h \rightarrow \infty} p^h = p$, $\lim_{h \rightarrow \infty} v(p^h) = \lim_{h \rightarrow \infty} u(x^h, p^h_S) = u(x, p_S) \leq u^*(S, p) \leq v(p)$. While $\lim_{h \rightarrow \infty} v(p^h) = \lim_{i \rightarrow \infty} v(p^i) \geq v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

For any sequence $\{p^i\}$ with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \leq p$. We can find a increasing sequence $p^{i'}$, s.t. $p^{i'} \leq p^i$ and $\lim_{i \rightarrow \infty} p^{i'} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{i'}) \geq \lim_{i \rightarrow \infty} v(p^i) \geq v(p)$, and since $\lim_{i \rightarrow \infty} v(p^{i'}) = v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

Finally, we prove that for any sequence $\{p^i\}$ such that $\lim_{i \rightarrow \infty} p^i = p$, we have that $\lim_{i \rightarrow \infty} v(p^i) = v(p)$. Construct two sequences $\{p^{1i}\}$ and $\{p^{2i}\}$, let $p_n^{1i} = \min\{p_n^i, p_n\}$, and $p_n^{2i} = \max\{p_n^i, p_n\}$, then $p^{1i} \leq p$, $p^{2i} \geq p$ and $p^{1i} \leq p^i \leq p^{2i}$, $\lim_{i \rightarrow \infty} p^{1i} = \lim_{i \rightarrow \infty} p^{2i} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{1i}) \geq \lim_{i \rightarrow \infty} v(p^i) \geq \lim_{i \rightarrow \infty} v(p^{2i})$. From above, $\lim_{i \rightarrow \infty} v(p^{1i}) = \lim_{i \rightarrow \infty} v(p^{2i}) = v(p)$, then $\lim_{i \rightarrow \infty} v(p^i) = v(p)$. Thus, the indirect utility function is continuous in price p .

We can similarly show that the maximal utility $u^*(S, p)$ is continuous in prices p , given group S . Indeed, consider a decreasing sequence $\{p^i\}$, with $p^i \geq p$ and $\lim_{i \rightarrow \infty} p^i = p$. $u^*(S, p^i) = \max_{x \in F(S, p^i)} u(x, p^i_S)$. $F(S, p^i) \subseteq F(S, p^j)$, $u(x, p^i_S) \leq u(x, p^j_S)$ for $i \leq j$, $p^i \geq p^j$. Thus, $u^*(S, p^i) \leq u^*(S, p^j)$, $u^*(S, p^i) \leq u^*(S, p)$, and $\lim_{i \rightarrow \infty} u^*(S, p^i) \leq u^*(S, p)$. Assume $u^*(S, p) = u(x, p_S)$, $x \in F(S, p)$. Since $F(S, p)$ is continuous in p , there exists a sequence of allocations $\{x^i\}_i$, s.t. $x^i \in F(S, p^i)$ for all i , $\lim_{i \rightarrow \infty} x^i = x(p)$. Since the utility function $u(x, t)$

is continuous in x and t , and $\lim_{i \rightarrow \infty} p_S^i = p_S$, then $\lim_{i \rightarrow \infty} u(x^i, p_S^i) = u(x(p), p_S)$. Since $u^*(S, p^i) \geq u(x^i, p_S^i)$, then $\lim_{i \rightarrow \infty} u^*(S, p^i) \geq \lim_{i \rightarrow \infty} u(x^i, p_S^i) = u(x(p), p_S) = u^*(S, p)$. This inequality together with the inequality above imply that $\lim_{i \rightarrow \infty} u^*(S, p^i) = u^*(S, p)$.

For an increasing sequence $\{p^i\}$, with $p^i \leq p$ and $\lim_{i \rightarrow \infty} p^i = p$. By definition of $u^*(S, p)$, $u^*(S, p) \leq u^*(S, p^i)$, there exists limit of the sequence $\lim_{i \rightarrow \infty} u^*(S, p^i)$, and $u^*(S, p) \leq \lim_{i \rightarrow \infty} u^*(S, p^i)$. Assume $u^*(S, p^i) = u(x_i, p_S^i)$, $F(\mathcal{N}, \mathbf{0})$ is sequentially compact, there exists a convergent subsequence of x^i , name as x^k , s.t. $\lim_{k \rightarrow \infty} x^k = x$, $x^k \in F(S, p^k)$. OPF is continuous in prices p , so $x \in F(S, p)$. Since $\lim_{k \rightarrow \infty} x^k = x$ and $\lim_{k \rightarrow \infty} p^k = p$, $\lim_{k \rightarrow \infty} u^*(S, p^k) = \lim_{k \rightarrow \infty} u(x^k, p_S^k) = u(x, p) \leq u^*(S, p)$. Thus, we have $\lim_{i \rightarrow \infty} u^*(S, p^i) = u^*(S, p)$.

By repeating the same strategy, used in the continuity of the indirect utility function, that bounds an arbitrary converging sequence with monotonic convergent sequences, we have that for any sequence p^i such that $\lim_{i \rightarrow \infty} p^i = p$, $p^i \leq p$, there is $\lim_{i \rightarrow \infty} u^*(S, p^i) = u^*(S, p)$. And for any sequence p^i , with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \geq p$, there is $\lim_{i \rightarrow \infty} u^*(S, p^i) = u^*(S, p)$. Hence, since any convergent sequence $\{p^i\}$ to p can be split into two convergent sub-sequences with values above or below p , we have that $\lim_{i \rightarrow \infty} u^*(S, p^i) = u^*(S, p)$. Hence, the function $u^*(S, p)$ is continuous in prices p . ■

Lemma 3 (Monotonicity of the Indirect Utility Function)

- a. If the problem (u, F) is monotonic, then the indirect utility function v is monotonic in prices. That is, if $p' < p$ and $v(p) > u((0, \dots, 0), 0)$ then $v(p') > v(p)$.
- b. Consider a utility maximizing group S at prices p , intermediary $n \in S$ and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. If $v(p) > u((0, \dots, 0), 0)$, then $v(p') > v(p)$.
- c. Consider any group S , intermediary $n \in S$ and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. If $u^*(S, p) > u((0, \dots, 0), 0)$, then $u^*(S, p') > u^*(S, p)$.
- d. Either of the following conditions is sufficient for the monotonicity of the problem (u, F) :
 - u is strictly monotonic on the total price paid by the planner. That is, $u(x, t) > u(x, \tilde{t})$ for any $t < \tilde{t}$ and $x \in A$.
 - The outcome possibility function F is strongly monotonic in prices.

Proof. First note that part a is clearly implied by part b.

To prove part b, assume S is a utility maximizing group at prices p and such that $v(p) > u((0, \dots, 0), 0)$. Let x be a utility maximizing allocation in $F(S, p)$. Then, $v(p) = u(x(p), p_S)$. Given that the problem (u, F) is monotonic, for $n \in S$, and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$, there exists $y \in F(S, p')$ such that $u(y, p'_S) > u(x, p_S)$, so $v(p') \geq u(y, p'_S) > u(x, p_S) = v(p)$.

To prove part c, consider the group S and prices p such that $u^*(S, p) > u((0, \dots, 0), 0)$. Let x be a utility maximizing allocation in $F(S, p)$. Then, $u^*(S, p) = u(x, p_S)$. Given that the problem (u, F) is monotonic, for each $n \in S$ and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$, there exists $y \in F(S, p')$ such that $u(y, p'_S) > u(x, p_S)$, so $u^*(S, p') \geq u(y, p'_S) > u(x, p_S) = u^*(S, p)$.

In order to prove part *d*, suppose that $u(x, t)$ is monotonic in t . Let p, p' and S defined as above. By the monotonicity of the OPF, $F(S, p) \subseteq F(S, p')$. Thus, any allocation of resource $x \in F(S, p)$, is also feasible for prices p' , that is $x \in F(S, p')$. Let $y = x$, $u(x, p'_S) > u(x, p_S)$, the problem (u, F) is monotonic.

On the other hand, consider a price p and a utility maximizing group S at prices p . Consider intermediary $n \in S$ and prices p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. Let x be a utility maximizing allocation in $F(S, p)$ and assume that $u(x, p_S) > u((0, \dots, 0), 0)$. Since F is strongly monotonic in p , for $v(p) = u(x(p), p_S)$ and $x(p) \in F(S, p)$, there is $\epsilon > 0$, s.t. $B_\epsilon(x(p)) \subset F(S, p')$. Since the preferences represented by $u(x, t)$ are monotonic in A , then there exists $y \in B_\epsilon(x(p))$, s.t. $u(y, p_S) > u(x(p), p_S) \geq u(x, p_S)$. Therefore, $u(y, p'_S) \geq u(y, p_S) > u(x, p_S)$. ■

Lemma 4 (Conditions for Existence of SPNE)

Consider prices and allocation (p, b, x) that is a SPNE. Assume that the utility maximizing groups at prices p are $S_1(p), \dots, S_J(p)$. Then,

- a. $\bigcap_{j=1}^J S_j(p) = \emptyset$.
- b. If the problem (u, F) is monotonic in prices, then there exists a utility maximizing group, without loss of generality, assume it is $S_1(p)$, s.t. $\forall n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$, $p_n = 0$.
- c. $\forall n \in \mathcal{N} \setminus \bigcup_{j=1}^J S_j(p)$, it would not increase the planner's utility even if its price decreases to 0. That is, for $p'_n = 0$ and $p' = (p'_n, p_{-n})$, we have that $v(p) = v(p')$.

Conversely, if there exists a vector of prices p that satisfies conditions (a)-(c), then p is supported by a SPNE. (with $b(p) = S_1(p)$)

Proof. a. Suppose $\bigcap_{j=1}^J S_j(p) \neq \emptyset$, then there exists intermediary $n \in \bigcap_{j=1}^J S_j(p)$. The indirect utility function without using intermediary n is $v_{-n}(p_{-n})$. Since intermediary n is in every utility maximizing group, the utility maximizing group in $\mathcal{N} \setminus \{n\}$ would achieve lower utility at prices p_{-n} , thus $v_{-n}(p_{-n}) < v(p)$. By Lemma 2, the indirect utility function is continuous, there exists ϵ small enough, such that intermediary n increases its price by ϵ , $p'_n = p_n + \epsilon$, $p' = (p'_n, p_{-n})$, s.t. $v(p) \geq v(p') > v_{-n}(p_{-n})$. Intermediary n would still be paid by the planner after increasing price by ϵ , hence there is incentive for intermediary n to deviate, thus p cannot be a SPNE. Hence, $\bigcap_{j=1}^J S_j(p) = \emptyset$.

b. Suppose that we cannot find such a utility maximizing group $S(p)$ at price p that includes all intermediaries with positive price in $\bigcup_{j=1}^J S_j(p)$. Assume the planner allocates resource with intermediaries in $S_1(p)$, and intermediary $n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$ posts a price $p_n > 0$. Without loss of generality, assume that $n \in S_2(p)$. Since intermediary n is not used by planner, it receives 0 utility. Consider the price vector $p' = (p_n - \epsilon, p_{-n})$ for some $\epsilon > 0$ such that $p_n > \epsilon$. Since the problem (u, F) is monotonic, by Lemma 3, if $v(p) > u((0, \dots, 0), 0)$, $v(p') \geq u^*(S_2(p), p') > u^*(S_2(p), p) = v(p)$. Since $v(p) = v_{-n}(p_{-n}) = v_{-n}(p'_{-n})$, then intermediary n will be in the utility maximizing group at price p and selected by the planner at prices p' . Thus, the payoff of intermediary n increases from 0 to p'_n , which is a contradiction.

If $v(p) = u((0, \dots, 0), 0)$, then for any group $S \subseteq \mathcal{N}$, $u^*(S, p) \leq u((0, \dots, 0), 0)$, $S_1(p) = \mathcal{N}$ satisfies the condition.

c. For intermediary n not to be in any utility maximizing group $S_j(p)$ at price p , it will not be used by the planner. If it lowers its price and improves the maximal utility achieved by the planner, it has incentive to lower its price and get paid. To make sure this case will not happen in a SPNE, it requires that even when the price decreases to 0, the maximal utility of the planner would not increase. Thus, if $p'_n = 0$, $p' = (p'_n, p_{-n})$, then $v(p) = v(p')$.

In order to prove the converse, consider p satisfying conditions (a), (b) and (c) and assume that the planner chooses the group $S_1(p)$ in condition (a), paying all intermediaries in $\bigcup_{j=1}^J S_j(p)$ with positive price. Since $S_1(p)$ is a utility maximizing group, the planner will achieve the maximal utility under p . For intermediary n , $n \in \mathcal{N} \setminus \bigcup_{j=1}^J S_j(p)$, it has no incentive to increase price based on the monotonicity of problem, the utility from groups of intermediaries including n will not be maximal for planner, n will not be paid. At the same time, condition (c) shows n has no incentive to decrease price. If $n \in \bigcup_{j=1}^J S_j(p)$, intermediary n has no incentive to increase its price by condition (a), because the planner will use a different utility maximizing group of intermediaries if its price is higher. This intermediary will not decrease its price by condition (b), since by lowering the price the intermediary receives less profit. Thus, any price p satisfying conditions (a),(b),(c) is a SPNE. ■

Proof of Lemma 1

Proof. Consider the intermediation problem (u, F) and let $(b(p), x(p))$ be a strategy of the planner that maximizes u given p . Let $S_1(p), \dots, S_J(p)$ be the utility maximizing groups at prices p . Without loss of generality, we assume that the strategy of planner $(b(p), x(p))$ satisfies the following two tie-breaking rules: (1) If \emptyset is a utility maximizing group at prices p , let $\underline{p}_n = \min\{q_n \in [0, P_n] | v(q_n, p_{-n}) = u((0, \dots, 0), 0)\}$ be the minimal price of agent n at which the empty coalition is efficient (by continuity of the indirect utility function such price exists). We require that if $p_n = \underline{p}_n$ then $n \in b(p)$, whereas if $p_n > \underline{p}_n$ then $n \notin b(p)$. (2) if \emptyset is not a utility maximizing group at prices p and there exists a utility maximizing group $S_1(p)$ such that $p_n = 0, \forall n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$ then $b(p) = S_1(p)$ (if there are multiple groups satisfying this condition then the planner chooses one among them). These tie-breaking rules guarantees that inaction is taken when it is optimal for the planner to do so. Furthermore, when there exists one or more utility maximizing groups containing all intermediaries who post positive prices, then the planner chooses one of such groups.

Let \mathcal{G} be the simultaneous move game of the intermediaries, where the strategy of intermediary n is p_n and the payoff to intermediary n is $V^n(p) = V^n(p, b(p), x(p)), \forall n \in \mathcal{N}$, given $(b(p), x(p))$. Clearly, a Nash equilibrium price vector p^* in game \mathcal{G} is a SPNE of the resource transmission game generated by (u, F) . By price-satiation, the strategy space $[0, P_n]$ of intermediary n satisfies $P_n < +\infty$.

In order to prove that the game \mathcal{G} has a Nash equilibrium, we verify the conditions of Theorem 3.1 in Reny[34]. Indeed, the strategy is non-empty, compact, convex subset are assumed. We show that the utility function $V^n(p)$ is quasi-concave in p_n , and that the game \mathcal{G} is better reply secure.

Step 1. The payoff function $V^n(p)$ of intermediary n is quasi-concave in p_n .

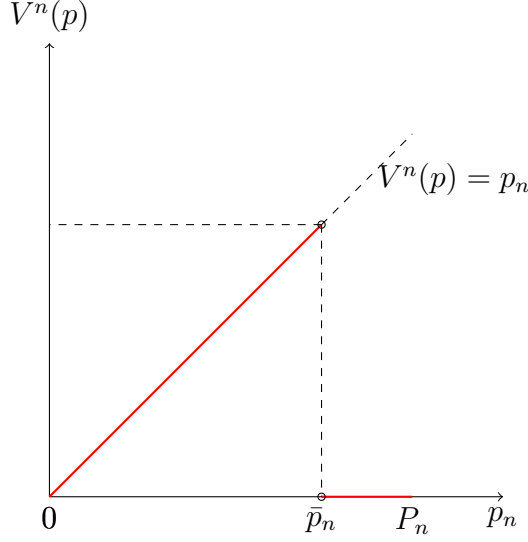


Figure 2: The payoff function $V^n(p)$ is quasi-concave in prices.

First, we show that the maximal utility of the group of intermediaries S at prices p (denoted by $u^*(S, p)$): (i) is decreasing in p_n for $n \in S$, and (ii) is independent of p_n for $n \notin S$.

Consider prices p and p' , with $p_{-n} = p'_{-n}$, $p_n < p'_n$ and $n \notin S$. In order to show (i), for $n \in S$, since the problem (u, F) is monotonic, from Lemma 3, if $u^*(S, p') > u^*((0, \dots, 0), 0)$, $u^*(S, p') < u^*(S, p)$. In order to show (ii), for $n \notin S$, $F(S, p) = F(S, p')$ from assumption (d) of OPF in definition 1. Furthermore, $u(x, p_S) = u(x, p'_S)$ since $p_S = p'_S$, $\forall x \in F(S, p)$. Hence, $u^*(S, p) = u^*(S, p')$.

Assume the upper bound of the intermediary n 's price is large enough that $u^*(S, p) < 0$ for $p = (P_n, \mathbf{0}_{-n})$ and $\forall S$ with $n \in S$, which means price p_n will not be paid if the price is too high.

Given the strategy of planner $(b(p), x(p))$, a utility maximizing group would be chosen. For any group of intermediaries S and T , $n \in S$ and $n \notin T$. As price p_n increases, $u^*(S, p)$ decreases and becomes negative if $p_n = P_n$, while $u^*(T, p)$ does not change as p_n changes. Given p_{-n} , define \bar{p}_n , s.t. $\max_S u^*(S, (\bar{p}_n, p_{-n})) = \max_T u^*(T, p)$, otherwise $\bar{p}_n = 0$, if $\max_S u^*(S, (0, p_{-n})) \leq \max_T u^*(T, p)$. When $p_n < \bar{p}_n$, $u^*(S, p) > u^*(T, p)$, for some S and $\forall T$. When $p_n > \bar{p}_n$, $u^*(S, p) \leq u^*(T, p)$ for some T and $\forall S$. Thus, the payoff function $V^n(p) = p_n$ when $p_n < \bar{p}_n$, and $V^n(p) = 0$ when $p_n > \bar{p}_n$. The function $V^n(p)$ is shown in as shown in Figure 2. If $p_n = \bar{p}_n$, then the planner is indifferent between choosing or not intermediary n .

In order to show the quasi-concavity of $V^n(p)$, consider constant $c \geq 0$. If $n \in b(\bar{p}_n, p_{-n})$, $V^n(\bar{p}_n, p_{-n}) = \bar{p}_n$. The upper contour set $\{p_n | V^n(p) \geq c\}$ equals $[c, \bar{p}_n]$ when $c \leq \bar{p}_n$, and $\{p_n | V^n(p) \geq c\} = \emptyset$ when $c > \bar{p}_n$. If $n \notin b(\bar{p}_n, p_{-n})$, $V^n(\bar{p}_n, p_{-n}) = 0$. The upper contour set $\{p_n | V^n(p) \geq c\}$ equals $[c, \bar{p}_n]$ when $c \leq \bar{p}_n$, and $\{p_n | V^n(p) \geq c\} = \emptyset$ when $c > \bar{p}_n$. Thus, the upper contour set $\{p_n | V^n(p) \geq c\}$ is convex for any constant c .

Step 2. The game \mathcal{G} for intermediaries is better reply secure. That is, for any prices p , which

is not a Nash equilibrium, there exists intermediary n who can secure a payoff strictly above $V^n(p)$ at p .

Assume the utility maximizing group of intermediaries at prices p are $S_1(p), \dots, S_J(p)$. From Lemma 2, the maximal utility $u^*(S, p)$ is continuous in p . The problem (u, F) is monotonic, from Lemma 4, the price vector p satisfies condition (a) to (c) if it is a Nash equilibrium. So, if prices p is not Nash Equilibrium, at least one condition is not satisfied.

If condition (a) is not satisfied, then there exists intermediary $n \in \bigcap S_j(p)$. This implies that $u^*(S_j(p), p) > u^*(T, p), \forall T \neq S_j(p), \forall j$. Since $u^*(S, p)$ is continuous in p , for p'_{-n} in a small neighborhood of p_{-n} , $u^*(S_j(p), (p_n, p'_{-n})) > u^*(T, (p_n, p'_{-n})), \forall T$. Thus, there exists $p'_n > p_n$, s.t. $u^*(S_j(p), (p'_n, p'_{-n})) > u^*(T, (p'_n, p'_{-n}))$ for some $S_j(p)$. Thus, some group of intermediaries $S_j(p)$ with $n \in S_j(p)$ is utility maximizing group for prices $p' = (p'_n, p'_{-n})$, intermediary n will secure payoff $V^n(p') = p'_n > p_n = V^n((p_n, p'_{-n}))$.

If condition (b) is not satisfied, then there exists intermediary $n \in \bigcup S_j(p)$ who is not used by the planner, and $p_n > 0$. Since problem (u, F) is monotonic in prices, there is $0 < p'_n < p_n$, s.t. for some group of intermediaries $S_j(p)$ where $n \in S_j(p)$, $u^*(S_j(p), (p'_n, p_{-n})) > u^*(T, (p'_n, p_{-n})), \forall T, n \notin T$, because the maximal utility of group T without n does not change as price p_n changes to p'_n , while $u^*(S_j(p), p)$ increases and $S_j(p)$ is utility maximizing group at prices p . Assume the group of intermediaries S , where $n \in S$, achieves maximal utility at prices (p'_n, p_{-n}) , $v((p'_n, p_{-n})) = u^*(S, (p'_n, p_{-n}))$, then for p'_{-n} close enough to p_{-n} , we have $u^*(S, (p'_n, p'_{-n})) > u^*(T, (p'_n, p'_{-n})), \forall T, n \notin T$. Thus, for neighborhood of p_{-n} , intermediary n chooses price p'_n would secure payoff $V^n(p') = p'_n > 0 = V^n((p_n, p'_{-n}))$.

If condition (c) is not satisfied, then there exists intermediary n who could decrease the price from p_n to $p'_n > 0$ to be used by the planner, so there is some utility maximizing group S at prices (p'_n, p_{-n}) with $n \in S$, and $u^*(S, (p'_n, p_{-n})) > u^*(T, (p'_n, p_{-n})), \forall T, n \notin T$. Thus, for prices p'_{-n} in the neighborhood of p_{-n} , $u^*(S, p') > u^*(T, p')$. Thus, intermediary n chooses prices p'_n would secure payoff $V^n(p', b, x) = p'_n > 0 = V^n((p_n, p'_{-n}))$.

Thus, for all prices p not an Nash equilibrium, some intermediary n can secure a strictly higher payoff. The game \mathcal{G} is better reply secure.

Hence, game \mathcal{G} has at least one pure strategy Nash equilibrium, and there exists a SPNE in the resource transmission game generated by problem (u, F) . ■

Proof of Theorem 1

Proof. \Leftarrow Suppose that $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset, \forall n$. Consider intermediary n who posts price $p_n > 0$. Let $p = (p_n, \mathbf{0}_{-n})$. Therefore, for any group S^n such that $n \in S^n$, $u^*(S^n, p) \leq u^*(S^n, \mathbf{0}) \leq u^*(S_j(\mathbf{0}), \mathbf{0})$ for any j . Since $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$ then there exists group $S_j(\mathbf{0})$ such that $n \notin S_j(\mathbf{0})$. Hence intermediary n will not be chosen by the planner at p . Note that in the case where $u^*(S^n, p) = u^*(S_j(\mathbf{0}), \mathbf{0})$, the planner chooses a group of intermediaries who post zero prices even though it is indifferent with some group of intermediaries with positive price.

\Rightarrow Let $S_1(p), \dots, S_{J'}(p)$ be the set of utility maximizing groups at prices p . First, we show that if $\bigcap_{j=1}^{J'} S_j(p) \neq \emptyset$, then p is not a SPNE. Indeed, pick $n \in \bigcap_{j=1}^{J'} S_j(p)$. Without using intermediary n , given the prices of other intermediaries p_{-n} , the indirect utility function without intermediary n (Definition 5) is $v_{-n}(p_{-n})$. Since n is used in all utility maximizing groups $S_j(p)$, then $v(p) > v_{-n}(p_{-n})$. Consider the decreasing sequence of prices $\{p^i\}$ such

that $p_n^i = p_n + \frac{1}{i}$ and $p_{n'}^i = p_{n'}$ for any $n' \neq n$. Since $p^i \geq p$ and $\lim_{i \rightarrow \infty} p^i = p$, by Lemma 2, $\lim_{i \rightarrow \infty} v(p^i) = v(p)$. Moreover, since $v(p) > v_{-n}(p_{-n})$, then there exists i large enough such that $v(p) \geq v(p^i) > v_{-n}(p_{-n})$. Thus, at the price vector p , intermediary n can increase his price by $\frac{1}{i}$ and get higher profit.

Second, let p be an IFE. We show that $S = S_j(p)$ is also a utility maximizing group at prices $\mathbf{0}$. Since p is IFE, there exists a planner-optimal allocation x such that $x \in F(S, p)$. Furthermore, $u(x, 0) = u^*(S_k(\mathbf{0}), \mathbf{0})$ for any k . Since $x \in F(S, p)$ and $F(S, p) \subseteq F(S, \mathbf{0})$ by the monotonicity of F , then $x \in F(S, \mathbf{0})$. Thus, x is feasible and utility maximizing allocation in $F(S, \mathbf{0})$. Therefore, $u^*(S, \mathbf{0}) = u(x, 0) = u^*(S_k(\mathbf{0}), \mathbf{0})$. Hence, S is a utility maximizing group at prices $\mathbf{0}$.

Finally, if p is an intermediation free equilibrium, then it is a SPNE. Thus, by the first step above, $\cap_{i=1}^J S_i(p) = \emptyset$. By the second step, $\cap_{i=1}^J S_i(\mathbf{0}) \subseteq \cap_{i=1}^J S_i(p)$. Hence, $\cap_{i=1}^J S_i(\mathbf{0}) = \emptyset$ ■

Proof of Remark 2

Proof. a. $F(S, p) = F(S)$, $u^*(S, p) = \max_{x \in F(S)} u(x, p_S) = \max_{x \in F(S)} \alpha(x)\beta(p_S)$. $u^*(S, \mathbf{0}) \leq u^*(T, \mathbf{0})$, which is $\max_{x \in F(S)} \alpha(x)\beta(0) \leq \max_{x \in F(T)} \alpha(x)\beta(0)$, equivalent with $\max_{x \in F(S)} \alpha(x) \leq \max_{x \in F(T)} \alpha(x)$. Then, $\max_{x \in F(S)} \alpha(x)\beta(p_S) \leq \max_{x \in F(T)} \alpha(x)\beta(p_T)$ for $p_S = p_T$. Therefore, $u^*(S, p) \leq u^*(T, p)$ whenever $p_S = p_T$.

b. $u(x, p) = u(x)$, $u^*(S, p) = \max_{x \in F(S, p)} u(x, p_S) = \max_{x \in \gamma(S)\delta(p_S)} u(x)$. $u^*(S, \mathbf{0}) \leq u^*(T, \mathbf{0})$, means $\max_{x \in \gamma(S)\delta(0)} u(x) \leq \max_{x \in \gamma(T)\delta(0)} u(x)$. Assume x^1 and x^2 solves $\max_{x \in \gamma(S)\delta(0)} u(x)$ and $\max_{x \in \gamma(T)\delta(0)} u(x)$ respectively, there is $u(x^1) \leq u(x^2)$. Since the preferences are homothetic, $u(\delta(t)x^1) \leq u(\delta(t)x^2)$, and $\delta(t)x^1, \delta(t)x^2$ solves the problem $\max_{x \in \gamma(S)\delta(t)} u(x)$ and $\max_{x \in \gamma(T)\delta(t)} u(x)$, thus $u^*(S, p) \leq u^*(T, p)$, with $t = p_S = p_T$. ■

Proof of Theorem 2

Proof.

The converse has already been shown in the text. To prove the forward part, we will use Lemmas 3 and 4.

Similar with Lemma 1, assume that the strategy of planner $(b(p), x(p))$ satisfies the following two tie-breaking rules: (1) If \emptyset is a utility maximizing group at prices p , let $\underline{p}_n(p_{-n}) = \min\{q_n \in [0, P_n] | v(q_n, p_{-n}) = u((0, \dots, 0), 0)\}$ be the minimal price of agent n at which the empty coalition is efficient (by continuity of the indirect utility function such price exists). We require that if $p_n = \underline{p}_n(p_{-n})$ then $n \in b(p)$, whereas if $p_n > \underline{p}_n(p_{-n})$ then $n \notin b(p)$. (2) if \emptyset is not a utility maximizing group at prices p and there exists a utility maximizing group $S_1(p)$ such that $p_n = 0, \forall n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$, then $b(p) = S_1(p)$ (if there are multiple groups satisfying this condition then the planner chooses one among them).

Recall that $u^*(S, p)$ is the maximal utility at prices p and group S . Let $\bar{u} = u^*(S_1(\mathbf{0}), \mathbf{0}) = \dots = u^*(S_J(\mathbf{0}), \mathbf{0})$ be the maximal utility achieved by the planner when the prices are zero. Assume there is a robust SPNE with $p \neq \mathbf{0}$. Then, $\forall p_n > 0$, intermediary n is used in the utility maximizing group of intermediaries at price p . From part (b) of Lemma 4, there exists a utility maximizing group S_1 at price p , such that all the intermediaries with positive prices are in the group S_1 , that is, $\forall p_n > 0$, intermediary $n \in S_1$. The maximal utility of the planner for the use of group S_1 is $u^*(S_1, p)$. Since $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, for any intermediary k with $p_k > 0$, there exists $S_j(\mathbf{0})$ such that $k \notin S_j(\mathbf{0})$. Let $S_2 = S_j(\mathbf{0})$. $p_{S_1} = \sum_{p_n > 0} p_n > \sum_{n \in S_2} p_n = p_{S_2}$.

S_2 is utility maximizing group at prices $\mathbf{0}$. $\bar{u} = u^*(S_2, \mathbf{0}) \geq u^*(S_1, \mathbf{0})$.

Since intermediary $k \in S_1$ and $k \notin S_2$, there are two cases: $S_2 \subset S_1$ or $S_2 \setminus S_1 \neq \emptyset$.

If $S_2 \subset S_1$, let the prices p^1 satisfy $p^1_{[S_2]} = p_{[S_2]}$ and $p^1_{[\mathcal{N} \setminus S_2]} = 0$, thus $u^*(S_2, p^1) = u^*(S_2, p)$. Then, $p_{S_2} = p^1_{S_2} = p^1_{S_1}$. From cross-monotonicity, $u^*(S_2, p^1) \geq u^*(S_1, p^1)$. At the same time, $p^1_{[S_1]} \leq p_{[S_1]}$ and $p^1_{[S_1]} \neq p_{[S_1]}$. The problem (u, F) is monotonic, by Lemma 3, if $u^*(S_1, p) > u((0, \dots, 0), 0)$, $u^*(S_1, p^1) > u^*(S_1, p)$ holds. Thus, $u^*(S_2, p) > u^*(S_1, p)$, which contradicts that S_1 is a utility maximizing group at price p . If $v(p) = u((0, \dots, 0), 0)$, then $\forall n, p_n > 0, p_n \geq \underline{p}_n(p_{-n})$, because $\underline{p}_n(p_{-n}) = \min\{q_n \in [0, P_n] | v(q_n, p_{-n}) = u((0, \dots, 0), 0)\}$, and $n \in b(p)$. Given the tie-breaking rules above, if $p_n > \underline{p}_n(p_{-n})$, intermediary n will not be paid, so $p_n = \underline{p}_n(p_{-n})$. Problem (u, F) is monotonic, $\forall 0 < p'_n < p_n$, there is $v(p'_n, p_{-n}) > u((0, \dots, 0), 0)$. Thus, utility strictly increases as p_n decreases, $u^*(S_1, p^1) > u^*(S_1, p)$ holds and $u^*(S_2, p) > u^*(S_1, p)$.

If $S_2 \setminus S_1 \neq \emptyset$, assume intermediary $i \in S_2$ and $i \notin S_1$, let prices p^2 satisfy $p^2_{[S_1]} = p_{[S_1]}$ and $p^2_i = p_{S_1} - p_{S_2}$ and $p^2_{[\mathcal{N} \setminus (S_1 \cup \{i\})]} = 0$. Then there is $p^2_{S_1} = p^2_{S_2}$. By cross-monotonicity, $u^*(S_2, p^2) \geq u^*(S_1, p^2) = u^*(S_1, p)$. At the same time, $p^2_{[S_2]} \geq p_{[S_2]}$ and $p^2_{[S_2]} \neq p_{[S_2]}$, thus $u^*(S_2, p) > u^*(S_2, p^2)$ by monotonicity of the problem (u, F) . So $u^*(S_2, p) > u^*(S_1, p)$ which is a contradiction with S_1 being a utility maximizing group at price p . Hence, $p = \mathbf{0}$ is the unique robust SPNE. ■

Proof of Remark 3

Proof. Cross-monotonicity is necessary for Theorem 2 to hold.

Consider the planner's utility function $u(x, t) = \tilde{u}(x)$ that is independent of the prices paid to the intermediaries. Also, consider the OPF F that is strongly monotonic in prices such that for intermediaries 1 and 2, $F(\{1\}, \mathbf{0}) = F(\{2\}, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Moreover, for some price vector $p = (p_1, p_2, 0, \dots, 0)$ where $p_1, p_2 > 0$, we have that $F(S, p) \subseteq F(\{1, 2\}, p) = F(\{1\}, p) = F(\{2\}, p)$ for any $S \subseteq \mathcal{N}$. First note that problem (u, F) is monotonic because F is strongly monotonic in prices. However, (u, F) is not cross-monotonic. To see this, assume that (u, F) is cross-monotonic. Then, $u^*(\{1\}, \mathbf{0}) = u^*(\{1, 2\}, \mathbf{0})$ implies that $u^*(\{1\}, p) = u^*(\{1, 2\}, (p_1, \mathbf{0}_{-1}))$. Furthermore, $u^*(\{1, 2\}, (p_1, \mathbf{0}_{-1})) > u^*(\{1, 2\}, p)$ by strong monotonicity in prices of F . Hence, $u^*(\{1\}, p) > u^*(\{1, 2\}, p)$, which contradicts $F(\{1, 2\}, p) = F(\{1\}, p)$. Finally, note that $\mathbf{0}$ and p are prices that are robust SPNE in the problem (u, F) , since group $\{1, 2\}$ is a utility maximizing group at both prices, hence when such group is chosen by the planner no intermediary has the incentive to deviate by strong monotonicity in prices of F .

Monotonicity in prices is necessary for Theorem 2 to hold.

Consider the planner's utility function $u(x, t) = \tilde{u}(x)$ that is independent of the prices paid to the intermediaries. Also, consider an OPF F that satisfies the following conditions: (1) $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$, $\forall n$; (2) $F(\mathcal{N}, \mathbf{0}) = F(S, p)$ for some prices $p = (p_1, \mathbf{0}_{-1})$ such that $p_1 > 0$ and a group S such that $1 \in S$; (3) $\forall n \neq 1$, for prices p^n with $p^n = (p_1, 0, \dots, p_n, 0, \dots, 0)$, $p_n > 0$, $x^*(\mathcal{N}, \mathbf{0}) \cap F(T, p^n) = \emptyset \forall T$ with $n \in T$, and $F(\mathcal{N} \setminus \{n\}, p^n) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$; finally, (4) for any price vector $p' = (p'_1, \mathbf{0}_{-1})$, $p'_1 > p_1$, $x^*(\mathcal{N}, \mathbf{0}) \cap F(T, p') = \emptyset, \forall T$ with $1 \in T$. The problem (u, F) meeting these conditions is not monotonic in prices since

u does not depend on t and F is not strongly monotonic in prices by condition (2).

We now see that the problem (u, F) has multiple robust SPNE. Indeed, first notice that since condition (1) is satisfied, $\mathbf{0}$ is an IFE by Theorem 1, and thus it is a robust SPNE. We now show that $p = (p_1, \mathbf{0}_{-1})$ is also a robust SPNE. Indeed, by condition (2), S is a utility maximizing group at prices $\mathbf{0}$. Assume that the planner chooses S and pays p_1 to intermediary 1, and the maximal utility that the planner could achieve at price $\mathbf{0}$ is \bar{u} . Then, intermediary 1 has no incentive to decrease its price. On the other hand, if intermediary 1 increases its price to p'_1 , from condition (4), the planner cannot get any utility maximizing allocation when choosing a group that contains intermediary 1. However, at price p' the planner can get utility \bar{u} by choosing group $\mathcal{N} \setminus \{1\}$, since by condition (1), $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{1\}, \mathbf{0}) = F(\mathcal{N} \setminus \{1\}, p')$. Thus, the planner will not choose intermediary 1 if his price increases to p'_1 . Alternatively, consider the case where intermediary $n \neq 1$ deviates to a price p_n . By condition (3), the planner will have utility less than \bar{u} using any group with intermediary n and get \bar{u} with $\mathcal{N} \setminus \{n\}$. Thus, if intermediary n charges a positive price he will not be used by planner. So no intermediary has incentive to deviate from $p = (p_1, 0, \dots, 0)$, and hence p is a robust SPNE. ■

Proof of Corollary 1

Proof. a. We show that $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$ is satisfied. For any intermediary n , let $S_n = \mathcal{N} \setminus \{n\}$. From the condition of minimally competitive outcome, $F(\mathcal{N}, \mathbf{0}) = F(S_n, \mathbf{0})$, for any n . $\max_{x \in F(S_n, \mathbf{0})} u(x, 0) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0)$, thus $S_n = \mathcal{N} \setminus \{n\}$ is a utility maximizing group. Then $\bigcap_n S_n = \emptyset$, so the intersection of utility maximizing group at prices $\mathbf{0}$ is $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$. From Theorems 1 and 2, there exists IFE and unique robust SPNE.

b. From the definition of duplicate OPF, $F(\mathcal{N}, \mathbf{0}) = F(\{1, \dots, k\}, \mathbf{0})$. Hence, for any intermediary i , $F(\mathcal{N} \setminus \{i\}, \mathbf{0}) = F(\{1, \dots, k\}, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Similar with part (a), from Theorem 1 and 2, there exists IFE and unique robust SPNE.

c. There exists a group of intermediaries S s.t. $F(S, \mathbf{0}) = F(\mathcal{N} \setminus S, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Let $S_1 = S$ and $S_2 = \mathcal{N} \setminus S$. There is $\max_{x \in F(S_1, \mathbf{0})} u(x, 0) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0) = \max_{x \in F(S_2, \mathbf{0})} u(x, 0)$. So S_1 and S_2 are utility maximizing group at prices $\mathbf{0}$. $S_1 \cap S_2 = \emptyset$, from Theorem 1 and 2, there exists IFE and unique robust SPNE. ■

B Appendix: Applications

The generality of our study provides a unified framework for the study of different literatures that seem disconnected, ranging from resource allocation problems in networks to minimal cost spanning tree models. Herein, we briefly introduce these applications while the following sections discuss each of them in detail.

Resource Transmission in Networks under Proportional Constraints (Sec. B.1):

Consider a planner interested in transmitting a divisible resource to agents (such as money). The planner has preferences over the different allocations of the resource to the agents. The planner can reach the agents via a group of intermediaries that may differ in the *types* of agents they can reach as well as the *quality* in which they can reach the agents. The types of agents that intermediaries reach are represented by a network. The quality in

which intermediaries reach the agents can be interpreted as the effective transmission of the resource from the intermediaries to the agents. This is represented by the total amount of the resource that an intermediary sends to the agents per unit of resource received, as well as by the proportions in which every agent receives a resource relative to another from a given intermediary.

This model can be applied to the transmission of advertising money in companies. A company looking to promote their product can use different media (the intermediaries) to reach the advertising target of their product; such intermediaries include TV channels, radio stations, Internet websites, newspapers, etc. The quality of the connections is important because, within the media, there are different channels that target to specific demographics of agents and may influence the planner’s objective differently. Alternatively, this model can incorporate the allocation of government’s money to people in need via charities. The government may decide to send the money via charities that will charge an indirect cost for the use of their services. The connections of the charities as well as their quality are exogenous information that the planner cannot control, and they are typically taken into account when making a decision on how to allocate the resources.

Resource Transmission in Networks under Unit-Capacities (Sec. B.2): Consider a planner interested in distributing a fix amount of a divisible resource to agents via a set of links owned by intermediaries. Multiple layers of intermediation are possible, and thus the planner might need to contract more than one intermediary to reach an agent. We assume that links have unit capacities¹⁰, which decrease the amount transmitted to the agents by the product of the capacity of the links used.

A particular case of this problem occurs when intermediaries are directly connected to agents and have ‘waste-constraints’ where intermediaries are directly connected to a subset of agents but only transmit a portion of the amount sent through them. Such is the case of universities or charities, where an overhead cost is charged for every dollar sent to them, and the planner can choose where every charity spends the resources —unlike in the case of proportional constraints, where the charities have exogenous priorities. The problem can also be applied to more complex layers of intermediation arising in network flow problems. For instance, when there is ground water that must be distributed to agents via private canals (intermediaries) that have an evaporation loss or other conveyance losses¹¹ that are proportional to the amount of water transmitted and might be different across canals. The owners of the canals may charge the planner for the use of their canals, and therefore the planner should consider the trade-offs between allocating resources to cheap canals with high conveyance losses as opposed to more efficient but relatively more expensive canals.

Resource Transmission in Networks under Total-Capacities (Sec. B.3): Consider a planner interested in distributing a fix amount of good to agents via a network of intermediaries. Intermediaries, who own the links, are constrained by the total capacity of every link. As above, multiple layers of intermediation are possible.

¹⁰Bloch and Dutta[4] study a model with unit capacities, where there is information transmission with different strengths of links.

¹¹Conveyance losses are typical in these models, and typically depend as a proportion of the length of the canal and structure. Jandoc, Juarez and Roumasset[21] study the optimal allocation of water networks in the presence of these losses.

This model can be applied to the distribution of resources when natural disasters occur. For instance, an organization interested in transmitting the resources to regions in need may be faced with transportation capacities (such as cargo in ships and planes). Multiple layers of intermediation might be required as goods sent to remote regions might require more than one mode of transportation. This model also has applications to the transmission of data in the internet. Data transmitted in networks often goes through intermediaries which charge for the use of their links. These links are often capacity constrained and might require the user of the link to pay in order for the goods to flow in the network.¹²

Minimal Cost Trees and Related Models (Sec. B.4): The generality of our model also encompasses problems of network building that might not be explicitly used for the transmission of a divisible good. Such is the case for a planner seeking to build a minimum cost spanning tree that connect agents (nodes) using links in a network owned by intermediaries. When intermediaries post prices for the use of their links, the planner can choose any set that connects the agents at the minimal cost. Applications of this model include the construction of electricity and water networks.

B.1 Resource Transmission in Networks under Proportional Constraints

Consider the case where there are fixed links between the intermediaries $\mathcal{N} = \{1, \dots, N\}$ and the agents $\mathcal{M} = \{1, \dots, M\}$. Every intermediary is connected to a group of agents and can transmit resources to the agents that it is connected with some fixed quality (ability), this is denoted by the *sharing-rate*. Let q_{nm} be the sharing-rate of intermediary n connected to agent m , where $q_{nm} \geq 0$ for each intermediary n . The matrix of sharing-rates is $Q = (q_{11}, \dots, q_{NM})_{N \times M}$, and $Q_n = (q_{n1}, \dots, q_{nM})$ is the ability of intermediary n to transmit the resource to the agents. We assume that if there is no link between intermediary n and agent m , then $q_{nm} = 0$. The sharing-rate distinguishes the way in which intermediaries transmit resources to agents per unit of money given.¹³ Two intermediaries connected to the same group of agents might have different impacts on the agents, and thus one might be better aligned than the other to the planner's preferences.

The planner has a utility function $u(x, p) = u(x)$ that is independent of the price paid to the intermediaries. That is, the planner cares only about the final resource transmitted to the agents in \mathcal{M} . Assume that the total resource available for the planner to transmit is I .

¹²This can be seen in the recent dispute between Time Warner Cable Company (TWC) vs Netflix and other streaming devices, where TWC was interested in controlling the quality of streaming movies due to capacity constraints on its network. A recent agreement on the payment by Netflix to TWC has been reached.

¹³One application of this model includes in the allocation of resources to charities who have a pre-determined set of priorities among agents. When $\sum_{m=1}^M q_{nm} < 1$, we can interpret the intermediary (charity) as being inefficient. Such inefficiencies happen often in charities (and universities) where every dollar spent is often decreased due to indirect cost which serves to pay for the administration. The case of $\sum_{m=1}^M q_{nm} > 1$ implies that a dollar transmitted using that intermediary increases, for instance when charities or universities offer matching funds from donors. Previous results in the transmission of resource in networks do not distinguish in the quality of the links or assume that the sharing-rate is equal across intermediaries.

Given the matrix of sharing rates Q , the outcome possibility function is

$$F(S, p) = \left\{ \sum_{n \in S} Q_n y_n \mid \sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n \text{ and } y_n \geq 0 \right\} \text{ if } \sum_{n \in S} p_n \leq I$$

$$F(S, p) = \{(0, \dots, 0)\} \text{ if } \sum_{n \in S} p_n > I$$

That is, the possibility set of a group S when posted prices are p is the transmission of not more than $I - \sum_{n \in S} p_n$ units of the resource using the abilities given by Q of the intermediaries in S .

Example 1 (Perfect Substitute Utility Function)

Consider a planner with utility function $u(x) = \sum_{m=1}^M \alpha_m x_m$, where α_m is the weight of the final resource allocated to agent m . Given the sharing-rates $\{q_{nm}\}_{\{n \in \mathcal{N}, m \in \mathcal{M}\}}$, the marginal utility of resource allocated to intermediary n is constant and given by $MU_n = \sum_{m=1}^M \alpha_m q_{nm}$. Without loss of generality we rename the intermediaries based on a non-increasing order of their marginal utility, that is $MU_1 \geq MU_2 \geq \dots \geq MU_N$.

When $MU_1 = \dots = MU_k > MU_{k+1}$ and $k \geq 2$, the planner is indifferent between allocating the resources to any of the intermediaries from $\{1, \dots, k\}$ when their prices are zero. If only one intermediary from $\{1, \dots, k\}$ has a price zero, then he can raise the price to slightly below the second lowest price posted by a different intermediary. Alternatively, if no intermediary from $\{1, \dots, k\}$ has zero price, then at most one of them will be chosen, and the ones who are not chosen have the incentive to decrease their price. Therefore, a SPNE requires that at least two intermediaries from $\{1, \dots, k\}$ have price zero. It is easy to verify that every price allocation such that $p_i = p_{i'} = 0$, for some $i, i' \in \{1, \dots, k\}$ and $p_n \geq 0$, $\forall n \neq i, i'$ is a SPNE. Thus, in this example there are multiple IFEs.

When $MU_1 > MU_2$, the intermediary 1 has some market power to price above zero and continue being chosen. In a SPNE, $p_2 = 0$ and $p_1 = I(1 - \frac{MU_2}{MU_1})$, $p_n \geq 0$, $\forall n \geq 3$ and intermediary 1 is chosen to transmit $I - p_1$ units of resource. The planner's utility would be $I \cdot MU_2$, which is welfare equivalent to the utility given by allocating all resources to the intermediary with the second highest marginal utility when he prices at 0. In particular, there is no IFE.

An alternative way to prove the existence of a robust SPNE is by computing the utility maximizing groups at $\mathbf{0}$ and applying Theorem 2 (since monotonicity and homotheticity of the preferences are clearly satisfied). Indeed, if $1, \dots, k$ are the intermediaries with marginal utility $MU_n = MU_1$, $\forall 1 \leq n \leq k$, then each of $\{1\}, \dots, \{k\}$ is a utility maximizing group at $\mathbf{0}$. Therefore, if $k > 1$ then $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, hence a unique robust SPNE exists. However, if $k = 1$, then $\bigcap_{j=1}^J S_j(\mathbf{0}) = \{1\}$, thus no IFE exists.

Example 2 (Symmetric Network)

Assume that the planner with utility function $u(x) = \min\{x_1, x_2, x_3\}$ cares about the agent who is allocated the least resources. The network in Figure 3 represents the connections

from intermediaries to agents given by the matrix of sharing-rates $Q = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

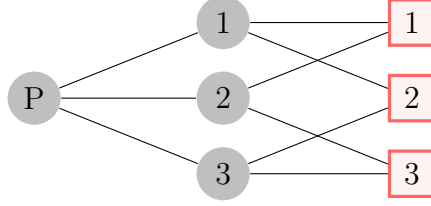


Figure 3: Network with three symmetric intermediaries

Every intermediary is connected to two agents and would always send the resource equally to the agents connected. Note that if the planner only uses two intermediaries, the optimal allocation is to transmit half the resources through each intermediary. Thus, the agent connected to both intermediaries would get half of the resource and each of the other two agents would get one quarter of the resource transmitted. In this case, the resource cannot be allocated equally to three agents and results in a waste of resources and an inefficiency for the planner. Thus, the planner-optimal allocation can only be achieved by using the three intermediaries in conjunction. Hence, every intermediary has some market power to post a positive price in equilibrium.

There is a symmetric equilibrium where every intermediary posts price $\frac{I}{6}$, the planner would use all the intermediaries $b(p) = \{1, 2, 3\}$, and the allocation of resource to agents is $x(p) = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$.

There is another equilibrium price allocation which results when every intermediary posts price equal to total resource I , that is $p = (I, I, I)$, and the planner pays one of the intermediaries (say, intermediary 1, $b(p) = \{1\}$) all the resource without transmitting anything, which means $x = (0, 0, 0)$. In this equilibrium, there is no incentive for intermediary 1 to deviate since it gets all the resource. For intermediary 2 or 3, even if one decreases his price, the planner cannot get positive utility because one intermediary is not connected to all the agents and at least one agent would receive 0 resource. Thus, paying all resource to intermediary 1 is still a best strategy for planner. The SPNE with planner's utility equal to 0 exists because intermediaries 2 and 3 cannot cooperate by lowering their prices simultaneously.

There is an easier way to verify that no IFE exists in this case. Indeed, note that the only utility maximizing group at the vector of prices $\mathbf{0}$ is $\{1, 2, 3\}$. Hence, the necessary conditions to guarantee an IFE in Theorem 1 do not hold.

Let $\text{conv}(Q) = \{\sum_{n=1}^N \lambda_n Q_n \mid \sum_{n=1}^N \lambda_n = I, \lambda_n \geq 0, \forall n\}$ be the convex hull of the sharing rates Q_1, \dots, Q_N of intermediaries. The points in $\text{conv}(Q)$ are the feasible allocations of the resource to agents subject to the constraints Q given by the intermediaries. Let Q_{-n} be the matrix where the row Q_n is removed from Q . Let $\text{conv}(\mathbf{0}, Q)$ be the convex hull of Q and the vector of zeros. Let $x^*(Q, u) = \{x \in \text{conv}(Q) \mid u(x) \geq u(x'), \forall x' \in \text{conv}(Q)\}$ be the set of allocations to the agents that maximize the planner's utility. Note that, when the planner's preferences are convex the set $x^*(Q, u)$ is a convex set. Moreover, when the planner's preferences are strictly convex the set $x^*(Q, u)$ contains a unique point.

The next result follows from the two main Theorems in the paper. We need to recognize that, due to the restrictions of the model, the assumptions in Theorem 2 regarding monotonicity and cross-monotonicity of a problem can be simply implied by the strong

monotonicity and homotheticity of the planner’s preferences, respectively.

Corollary 2

- a. Given the sharing rates of intermediaries Q_1, \dots, Q_N , there exists an IFE (or $\mathbf{0}$ is the unique robust SPNE) for any strongly monotonic¹⁴ and homothetic preferences of the planner if and only if for every intermediary n , $Q_n \in \text{conv}(\mathbf{0}, Q_{-n})$.
- b. Suppose that preferences of the planner are homothetic, strongly monotonic and strictly convex. An IFE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if the utility maximizing allocation $x^*(Q, u)$ belongs to the intersection of $\bigcap_{n \in \mathcal{N}} \text{conv}(Q_{-n})$.

Part (a) provides conditions for the existence of an IFE for any strongly monotonic and homothetic preferences of the planner. Such conditions imply that the ability Q_n to transmit the resource by intermediary n can be replicated by a subset of other intermediaries. On the other hand, part (b) focuses on a specific utility function u of the planner that is monotonic and strictly convex. It requires that the utility maximizing allocation belongs to $\text{conv}(Q_{-n})$ for any n . Thus, no intermediary is unique, as his ability can be replicated by the ability of others.

B.2 Resource Transmission in Networks under Unit-Capacities

We consider the problem of intermediation with unit-capacity constraints. A finite directed network $G = (V, E)$ without cycles that connects a single source P and sinks $\mathcal{M} = \{1, \dots, M\} \subset V$ is interpreted as connecting the planner with agents \mathcal{M} . The link $e \in E$ has a unit-capacity constraint c_e , which means that every unit of resource transmitted using link e would receive at most c_e units. Consider the case where the planner is endowed with I units of resource to distribute to the agents. Thus, for instance, if I units of good are transmitted in the sequence of links with unit capacities c_1, \dots, c_l , then $c_1 \cdots c_l I$ is the maximal amount of resource that reaches its destination.

Assume the intermediaries in the set $\mathcal{N} = \{1, \dots, N\}$ own the links in the network. Let $E = \{E_1, \dots, E_N\}$ be a partition of the links E , where E_n represents the links owned by intermediary n .¹⁵

The planner has preferences over allocations in \mathbb{R}_+^M denoted by a utility function $u : \mathbb{R}_+^M \rightarrow \mathbb{R}$ that is independent of the prices p . Thus, for instance, if the planner only cares about the total allocation to the agents, then $u(x) = \sum_m x_m$, but in general the planner might care about the worst individual $u(x) = \min_{m \in \mathcal{M}} x_m$ or some other utility function.

Assume the intermediaries post prices $p = (p_1, \dots, p_N)$ for the use of their links.¹⁶ Given

¹⁴The preferences represented by a utility function u are strongly monotonic if for any x and x' such that $x \geq x'$ and $x \neq x'$, $u(x) > u(x')$. While we use strong monotonicity in Corollaries 2 and 3, the same results apply for some non-monotonic preferences such as those represented by a perfect complements utility function $u(x) = \min_{i \in \mathcal{N}} x_i$.

¹⁵The canonical case of this model occurs when every intermediary owns one link. Another traditional case occurs when intermediaries own the span of links emanating from nodes. Moreover, the model where there is a single agent and every link has capacity 1 is discussed in Choi, Galeotti and Goyal[10]. Their results from Theorem 1 can be easily obtained from our Corollary 3 below.

¹⁶We focus on the case where each intermediary posts a single price for the use of all his links. We do not study the case of multiple pricing, but it is also an interesting case.

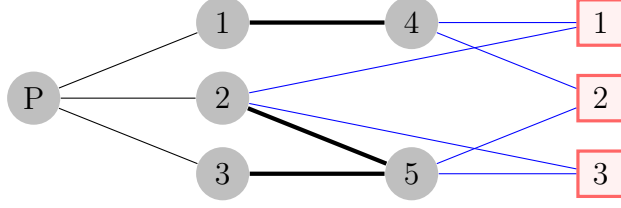


Figure 4: Network with Multiple Layers of Intermediation

the prices, the planner decides on the group of intermediaries to contract by paying the prices posted, and distributes the rest of the resources. Thus, for instance, if the planner is selecting group S , then he pays a total price of $\sum_{n \in S} p_n$ for the use of links in S , and $I - \sum_{n \in S} p_n$ units of resource are left for transmission to the agents.

For intermediaries $S \subset \mathcal{N}$ and agent $m \in \mathcal{M}$, let $PG(S, m)$ be the paths in G connecting the planner with agent m in the network where the capacities of the intermediaries in $\mathcal{N} \setminus S$ are zero. For a given path w with unit capacities $(c_1 \dots c_l)$ on the links, let $c(w) = c_1 \dots c_l$ be the unit capacity of the path. Given an agent m and intermediaries $S \subset \mathcal{N}$, let $\bar{c}^m(S) = \max_{w \in PG(S, m)} c(w)$ be the maximum unit capacity of the paths that connect agent m with the planner in the network. Note that since there is a finite number of paths, $\bar{c}^m(S)$ is easily computable. Given the group of intermediaries S , the maximal unit capacity is $\bar{c}^m(S)$ for agent m . Let $x^{m, S} = (0, \dots, 0, \bar{c}^m(S), 0, \dots, 0) \in \mathbb{R}_+^M$ be the vector representing the maximal transmission to the agent m using intermediaries in S . The OPF for group S and vector of prices p is

$$F(S, p) = \{x \in \mathbb{R}_+^M \mid x \leq \sum_{m=1}^M \lambda^m x^{m, S} (I - p_S), \sum_{m=1}^M \lambda^m = 1, \lambda^m \geq 0, \forall m\} \text{ if } \sum_{n \in S} p_n \leq I$$

$$F(S, p) = \{(0, \dots, 0)\} \text{ if } \sum_{n \in S} p_n > I$$

Definition 12 (Non-zero Corners Utility Function)

The utility function has non-zero corners if for any $x \in \mathbb{R}_+^M$ such that $x_m = 0$ for some m , then $u(x) = 0$; and if $x > (0, \dots, 0)$, then $u(x) > 0$. The preferences of the planner are non-zero corners if there exists a non-zero corners utility function that represents such preferences.

The perfect complements utility function $u(x) = \min\{x_1, \dots, x_M\}$ and the Cobb-Douglas utility function $u(x) = \prod_{m=1}^M x_m^{\alpha_m}$ satisfy non-zero corners. Given that the preferences of planner are homothetic, the problem (u, F) is monotonic if the preferences are strongly monotonic or the utility function has non-zero corners (we prove this in the proof of Corollary 3 below).

Example 3

Consider the network in Figure 4. The intermediaries $\mathcal{N} = \{1, \dots, 5\}$ are represented by the middle nodes in the network. Each of them own the links that originate from their node.

The agents $\mathcal{M} = \{1, 2, 3\}$ are in the final layer of network. The black (thick) links have a unit capacity of 1, while the blue links have a unit capacity $c_j = 0.5$. The planner has a perfect complement utility function $u(x) = \min\{x_1, x_2, x_3\}$ over the final allocation of the resource to the agents in \mathcal{M} .

In this example no intermediary is fundamental. That is, the unit capacity of resource transmission to agent m with all intermediaries except n is $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N}) = 0.5$, $\forall m, n$. There is an IFE and unique robust SPNE, $p = \mathbf{0}$, $b(p) = \{1, 2, 4\}$, $x(p) = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$.

Consequences of Theorems 1 and 2 in the problem of resource transmission under unit-capacities are described below.

Corollary 3

- a. Suppose that for any agent $m \in \mathcal{M}$ and intermediary $n \in \mathcal{N}$ we have that $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$. Then, for any homothetic preferences of the planner, the price vector $p = \mathbf{0}$ is an IFE and unique robust SPNE. Conversely, if for any strongly monotonic utility function of the planner there exists an IFE (or $\mathbf{0}$ is the unique robust SPNE) then $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ for any agent m and intermediary n .
- b. Suppose the planner's utility function is homothetic and has non-zero corners. An IFE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

This corollary establishes the sufficient conditions for the existence of an IFE and for the prices $\mathbf{0}$ to be the unique robust SPNE. These conditions require that the maximal unit capacity that can be transmitted to an agent in the network should not change when any intermediary is removed. Part (a) shows that this property is necessary if we want the existence for any monotonic utility function of the planner. On the other hand, part (b) shows that the same condition is necessary when we restrict to a single set of preferences of the planner that satisfy non-zero corners.

B.3 Resource Transmission in Networks under Total-Capacities

We consider the case of intermediation with total capacity constraints on the links. A finite directed network $G = (V, E)$ without cycles that connects a single source P and sinks $\mathcal{M} = \{1, \dots, M\}$ is interpreted as connecting the planner with agents \mathcal{M} . Every link $l \in E$ in the network has a capacity constraint c_l , which is the maximal capacity that can be transmitted in that link.¹⁷ Assume the intermediaries in the set $\mathcal{N} = \{1, \dots, N\}$ own the links in the network. Let $E = (E^1, \dots, E^N)$ be a partition of G , where E^n represents the links owned by intermediary n . The planner is endowed with I units of the resource and has preferences over the final allocations of the agents, denoted by a utility function $u(x) : \mathbb{R}_+^M \rightarrow \mathbb{R}$. Unlike in the case of unit-capacities discussed above, the links have total capacities, therefore if I units of good are transmitted in the sequence of links with

¹⁷Similar models with capacity constraints in links have been studied in the literature, for instance Bochet, Ilkilic, Moulin and Sethuraman (2012) discuss the transmission of a divisible resources from suppliers to demanders in a network with similar capacity constraints over the links.

total capacities (c_1, \dots, c_l) , then $\min\{c_1, \dots, c_l, I\}$ reach their destination. The allocation of resource follows the same posting-price mechanism as in the case of unit-capacities.

Given an agent m and intermediaries $S \subset \mathcal{N}$, let $\bar{c}^m(S, I)$ ¹⁸ be the maximal amount of resource that can be transmitted to agent m using the links owned by intermediaries in S ¹⁹ when I units are available for transmission. Notice $\bar{c}^m(S, I)$ is easily computable in the network, for instance the simple Ford-Fulkerson algorithm ([11]) computes the max-flow in a network. Let $x^{m,S} = (0, \dots, 0, \bar{c}^m(S, I), 0, \dots, 0) \in \mathbb{R}_+^M$ be the vector representing the maximal transmission to the agent m using intermediaries in S . The OPF for group S and vector of prices p is

$$\tilde{F}(S, p) = \left\{ x \in \mathbb{R}_+^M \mid x \leq \sum_{m=1}^M \lambda^m x^{m,S} (I - p_S), \sum_{m=1}^M \lambda^m = 1, \lambda^m \geq 0, \forall m \right\} \text{ if } \sum_{n \in S} p_n \leq I$$

$$\tilde{F}(S, p) = \{(0, \dots, 0)\} \text{ if } \sum_{n \in S} p_n > I$$

Unlike the previous two applications, the OPF in this example is not additive (see footnote 9). This can be readily seen in an example of two links l_1, l_2 owned by different intermediaries, where (l_1, l_2) is the only path connecting the planner to a single agent. Each link has capacity 1. If the planner selects l_1 or l_2 , then he cannot transmit anything to the agent. However, if the planner selects l_1 and l_2 , then he can transmit 1 unit.

Furthermore, unlike in the previous two applications, the OPF \tilde{F} is not homothetic in the resource I . Thus, an increase in the amount of the resource I may change the multiplicity of equilibria and welfare of the planner at equilibrium, as illustrated in the following example.

Example 4

Consider a graph with three parallel links directly connecting the planner with a single agent. The links are owned by different intermediaries and have capacities 10, 10 and 11, respectively. The planner cares about transmitting the maximal amount of the resource to the agent (i.e., $u(x) = x$). If the planner has $I = 18$ units of resource, then every pair of links can transmit the full resource (thus every pair of intermediaries would maximize the utility). The prices $p = (0, 0, 0)$, $b(p) = \{1, 2\}$ and $x(p) = 18$ is an IFE and a unique robust SPNE. At the same time, prices $p = (8, 8, 8)$, $b(p) = \{3\}$ and $x(p) = 10$ is also SPNE, since intermediaries 1 and 2 cannot coordinate to lower the prices and get higher utility.

If the planner has $I = 40$ units of resource, there is a SPNE with $p = (0, 0, 20)$ and intermediaries 1 and 3 (or 2 and 3) being used. Note that in this equilibrium, intermediary 3 has a link with a larger capacity constraint than intermediaries 1 and 2, but he posts a positive price and gets a larger benefit than intermediaries 1 and 2. There are multiple SPNE, for example $p = (30, 30, 30)$ and only intermediary 3 being used. However, there is a unique robust SPNE.

¹⁸Unlike in the previous section, the results under total capacity depend on the total resource I , see below.

¹⁹Alternatively, we can re-interpret this as saying that the capacities of all the links owned by the intermediaries in $\mathcal{N} \setminus S$ are changed to zero.

This example also shows that when resource I increases, the planner's utility at the equilibrium may not increase and the increase resource is paid to intermediaries.

This example also illustrates that the problem (u, \tilde{F}) is not monotonic, hence the results in Theorem 2 might not apply. Indeed, once the full capacity of the network has been reached, a strict increase in one of the prices may not strictly decrease the OPF. Therefore, the results in Theorem 2 may not apply. Consequences of Theorems 1 in the problem of resource transmission under total-capacities are described below.

Corollary 4

- a. For any monotonic utility function u there exists an IFE if and only if the full transmission of the resource to any agent m without using the links of intermediary n is possible, that is $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ for any agent m and intermediary n .
- b. Suppose that the planner's utility function has non-zero corners. In the problem without capacities, i.e., capacities are infinity for every link, an IFE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if there is no intermediary who owns link(s) on every path from the planner to some agent.

We use a simple argument of the max-flow min-cut Theorem to prove part *a*. A particular case of part *b* is discussed in Choi, Galeotti, Goyal[10], which proves the case that connects sellers and buyers, and they generate a surplus of 1 if they connect, and a surplus of 0 if they do not connect.

B.4 Separable Utility: Minimum Cost Spanning Trees and Related Models

In this section we restrict our attention to intermediation problems (u, F) with a separable utility function, $u(x, p_S) = u(x) - p_S$, and an outcome possibility function that is independent of the price p , $F(S, p) = F(S)$. Intermediation problems with such structure capture more stylistic settings previously discussed in the literature, as shown below.

Example 5 (MCST and Related Models, Moulin and Velez[33])

Let $\mathcal{B} = \{B_1, \dots, B_c\} \subset 2^N$ be a collection of acceptable subsets of intermediaries such that if $B_i \in \mathcal{B}$ and $B_i \subset D$ then $D \in \mathcal{B}$. Consider the outcome space $A = \mathbb{R}$, the utility of the planner $\bar{u}(x, p_S) = x - p_S$ and OPF equal to $F(S, p) = [0, 1]$ if $S \in \mathcal{B}$ and $F(S, p) = \{0\}$ if $S \notin \mathcal{B}$. Thus, the planner has a quasilinear utility function with numeraire good equal to the total price paid. The OPF has a positive element only when it is part of an acceptable set.

For instance, if \mathcal{B} contains at least two individual intermediaries, say $\{i\}$ and $\{j\}$ are acceptable, then at a SPNE, the planner gets utility 1 and pays no money for the intermediaries. This is similar to a Bertrand competition model, where intermediaries lower their prices to zero in hopes to be chosen by the planner.

One particular case of this setting occurs in the minimal cost spanning tree (MCST) discussed in Moulin and Velez[33], where the links E in a network connecting a set of nodes \mathcal{M} are owned by the group of intermediaries \mathcal{N} . Let (E_1, \dots, E_N) be a partition of the set of links E , where E_n represents the links owned by intermediary n . The set of

acceptable intermediaries $\mathcal{B} \subset 2^{\mathcal{N}}$ contain the groups of intermediaries whose links connect to all nodes in \mathcal{M} . Note this might not necessarily be a spanning tree. In the case where every intermediary owns exactly one link, the set \mathcal{B} contains all spanning trees.

Other related models of interconnection in trees can be similarly encompassed by this analysis, including the Steiner tree problem where the shortest interconnect for a given set of objects is found.

Let $u_S = \max_{x \in F(S)} u(x)$ be the maximal utility achieved when using the intermediaries in S and $\bar{u} = u_{\mathcal{N}} = \max_{x \in F(\mathcal{N})} u(x)$ be the maximal utility achieved when using all the intermediaries. The straightforward consequence of Theorems 1 and 2 are discussed below.

Corollary 5

- a. Consider an intermediation problem (u, F) with a separable utility function, $u(x, p_S) = u(x) - p_S$, and an outcome possibility function that is independent of the prices p , $F(S, p) = F(S)$. An IFE exists (or $\mathbf{0}$ is the unique robust SPNE) exists if and only if the group of intermediaries who achieve the maximal utility, $\mathcal{S} = \{S_i \subseteq \mathcal{N} | u_{S_i} = \bar{u}\}$, satisfy $\bigcap_{S_i \in \mathcal{S}} S_i = \emptyset$.
- b. For the model in Example 5, an IFE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if the intersection of the acceptable sets is empty, that is $\bigcap_{B_i \in \mathcal{B}} B_i = \emptyset$. Furthermore, in the MCST problem an IFE exists (or $\mathbf{0}$ is the unique robust SPNE) if and only if for every node $m \in \mathcal{M}$ there are at least two intermediaries with links to node m .

Proofs of Results in Appendix B

Proof of Corollary 2

Proof. We prove that the problem (u, F) is monotonic and cross-monotonic.

Recall that the preferences of the planner are independent of price $u(x, p) = u(x)$, strongly monotonic and homothetic in x . Consider a price p and group S such that $\sum_{n \in S} p_n \leq I$. Then, the OPF equals $F(S, p) = \{\sum_{n \in S} Q_n y_n | \sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n \text{ and } y_n \geq 0\}$. Consider prices p' , s.t. $p'_n < p_n$, $p'_{-n} = p_{-n}$ for $n \in S$. For any $x \in F(S, p)$, assume $x = \sum_{n \in S} Q_n y_n$, $\sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n < I - \sum_{n \in S} p'_n$. Thus, for any y' , s.t. $\sum_{n \in S} y'_n \leq I - \sum_{n \in S} p'_n$, $x' = \sum_{n \in S} Q_n y'_n \in F(S, p')$, so there exists $x' \in F(S, p')$ such that $x' > x$. By monotonicity of u , $u(x') > u(x)$. Thus the problem (u, F) is monotonic in prices p .

In order to show that the problem (u, F) is cross-monotonic, note that

$$\begin{aligned} F(S, p) &= \left\{ \sum_{n \in S} Q_n y_n \mid \sum_{n \in S} y_n \leq I - \sum_{n \in S} p_n \text{ and } y_n \geq 0 \right\} \\ &= \left(I - \sum_{n \in S} p_n \right) \left\{ \sum_{n \in S} Q_n y_n \mid \sum_{n \in S} y_n \leq 1 \text{ and } y_n \geq 0 \right\}. \end{aligned}$$

Hence, remark 2 (b) is satisfied.

- a. First, if $\forall n, Q_n \in \text{conv}(\mathbf{0}, Q_{-n})$, then the OPF $F(\mathcal{N}, \mathbf{0}) = F(\mathcal{N} \setminus \{n\}, \mathbf{0})$. Thus, for each n there exists a utility maximizing group S_n at prices $\mathbf{0}$, s.t. $n \notin S_n$. Therefore, $\bigcap_{n \in \mathcal{N}} S_n = \emptyset$. So, from Theorem 1 and 2, there exists IFE and unique robust SPNE.

Second, if there exists IFE for any monotonic and homothetic preferences. Suppose there is intermediary n , s.t. $Q_n \notin \text{conv}(\mathbf{0}, Q_{-n})$. Consider the utility function $u(x) = \min\{\frac{x_1}{q_{n1}}, \dots, \frac{x_M}{q_{nM}}\}$, then the indirect utility function satisfies $v(\mathbf{0}) > v_{-n}(\mathbf{0}_{-n})$. Hence, $p = \mathbf{0}$ is not equilibrium price allocation. Given prices $\mathbf{0}_{-n}$, intermediary n has incentive to deviate and post positive price $p'_n > 0$ with $p' = (p'_n, \mathbf{0}_{-n})$ and $v(p') > v_{-n}(\mathbf{0}_{-n})$. Thus, intermediary n would a higher payoff, which is a contradiction. Hence, $Q_n \in \text{conv}(\mathbf{0}, Q_{-n}), \forall n$.

b. When the planner has strictly convex preferences, there is a unique point $x \in F(\mathcal{N}, \mathbf{0})$ that maximizes the utility at prices $\mathbf{0}$. Let $x = x^*(Q, u)$. Note that $x \in \bigcap_{n \in \mathcal{N}} \text{conv}(Q_{-n})$ is equivalent for the group $S_n = \mathcal{N} \setminus \{n\}$ to be a utility maximizing group at prices $\mathbf{0}$. Hence, $\bigcap_{n \in \mathcal{N}} S_n = \emptyset$. Thus, from Theorem 1 and 2, there exists IFE and unique robust SPNE.

For the converse, if an IFE exists, then by part a, $Q_n \in \text{conv}(0, Q_{-n})$ for every intermediary n . Thus, $F(N, \mathbf{0}) = \text{conv}(0, Q) = \text{conv}(0, Q_{-n})$. Therefore, $x^*(Q, u) \in \text{conv}(0, Q_{-n})$ for all n . Hence, $x^*(Q, u) \in \bigcap_{n \in \mathcal{N}} \text{conv}(0, Q_{-n})$. By monotonicity of u , $x^*(Q, u)$ is in the boundary of $\text{conv}(0, Q_{-n})$, then $x^*(Q, u) \in \bigcap_{n \in \mathcal{N}} \text{conv}(Q_{-n})$ ■

Proof of Corollary 3

Proof. a. For any agent m and intermediary n we have that $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$, which implies that $F(\mathcal{N} \setminus \{n\}, \mathbf{0}) = F(\mathcal{N}, \mathbf{0})$. Thus, group of intermediaries $S_n = \mathcal{N} \setminus \{n\}$ is utility maximizing group at prices $\mathbf{0}$, there exists $S_j(\mathbf{0}) = S_n$, so $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$.

We now show that the problem (u, F) is monotonic. Indeed, since preferences are strongly monotonic and homothetic, for price p and p' , with $p'_n < p_n, p'_{-n} = p_{-n}, n \in S, p_S \leq I$, then $p'_S < I$. Thus, there exists $y \in F(S, p')$, s.t. $y \geq x$ and $y \neq x$. Since the preferences are strongly monotonic, $u(y) > u(x)$.

In order to show that (u, F) is cross-monotonic, note that the preferences are homothetic and $F(S, p) = \gamma(S)\delta(p_S)$ for a set $\gamma(S) \subset \mathbb{R}^M$. Thus, it satisfies cross-monotonicity from Remark 2 (b). From Theorem 1 and 2, there exists IFE and unique robust SPNE.

Conversely, for any monotonic utility function, there exists an IFE (or unique robust SPNE). Suppose $\bar{c}^m(\mathcal{N} \setminus \{n\}) < \bar{c}^m(\mathcal{N})$, then if utility function $u(x) = x_m$, the planner only cares about the resource allocated to agent m , and deleting intermediary n would decrease the maximal unit capacity allocated to agent m . Thus intermediary n could post price $p_n > 0$ and get positive benefit. So there is no IFE.

b. Similar to part (a), the problem (u, F) is cross-monotonic due to the homotheticity of preferences. We now show that the problem (u, F) is monotonic when preferences are homothetic and the utility function is non-zero corners. Indeed, we simply prove that the outcome possibility function F is strongly monotonic in prices (which implies the monotonicity of (u, F) by Lemma 3). For price p and p' , with $p'_n < p_n, p'_{-n} = p_{-n}, n \in S$, for $x \in F(S, p), u(x) > u((0, \dots, 0))$, thus $x_m > 0, \forall m$, there are links to all agents with group S . $I - p_S < I - p'_S$, there exists $y \in F(S, p')$ and $y > x$.

Since preferences are homothetic and the utility function has non-zero corners, $\bar{c}^m(\mathcal{N} \setminus \{n\}) = \bar{c}^m(\mathcal{N})$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$, then we have that an IFE or a unique robust SPNE exists.

Conversely, suppose that x is an IFE (or a unique robust SPNE exists). By Theorem 2, $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, since the problem (u, F) is monotonic and cross-monotonic. Suppose that $x = (x_1, \dots, x_M) \in F(S, \mathbf{0})$. $x_m > 0$, for any m , so $x = \lambda^m x^{m,S} I$.

Assume $\exists m, n$ such that $\bar{c}^m(\mathcal{N} \setminus \{n\}) < \bar{c}^m(\mathcal{N})$. To prove $v(\mathbf{0}) > v_{-n}(\mathbf{0}_{-n})$, there is

$x \in F(\mathcal{N} \setminus \{n\}, \mathbf{0})$ s.t. $u(x) = v_{-n}(\mathbf{0}_{-n})$. Here to prove $\exists x' \in F(\mathcal{N}, \mathbf{0})$, s.t. $x' > x$ which means $x'_i > x_i, \forall i$. $x^i = (0, \dots, 0, \bar{c}^i(\mathcal{N}), 0, \dots, 0) \in \mathbb{R}^M$, assume $x_{-n}^i = (0, \dots, 0, \bar{c}^i(\mathcal{N} \setminus \{n\}), 0, \dots, 0) \in \mathbb{R}^M$. Since the preferences are non-zero corners, $x_i > 0, \forall i$. Thus there is $\lambda^i > 0$ with $\sum_i \lambda^i = 1$, s.t. $x = \sum_{i=1}^M \lambda^i x_{-n}^i I$. $\exists \epsilon > 0$ small enough, with $\lambda'^m = \lambda^m - \epsilon \geq 0$, s.t. $\lambda'^m \bar{c}^m(\mathcal{N}) > \lambda^m \bar{c}^m(\mathcal{N} \setminus \{n\})$. Let $\lambda'^i = \lambda^i + \frac{\epsilon}{M-1}, \forall i \neq m, \sum_{i=1}^M \lambda'^i = 1$, there is $\lambda'^i \bar{c}^i(\mathcal{N}) > \lambda'^i \bar{c}^i(\mathcal{N} \setminus \{n\})$. Then $x' = \sum_{i=1}^M \lambda'^i x_{-n}^i I > x$ and $x' \in F(\mathcal{N}, \mathbf{0})$. The preferences are monotonic and $x' > x$, then $v(\mathbf{0}) \geq u(x', 0) > u(x, 0) = v_{-n}(\mathbf{0}_{-n})$. At prices $p_{-n} = \mathbf{0}_{-n}$, from Lemma 2, intermediary n has incentive to deviate and charge positive price $p_n > 0$. So there is no IFE. ■

Proof of Corollary 4

Proof.

a. First, we prove $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I)$ for every intermediary n and agent m if and only if $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ for every intermediary n and agent m . To see that, if $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$, then $\bar{c}^m(\mathcal{N}, I) \geq \bar{c}^m(\mathcal{N} \setminus \{n\}, I) = I$ and $\bar{c}^m(\mathcal{N}, I) \leq I$, so $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I)$. To prove the converse, consider the problem of maximal flow from the planner to agent m , then $\bar{c}^m(\mathcal{N}, I)$ is the maximal flow. Thus, there exists intermediary n who owns a link in the minimal cut such that after deleting his link, the maximal flow decreases. Hence, $\bar{c}^m(\mathcal{N}, I) > \bar{c}^m(\mathcal{N} \setminus \{n\}, I)$, which is a contradiction.

Note that $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) = \bar{c}^m(\mathcal{N}, I) \forall m, n$ if and only if coalition $N \setminus n$ is a utility maximizing coalition at prices $\mathbf{0}$ for all n . Hence, by Theorem 1, an IFE exists.

Conversely, for any monotonic utility function, there exists an IFE (or unique robust SPNE). Suppose $\bar{c}^m(\mathcal{N} \setminus \{n\}, I) < \bar{c}^m(\mathcal{N}, I)$ and planner's utility function is $u(x) = x_m$, then if $p_{-n} = \mathbf{0}_{-n}$, since $u(\mathcal{N} \setminus \{n\}, \mathbf{0}) = \bar{c}^m(\mathcal{N} \setminus \{n\}, I) < \bar{c}^m(\mathcal{N}, I) = u^*(\mathcal{N}, \mathbf{0})$, intermediary n has incentive to post positive price $p_n > 0$, s.t. $p' = (p_n, \mathbf{0}_{-n})$ and $u^*(\mathcal{N} \setminus \{n\}, \mathbf{0}) < u^*(\mathcal{N}, p')$, intermediary n gets p_n . Thus, $\mathbf{0}$ is not an equilibrium.

b. Given the non-zero corner utility function, the planner needs to use path to each agent. If there exists intermediary n that owns link on every path to agent m , then $c^m(\mathcal{N} \setminus \{n\}, I) = 0 < c^m(\mathcal{N}, I) = I$, so posting a zero price for intermediary n is not optimal for him, hence there is no IFE. Conversely, if there is no intermediary n that owns link on every path connected to agent m , then $c^m(\mathcal{N} \setminus \{n\}, I) = c^m(\mathcal{N}, I) = I$ (due to the infinite capacities). From Theorem 1, there exists an IFE. ■

Proof of Corollary 5

Proof.

a. Since F is independent of the price, any group $S_j \subseteq \mathcal{N}$ such that $u_{S_j} = \bar{u}$ is a utility maximizing group $S_j(\mathbf{0})$ at prices $\mathbf{0}$. Then $\bigcap_j S_j = \emptyset$ is equivalent with $\bigcap_j S_j(\mathbf{0}) = \emptyset$. Furthermore, the utility function $u(x, t) = u(x) - t$ is strictly decreasing in total price paid t . Hence, from Lemma 3, the problem (u, F) is monotonic. $u^*(S, \mathbf{0}) = u_S \leq u^*(T, \mathbf{0}) = u_T$ if and only if $u^*(S, p) = u_S - p_S \leq u^*(T, p) = u_T - p_T$ with $p_S = p_T$; from Definition 9, the problem is cross-monotonic. Therefore, $\bigcap_{S_j \in \mathcal{S}} S_j = \emptyset$ if and only if there exists IFE and unique robust SPNE.

b. The monotonic and cross-monotonicity of the problem (u, F) is trivial. Note that for any group of intermediaries $B_k \in \mathcal{B}$, $u^*(B_k, \mathbf{0}) = 1 = u^*(\mathcal{N}, \mathbf{0})$. So B_k is a utility maximizing group at prices $\mathbf{0}$. From Theorem 1 and 2, there exists IFE and unique robust SPNE if and

only if $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$.

We now show that in the MCST, $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$ is equivalent to have every node linked via at least two intermediaries. If every node is linked to at least two intermediaries, then $\mathcal{N} \setminus \{n\}$ is an acceptable set, $\mathcal{N} \setminus \{n\} \in \mathcal{B}$. Hence $\bigcap_{B_k \in \mathcal{B}} B_k = \emptyset$. On the other hand, if m is uniquely linked to intermediary n , then $\mathcal{N} \setminus \{n\}$ is not acceptable, thus $u_{\mathcal{N}} = 1 > 0 = u_{\mathcal{N} \setminus \{n\}}$, intermediary n could charge positive price at equilibrium. Thus, there is no IFE. ■

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