

Free Intermediation in Resource Transmission^{*}

Lining Han [†] and Ruben Juarez[‡]

Department of Economics, University of Hawaii

February 28, 2018

Abstract

We provide a framework herein for the study of the allocation of a divisible resource from a planner to agents via intermediaries. Intermediaries simultaneously post fees for their services, and the planner optimally selects a subset of them to assist in the transmission of the resource. We provide necessary and sufficient conditions for the existence of a perfectly competitive equilibrium, where the intermediaries selected by the planner post fees equal to zero. Furthermore, these conditions are necessary and sufficient to guarantee the uniqueness of equilibrium when intermediaries who are not selected post fees equal to zero.

Keywords: Resource-sharing, Intermediation, Bertrand Competition

JEL classification: C70, D85

^{*}We are grateful for the comments received from Francis Bloch, Mihai Manea, Herve Moulin, two outstanding referees, advisory editors and participants at the Southwest Economic Theory Conference in Riverside, the Coalition Theory Networks Conference in Moscow, the Game Theory Festival in Stony Brook and the World Congress of the Game Theory Society in Maastricht.

[†]Email: hanl@hawaii.edu

[‡](Corresponding author) Postal address: 2424 Maile Way, Saunders Hall 542, Honolulu HI 96822. Email: rubenj@hawaii.edu

1 Introduction

There are many markets for which intermediaries play an essential role, the most common of which includes the transmission of goods and resources to agents. For instance, the allocation of government resources to agents often requires the use of private for-profit companies, called intermediaries, that are more closely connected to the agents than the government agency. Intermediaries, therefore, enable the government agency to target their agents more effectively. This top-down structure provides an opportunity for competition between intermediaries, with the potential for added benefits. Such benefits, however, are largely dependent on the way in which the intermediaries transmit the resources to the agents and the type of resource (i.e. divisible vs indivisible) that is to be distributed.

Although much attention has been paid to the case of intermediation for indivisible goods, few studies have focused on intermediation for divisible goods and resources. Herein, we introduce a general model of intermediation in which a planner is interested in transmitting a divisible resource (such as money) to agents. Although the planner is not directly linked to them, it can do so via a group of intermediaries. Different groups of intermediaries have the ability to transmit different allocations, and so intermediaries differ not only in relation to the agents they can reach, but also in the quality of their intermediation.

We focus on the case where intermediaries are private and independent entities can charge the planner for access to their agents. While intermediaries care about maximizing the amount paid by the planner, the planner has preferences over the different allocations of the resource to the agents, as well as the total amount paid to the intermediaries employed. We study the case of perfect information, where the planner and intermediaries are aware of each other's preferences and abilities.

In the first stage of our game, intermediaries independently and simultaneously report their fees for providing the planner with access to their agents. The fees might affect the transmission of the resource. In the second stage, the planner selects the intermediaries and feasible amounts of the resource allocated to each of them for transmission to agents. Intermediaries who are not selected do not get paid. The ultimate goal of the intermediary is to be contracted and maximize the price paid by the planner, and the goal of the planner is to distribute as much resource to the agents in a way that maximizes his preferences. We use a subgame perfect Nash equilibrium (SPNE) to describe the result of the strategic behavior between the planner and the intermediaries. The equilibrium price of intermediaries depends on the utility function of the planner and the abilities of the intermediaries to transmit the resource to the agents.

1.1 Overview of the Results

We describe the main results of our paper through a simple example. Consider a planner who is endowed with I units of money and seeks to transfer it to two agents. His preferences over the allocation of the money (x_1, x_2) are given by a perfect complement utility function $u(x_1, x_2) = \min\{x_1, x_2\}$. The intermediaries vary in terms of their ability to transmit the money to agents. This variation is captured by an *outcome possibility function (OPF)* F that assigns to every group of intermediaries and prices a set of potential outcomes from which the planner may choose. For this example, assume there are three intermediaries, and given the vector of prices $p = (p_1, p_2, p_3)$ of the intermediaries the OPF F is given by

$$F(\{1\}, p) = \{(x, 0) | 0 \leq x \leq \frac{6}{5}(I - p_1)\},$$

$$F(\{2\}, p) = \{(0, x) | 0 \leq x \leq \frac{6}{7}(I - p_2)\},$$

$$F(\{3\}, p) = \{(x, x) | 0 \leq x \leq \frac{I - p_3}{2}\},$$

$$F(S, p) = \text{conv}(\cup_{n \in S} F(\{n\}, (\sum_{i \in S} p_i)e^n)) \text{ for } S \subset \{1, 2, 3\},$$

where *conv* is the convex hull of the sets and $e^n \in \mathbb{R}^3$ is the vector equal to 1 on the n -th coordinate and zero otherwise.

When the intermediaries post prices equal to zero, the planner can maximize his utility by allocating $(\frac{I}{2}, \frac{I}{2})$ to the agents, using three potential groups of intermediaries (see Figure 1, left). The planner can transmit $\frac{5I}{12}$ and $\frac{7I}{12}$ via intermediaries 1 and 2, respectively. Alternatively, the planner can allocate all the money to intermediary 3. Moreover, the planner can also use intermediaries 1, 2, and 3 by selecting any convex combination of the above.

An important challenge is to identify conditions for the existence of a subgame perfect Nash equilibrium that is planner-optimal, where the intermediaries use post prices equal to zero (*free intermediation equilibrium (FIE)*). This equilibrium does not preclude those intermediaries who are not used to post a positive price. However, at an FIE, all intermediaries—regardless of whether they are used by the planner—earn zero profit. Thus, an FIE resembles a perfectly competitive equilibrium whereby the planner directly transmits the resource to the agents, as if there were no intermediaries. Theorem 1 shows that an FIE exists if and only if the intersection of the utility-maximizing groups at prices equal to zero (in our example, $\{1, 2\}$, $\{3\}$ and $\{1, 2, 3\}$) is empty.

Even when an FIE exists, other SPNEs may exist. In our example, at prices (p_1, p_2, p_3) , where $p_1 \geq I$, $p_2 \geq I$, and $p_3 = I$, the planner pays intermediary 3 an amount equal to I and

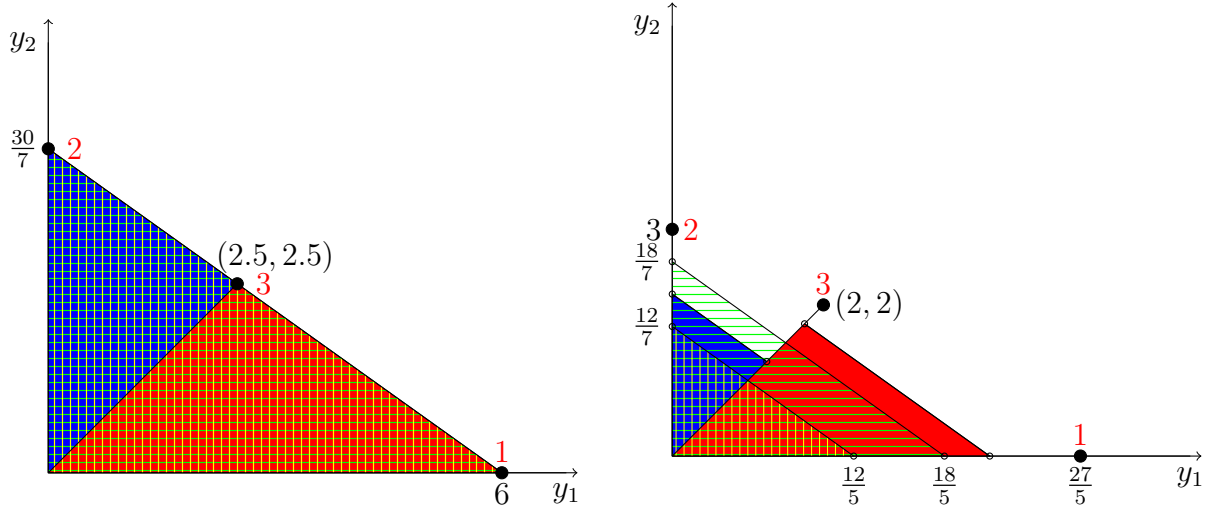


Figure 1: Left and right illustrate the outcome possibility function F when $I = 5$ and prices are $p = (0, 0, 0)$ and $p = (0.5, 1.5, 1)$, respectively. The sets $F(\{1\}, p)$, $F(\{2\}, p)$ and $F(\{3\}, p)$ correspond to the line connecting the origin with points 1, 2, and 3, respectively. The sets $F(\{2, 3\}, p)$, $F(\{1, 3\}, p)$, $F(\{1, 2\}, p)$, and $F(\{1, 2, 3\}, p)$ correspond to the shaded areas in blue, red, green/horizontal-lines, and yellow/vertical-lines, respectively.

transmits no money to the agents. This is an equilibrium, because neither intermediaries 1 nor 2 can decrease their price to undercut intermediary 3. Intermediary 3 has no incentive to decrease its price, because it is being selected.

The multiplicity of SPNE is undesirable, though, as it decreases the predictive power of equilibrium. We introduce a refinement to the SPNE, namely the *robust SPNE*, whereby the group of intermediaries that is not selected by the planner price at zero. In particular, a robust SPNE is a refinement of a coalition-proof Nash equilibrium for those intermediaries who are not selected by the planner. One can imagine that if intermediaries are not selected, then they will have the incentive to undercut their prices (individually or in groups), in order to get selected. Thus, a robust SPNE prevents group manipulation by the intermediaries who are not selected. Theorem 2 shows that if the problem (u, F) is monotonic and cross-monotonic,¹ the vector of zero prices is the unique robust SPNE, but if and only if the intersection of the utility-maximizing groups at prices equal to zero is empty (as in our example).

Our results work for a wide variety of planners' preferences and have implications on the large class of intermediation settings discussed in the supplementary material (Han and Juarez[9]).

¹Monotonicity occurs, for instance, when the planner is strictly worse-off when all prices increase. Cross-monotonic preferences include the case of homothetic preferences in prices, as well as a variety of other weaker conditions.

1.2 Related Literature

The allocation of divisible resources has been prolific, especially in the network literature (see Jackson[15] for a comprehensive survey in networks). Such literature includes normative, positive, and strategic considerations, for example Hougaard et al.[10, 11, 12, 14, 13], Megiddo[24], Moulin[26, 25], Moulin et al.[27], Bochet et al.[4], and Juarez et al.[16, 19, 20, 18, 17]. However, we study the problem of transmitting a divisible good with intermediaries, which creates substantial differences in the equilibria, strategies, and difficulty of the model. A closely related paper is Moulin and Velez[28], which studies the price of imperfect competition for the problem of a spanning tree. Moulin and Velez[28]’s equilibrium can be derived from our main results. Our companion paper, Han and Juarez[8], studies the transmission of a divisible resource when the abilities of the intermediaries are unknown to the planner. In addition, it characterizes a large class of strategy-proof mechanisms when the planner elicits the intermediaries’ abilities to transmit the resource to agents. Our work in this paper focuses on a more general game of perfect information that includes the model from Han and Juarez[8] as a particular case.

There is also a large and growing body of literature on the transmission of indivisible goods and services with intermediaries; for instance, Condorelli and Galeotti[6] survey strategic models of intermediation in networks, while Manea[22] studies a dynamic game on bilateral bargaining in a network with intermediation. Furthermore, Kotowski and Leister[21] consider intermediary traders in a network with an auction mechanisms and find stable and equilibrium networks, and Gale and Kariv[7] study a market with intermediaries and discover that pricing behavior converges to the competitive equilibrium in an experiment. Siedlarek[30] investigates the behaviors of intermediaries in a stochastic model of bargaining on different routes, and Blume et al.[3] consider the trading of divisible goods between sellers and buyers via intermediaries. Choi, Galeotti and Goyal[5] research, theoretically and experimentally, pricing in complex structures of intermediation, while Gale and Kariv[7], Siedlarek[30], Blume et al.[3], and Choi, Galeotti, and Goyal[5] study equilibria where the pay-offs of non-essential intermediaries tend to zero, as in the FIE introduced by our paper. The theoretical result in Choi, Galeotti, and Goyal[5] can be obtained as a particular case of our results.

Our model generalizes the classical Bertrand[1] price competition model and equilibrium in two dimensions. First, groups of intermediaries might have different abilities. Second, the planner not only cares about the price paid to the intermediaries, but also about the quality of the resource’s transmission. Simon and Zame[31] prove the existence of a (mixed-strategy) Nash equilibrium in discontinuous games, including the Bertrand competition game, when

the sharing rule is endogenous, and Reny[29] proves the existence of a pure-strategy Nash equilibrium in compact, quasi-concave and better-reply secure games. More recently, Bich and Laraki[2] have extended Reny's work to obtain tighter conditions for the existence of approximate equilibria. They have also demonstrated that many sharing rules, especially related to competition models like this paper, generate pure and mixed-strategy equilibria. Reny's result is used to prove the existence of equilibria in our setting.

2 The Model

Let $A = \mathbb{R}_+^M$ be the set of feasible outcomes, herein interpreted as the potential allocations of resource to the agents in $\mathcal{M} = \{1, \dots, M\}$. The planner is interested in choosing one of these outcomes but cannot select it directly. Instead, a group of $\mathcal{N} = \{1, \dots, N\}$ intermediaries is able to access subsets of the outcomes and set fixed prices $p = (p_1, \dots, p_N)$ for the use of their ability. Given a group of intermediaries $S \subset \mathcal{N}$, the aggregate price of group S is denoted by $p(S) = \sum_{n \in S} p_n$, and the projection of the vector of prices p over $\mathbb{R}_+^{|S|}$ is denoted by $p_S \in \mathbb{R}_+^{|S|}$. For simplicity, we denote $p_{-n} = p_{\mathcal{N} \setminus \{n\}}$. Given prices p and p' , we say that $p' < p$ if $p'_i < p_i$ for all $i \in \mathcal{N}$. One important price vector is the vector of zero prices $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}_+^N$. In order to avoid confusion, we reserve the vector $(0, \dots, 0) \in \mathbb{R}_+^M$ to represent the allocation of the agents, whereas $\mathbf{0}$ represents the vector of zero prices.

Definition 1 (Intermediation Problem)

An intermediation problem is a pair of functions (u, F) such that:

- $u : A \times \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the **planner's preferences** over the chosen outcome, as well as the aggregate price paid to the intermediaries chosen. We assume that u is continuous, monotonic in A , and non-increasing in \mathbb{R}_+ .²
- $F : 2^{\mathcal{N}} \times \mathbb{R}_+^N \rightarrow 2^A$ is an **outcome possibility function (OPF)** that assigns to every group of intermediaries and vector of prices a set of potential outcomes. We assume that F satisfies the following conditions:
 - a. $F(S, p)$ is a compact set for any group $S \in 2^{\mathcal{N}}$ and vector of prices $p \in \mathbb{R}_+^N$. Furthermore, $F(\emptyset, \mathbf{0}) = \{(0, \dots, 0)\}$.

²The utility function u is monotonic if, for any $x > y$ and t , we find that $u(x, t) > u(y, t)$. For the sake of brevity, we omit the other standard definitions for utility functions, but we do follow standard definitions from Mas-Colell et al.[23], Chapter 3.

- b. $F(S, p)$ is continuous in the vector of prices p for any group $S \in 2^{\mathcal{N}}$.³
- c. F is non-decreasing in the group of intermediaries at prices $\mathbf{0}$. That is, if $S \subseteq T$ then $F(S, \mathbf{0}) \subseteq F(T, \mathbf{0})$.
- d. F only depends on the prices of the chosen group. That is, $F(S, q) = F(S, p)$ for $q_S = p_S$ and for any $S \in 2^{\mathcal{N}}$.
- e. F is non-increasing in prices. That is, if $p \leq q$ then $F(S, q) \subseteq F(S, p)$ for any $S \in 2^{\mathcal{N}}$.

An intermediation problem is composed of two functions u and F . First, the function u represents the planner's preferences over the chosen outcome as well as the aggregate price paid to the intermediaries who are contracted. We assume the planner's utility does not decrease as more resources are allocated to the agents and the planner's utility is non-increasing on the total amount paid to the intermediaries.

Second, intermediaries vary in their ability to transmit the resource to the agents. These differences come from the group of intermediaries selected as well as the prices paid to them. This variation is described formally by an outcome possibility function F that assigns a set of potential outcomes to every group of intermediaries and price vector. We interpret $F(S, p)$ as the outcomes available for the planner to use, after he has contracted group S and paid prices $p(S)$. We have five assumptions regarding F . The first two assumptions are technical assumptions needed to guarantee the existence of equilibrium. In particular, $F(\emptyset, \mathbf{0}) = \{(0, \dots, 0)\}$ gives the planner the possibility of inaction. Selecting more intermediaries when prices are $\mathbf{0}$ should lead to no fewer feasible outcomes, which is the spirit of the third assumption. The fourth assumption guarantees that the ability of a group of intermediaries should only depend on themselves, and not on the prices posted by intermediaries outside the group. The last assumption, which relates to monotonicity, represents the fact that higher prices paid to intermediaries result in no more resources being available to transmit by the planner.

We study a two-stage perfect information game where, in the first stage, intermediaries choose simultaneously and independently prices p in order to gain access to their outcome set. In the second stage, after observing the price vector p charged by the intermediaries, the planner chooses a group of intermediaries $b(p) \subset \mathcal{N}$ and a feasible outcome $x(p) \in F(b(p), p)$.

³For a given $x \in A$ and $\epsilon > 0$, the ball with center x and radius ϵ is denoted by $B_\epsilon(x) = \{x' \in A \mid \|x - x'\| < \epsilon\}$. In order to define continuity formally, for a given sequence of sets $\{\mathcal{M}^i\}_i$, the closure $cl(\{\mathcal{M}^i\}_i)$ is defined as $x \in cl(\{\mathcal{M}^i\}_i)$, if and only if for any $\epsilon > 0$ there exists $\delta > 0$, such that $B_\epsilon(x) \cap \mathcal{M}^i \neq \emptyset$ for any $i > \delta$. $F(S, p)$ is continuous in p for any sequence of prices $\{p^i\}_i$ converging to p , that is $\lim_{i \rightarrow \infty} p^i = p$, we have that $F(S, p) = cl(\{F(S, p^i)\}_i)$.

Definition 2 (Resource Transmission Game)

Given an intermediation problem (u, F) , the **resource transmission game** is a sequential game of perfect information, such that:

- The strategy space of intermediary n is $[0, P_n]$, where $0 \leq P_n \leq +\infty$. The **strategy of intermediary n** is to set a price $p_n \in [0, P_n]$ that the planner has to pay, in order to use his ability. Let $p = (p_1, \dots, p_N)$ be the vector of strategies employed by the intermediaries.
- The **strategy of the planner** is a pair of functions $b : \mathbb{R}_+^N \rightarrow 2^{\mathcal{N}}$ and $x : \mathbb{R}_+^N \rightarrow A$, such that $x(p) \in F(b(p), p)$.
- The objective of each intermediary is to maximize the price paid by the planner. The **utility $V^n(p, b, x)$ of intermediary n** is $V^n(p, b, x) = p_n$ if $n \in b(p)$, and $V^n(p, b, x) = 0$ if $n \notin b(p)$. That is, only the selected intermediaries might acquire positive utility equating to their proposed prices.
- The planner only pays for the intermediaries used, and the **utility of the planner** equals $u(x(p), \sum_{n \in b(p)} p_n)$.

The rest of the paper imposes no restrictions on the strategy space of the intermediaries.

Given prices p and a group of intermediaries S , the set of *utility-maximizing allocations* at (S, p) is the set $x^*(S, p) \subset F(S, p)$, such that $x \in x^*(S, p)$, if and only if $u(x, p(S)) \geq u(x', p(S))$ for any $x' \in F(S, p)$. Since $F(S, p)$ is compact and u is continuous, the set $x^*(S, p)$ is non-empty. The *maximal utility* $u^*(S, p)$, given prices p for group S , equals $u(x, p(S))$ for $x \in x^*(S, p)$.⁴ Given prices p , the group $S(p) \subset \mathcal{N}$ is a *utility-maximizing group* at p if $u^*(S(p), p) \geq u^*(T, p)$ for any $T \in 2^{\mathcal{N}}$.

Definition 3 (Subgame Perfect Nash Equilibrium)

The strategies of the intermediaries, $p \in \mathbb{R}_+^N$, and planner, (b, x) , are a **subgame perfect Nash equilibrium (SPNE)** if (a) $V^n(p, b, x) \geq V^n(\tilde{p}_n, p_{-n}, b, x)$ for any $n \in \mathcal{N}$ and $\tilde{p}_n \in \mathbb{R}_+$, and (b) for any prices p , whereby the selected group $b(p)$ is a utility-maximizing group at p and $x(p) \in x^*(b(p), p)$.

When there is no confusion, an SPNE (p, b, x) will simply be referred to as the “vector of prices” p .

Our objective is to study conditions surrounding the intermediation problem, such that at equilibrium intermediaries have no market power to price above zero. This resembles

⁴The maximal utility $u^*(S, p) = \max_{x \in F(S, p)} u(x, p(S))$ depends on prices p rather than on $p(S)$, because $F(S, p)$ depends on p .

the case of a perfectly competitive market, in which the planner is able to transmit the full resource to the agents as if the planner were directly connected to the agents. We capture this notion in the definition of free intermediation equilibrium, where the final allocation implemented as an SPNE is the welfare equivalent for the planner if all the intermediaries price at zero.

Definition 4 (Free Intermediation Equilibrium)

A **free intermediation equilibrium (FIE)** (p, b, x) is a vector of strategies, such that (p, b, x) is an SPNE and $u(x(p), p(b(p))) = \max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0)$.

Note that an FIE requires that the allocation to agents $x(p)$ and prices paid to intermediaries selected $p_{b(p)}$ are *planner-optimal*, that is, they achieve the maximal utility $\max_{x \in F(\mathcal{N}, \mathbf{0})} u(x, 0)$. However, with an FIE, not all intermediaries need to be pricing at zero.

Definition 5 (Indirect Utility Function)

- The **indirect utility function** $v(p) = \max_{x \in F(S, p), S \in 2^{\mathcal{N}}} u(x, p(S))$ is the maximal utility that the planner can achieve given the prices p .
- The **indirect utility function without intermediary n** is denoted by $v_{-n}(p_{-n}) = v_{\mathcal{N} \setminus \{n\}}(p_{-n}) = \max_{x \in F(S, p), S \in 2^{\mathcal{N} \setminus \{n\}}} u(x, p(S))$.

Note that since the OPF F and utility function u are non-increasing in prices, the indirect utility function v is non-increasing in prices as well. Continuity of the indirect utility function is guaranteed, due mainly to the continuity of the utility function u and OPF F . This is proven in Lemma 2 and used in the main results.

3 Free Intermediation Equilibria

We formalize below a situation in which a strict decrease in the prices of the intermediaries leads to a strict increase in the utility of the planner.

Definition 6 (Monotonicity)

The problem (u, F) is **monotonic in prices** if, for any $S \subseteq \mathcal{N}$, intermediary $n \in S$, and prices p and p' such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$ we have that for any $x \in F(S, p)$ such that $u(x, p(S)) > u((0, \dots, 0), 0)$, there exists $y \in F(S, p')$ such that $u(y, p'(S)) > u(x, p(S))$.

Monotonicity in prices is satisfied when the planner is strictly better off, due either to paying less for intermediation or by having strictly better-off options available to choose from (Remark 1 in Han and Juarez [9]). When there is no confusion, we refer to the problem

(u, F) as *monotonic* instead of monotonic in prices. Monotonicity in prices implies that the indirect utility function of the planner is monotonic in prices (Lemma 3(a)).

While in situations where the planner uses an exogenous tie-breaking rule, equilibrium existence is not guaranteed, we build on a result posited by Reny[29], to prove that an equilibrium exists in our specification of the game that allows strategic tie-breaking for the planner.

Lemma 1 (Existence of SPNE)

There exists an SPNE in every resource transmission game generated by a monotonic problem (u, F) , such that:

- *the strategy space of every intermediary n has a finite maximum price P_n and*
- *the planner is (weakly) better-off not selecting any group instead of a group with an intermediary posting his maximal price. That is, for each group of intermediaries $S \subseteq \mathcal{N}$ and intermediary $n \in S$, $u(x, P_n) \leq u((0, \dots, 0), 0)$ for any $x \in F(S, (P_n, \mathbf{0}_{-n}))$.*

One important utility-maximizing group of intermediaries occurs when the posted prices are zero. The structure of such groups, especially with regard to their intersection, is important in understanding the existence of free intermediation equilibria.

Theorem 1 (Existence of FIE)

Let $S_1(\mathbf{0}), S_2(\mathbf{0}), \dots, S_J(\mathbf{0})$ be the utility-maximizing groups at prices $\mathbf{0}$. There is a free intermediation equilibrium, if and only if $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$.

The existence of an FIE implies that no intermediary belongs to all the utility-maximizing groups at zero prices. The intuition is that if a group of intermediaries belongs to this intersection, these intermediaries will have sufficient market power to price above zero, thus creating an equilibrium that is not planner-optimal. The extreme case occurs in the traditional Bertrand competition model, where symmetric producers with zero marginal cost of production compete in price and the unique equilibrium prices equal zero. However, when producers have different marginal costs of production, an equilibrium where producers price above zero is possible.

3.1 Robust SPNE

Multiplicity of equilibria may occur, as shown in Section 1.1. However, we can argue that some of the equilibria might not be as likely to occur because there are groups of intermediaries who may gain by offering their abilities at a lower price. In particular, intermediaries who are not chosen by the planner always have the incentive to undercut their prices in the

hope of being chosen. In this section we look at the robustness of SPNE, whereby intermediaries who are not used by the planner cannot jointly decrease their prices and affect the equilibrium. Formally, an SPNE is robust when those intermediaries who are not used by the planner charge prices equal to zero.

Definition 7 (A robust SPNE)

The subgame perfect Nash equilibrium (p, b, x) is **robust** if intermediaries who are not used by the planner post zero prices. That is, the SPNE (p, b, x) is robust whenever $n \notin b(p)$ implies $p_n = 0$.

A robust SPNE is an equilibrium refinement weaker than a collusion-proof Nash equilibrium, since intermediaries who are not selected by the planner cannot collude or gain by lowering their prices.⁵ Note that at a robust SPNE it is possible for intermediaries who are used by the planner to charge positive prices, and thus a robust SPNE might not be an FIE. Furthermore, it is possible for a robust SPNE not to exist, or for multiple robust SPNEs to exist. Our analysis in this section will focus on finding the conditions surrounding the intermediation problem to ensure the existence and uniqueness of the robust SPNE.

The intermediation problem is cross-monotonic whenever the ranking of groups at the maximal utility is maintained as prices change.

Definition 8 (Cross-Monotonic)

The problem (u, F) is **cross-monotonic** if $\max_{x \in F(S, \mathbf{0})} u(x, 0) \leq \max_{x \in F(T, \mathbf{0})} u(x, 0)$, in which case $\max_{x \in F(S, p)} u(x, p(S)) \leq \max_{x \in F(T, p)} u(x, p(T))$ for any p with $p(S) = p(T)$.

Cross-monotonicity is satisfied in a large class of intermediation problems, for instance when the utility function u is product separable and the OPF F is independent of prices, or when the OPF F is product separable and the utility function u is independent of prices and homothetic (Remark 2 in Han and Juarez [9]).

Theorem 2 (Uniqueness of a Robust SPNE)

Assume that the problem (u, F) is monotonic and cross-monotonic. $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, if and only if the price $\mathbf{0}$ is the unique robust SPNE.

This theorem complements the existing results of Theorem 1. Under the conditions of monotonicity in prices and the cross-monotonicity of the intermediation problem, and when no group of intermediaries belongs to any utility-maximizing group, there exists a uniquely robust SPNE.

⁵A robust SPNE does not prevent group manipulations by individuals who are selected by the planner. In fact, it is easy to see that a full coalition-proof Nash equilibrium does not exist for almost any intermediation problem.

The proof of the converse of this theorem is readily seen. Indeed, if the price $\mathbf{0}$ is a (robust) SPNE, then $\mathbf{0}$ is also an FIE. Therefore, according to Theorem 1, $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$. The other direction in the proof is substantially more difficult than the proof of Theorem 1. It requires a variety of intermediate results related to the continuity and monotonicity of the indirect utility function (Lemmas 2 and 3) as well as a result that provides conditions that an SPNE satisfies even when it is not an FIE (Lemma 4). The intuition of the proof is as follows. Suppose that $\mathbf{0}$ is not the unique robust SPNE. Then, there exists another price p that is a robust SPNE where $p \neq \mathbf{0}$. Suppose that S_1 is the selected group of intermediaries at price p and includes all the intermediaries with strictly positive price at p (by robust SPNE). Since $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, we can find another group of intermediaries S_2 which is a utility-maximizing group at $\mathbf{0}$, and where some intermediary with strictly positive price at p is not in S_2 (so, $p(S_1) > p(S_2)$). Let p' be a price vector such that $p'(S_2) = p(S_2) = p'(S_1)$. By cross-monotonicity, $u^*(S_2, p') \geq u^*(S_1, p')$. Furthermore, by monotonicity, $u^*(S_1, p') > u^*(S_1, p)$. Since $u^*(S_2, p) = u^*(S_2, p')$ because $p'(S_2) = p(S_2)$, $u^*(S_2, p) > u^*(S_1, p)$. This is a contradiction that S_1 is selected at price p .

4 Conclusion

This paper investigates how intermediation affects the resource transmission between a planner and agents. We build a game theoretical model to study the market power of intermediaries charging the planner a price for the use of their abilities to transmit the resource. We discover and describe the necessary and sufficient conditions for the existence of FIEs and the uniqueness of a robust SPNE.

This paper is a forerunner to the analysis of the transmission of a divisible resource from a planner to agents via intermediaries. Future work should include the case of imperfect information about the OPFs, competition between multiple planners, and more complex pricing structures that include variable-pricing instead of fixed pricing.⁶

Appendix: Proofs

Lemma 2 (Continuity of the Indirect Utility Function)

For any problem (u, F) , the indirect utility function $v(p)$ is continuous in p . Furthermore, the maximal utility $u^(S, p)$, given group S , is continuous in prices p .*

⁶In this regard, Han and Juarez[8] consider a more stylistic resource transmission game in a network, whereby the planner has incomplete information about the abilities of the intermediaries to transmit the good to agents. Strategy-proof mechanisms, whereby intermediaries report their abilities to the planner, are characterized.

Proof. We first show that the indirect utility function is continuous. We prove this in five steps, breaking down the problem into converging sequences bounded from above and below the limit price.

Consider a decreasing sequence $\{p^i\}$, s.t. $\lim_{i \rightarrow \infty} p^i = p$, $p^i \geq p$. The indirect utility function $v(p)$ is non-increasing in p , since u is non-increasing in the aggregate prices and F is non-increasing in prices. Thus, $v(p^i) \leq v(p)$ for any i . $\lim_{i \rightarrow \infty} v(p^i)$ exists for the monotonicity of $\{v(p^i)\}_i$. Therefore, $\lim_{i \rightarrow \infty} v(p^i) \leq v(p)$. From Definition 1, for any S , $F(S, p^i) \subseteq F(S, p)$. Let $S = S(p)$ be a utility-maximizing group and $x(p)$ be the optimal allocation at prices p , in which case $v(p) = u(x(p), p(S))$. Since $F(S, p)$ is continuous in p , there exists a sequence of allocations $\{x^i\}_i$, s.t. $x^i \in F(S, p^i)$ for all i , $\lim_{i \rightarrow \infty} x^i = x(p)$. Since the utility function $u(x, t)$ is continuous, and $\lim_{i \rightarrow \infty} p^i(S) = p(S)$, then $\lim_{i \rightarrow \infty} u(x^i, p^i(S)) = u(x(p), p(S))$. Since $v(p^i) \geq u(x^i, p^i(S))$, then $\lim_{i \rightarrow \infty} v(p^i) \geq \lim_{i \rightarrow \infty} u(x^i, p^i(S)) = u(x(p), p(S)) = v(p)$. This inequality, together with the inequality above, implies that $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

For any sequence $\{p^i\}$ with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \geq p$. We can find a decreasing sequence $p^{i'}$, s.t. $p^{i'} \geq p^i$ and $\lim_{i \rightarrow \infty} p^{i'} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{i'}) \leq \lim_{i \rightarrow \infty} v(p^i) \leq v(p)$, and since $\lim_{i \rightarrow \infty} v(p^{i'}) = v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

Consider an increasing sequence $\{p^i\}$, with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \leq p$. There exist limits for the monotonic decreasing sequence $v(p^i)$ and $v(p^i) \geq v(p)$, so $\lim_{i \rightarrow \infty} v(p^i) \geq v(p)$. Assume $v(p^i) = u^*(S^i, p^i)$, the group of intermediary $S^i \in 2^{\mathcal{N}}$, there exists group S , s.t. $\{p^k\}$ which is a subsequence of $\{p^i\}$, $S^k = S$ and $\lim_{k \rightarrow \infty} u^*(S, p^k) = \lim_{i \rightarrow \infty} v(p^i)$. Assume x^k is the utility-maximizing allocation in $F(S, p^k)$, $u^*(S, p^k) = u(x^k, p^k(S))$, $x^k \in F(S, p^k) \subset F(\mathcal{N}, \mathbf{0}) \subset \mathbb{R}_+^M$. $F(\mathcal{N}, \mathbf{0})$ is compact, so $F(\mathcal{N}, \mathbf{0})$ is sequentially compact, there exists a convergent subsequence of x^k , name as x^h , s.t. $\lim_{h \rightarrow \infty} x^h = x$, $x^h \in F(S, p^h)$. OPF is continuous in prices p , so $x \in F(S, p)$. Since $\lim_{h \rightarrow \infty} x^h = x$ and $\lim_{h \rightarrow \infty} p^h = p$, $\lim_{h \rightarrow \infty} v(p^h) = \lim_{h \rightarrow \infty} u(x^h, p^h(S)) = u(x, p(S)) \leq u^*(S, p) \leq v(p)$. While $\lim_{h \rightarrow \infty} v(p^h) = \lim_{i \rightarrow \infty} v(p^i) \geq v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

For any sequence $\{p^i\}$ with $\lim_{i \rightarrow \infty} p^i = p$, $p^i \leq p$. We can find an increasing sequence $p^{i'}$, s.t. $p^{i'} \leq p^i$ and $\lim_{i \rightarrow \infty} p^{i'} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{i'}) \geq \lim_{i \rightarrow \infty} v(p^i) \geq v(p)$, and since $\lim_{i \rightarrow \infty} v(p^{i'}) = v(p)$, we have $\lim_{i \rightarrow \infty} v(p^i) = v(p)$.

Finally, for any sequence $\{p^i\}$ s.t. $\lim_{i \rightarrow \infty} p^i = p$, we prove that $\lim_{i \rightarrow \infty} v(p^i) = v(p)$. Construct two sequences $\{p^{1i}\}$ and $\{p^{2i}\}$, and let $p_n^{1i} = \min\{p_n^i, p_n\}$ and $p_n^{2i} = \max\{p_n^i, p_n\}$, in which case $p^{1i} \leq p$, $p^{2i} \geq p$ and $p^{1i} \leq p^i \leq p^{2i}$, $\lim_{i \rightarrow \infty} p^{1i} = \lim_{i \rightarrow \infty} p^{2i} = p$. Thus, $\lim_{i \rightarrow \infty} v(p^{1i}) \geq \lim_{i \rightarrow \infty} v(p^i) \geq \lim_{i \rightarrow \infty} v(p^{2i})$. From the above, $\lim_{i \rightarrow \infty} v(p^{1i}) = \lim_{i \rightarrow \infty} v(p^{2i}) = v(p)$, then $\lim_{i \rightarrow \infty} v(p^i) = v(p)$. Thus, the indirect utility function is continuous in price p .

By repeating the same strategy used above, we can demonstrate that the maximal utility $u^*(S, p)$ is continuous in prices p , given group S . ■

Lemma 3 (Monotonicity of the Indirect Utility Function)

For any monotonic problem (u, F) :

- a. The indirect utility function v is monotonic in prices. That is, if $p' < p$ and $v(p) > u((0, \dots, 0), 0)$, then $v(p') > v(p)$.
- b. Consider a utility-maximizing group S at prices p , intermediary $n \in S$, and prices p' , such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. If $v(p) > u((0, \dots, 0), 0)$, then $v(p') > v(p)$.
- c. Consider any group S , intermediary $n \in S$, and prices p' , such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$. If $u^*(S, p) > u((0, \dots, 0), 0)$, then $u^*(S, p') > u^*(S, p)$.

Proof. First note that part *a* is clearly implied by part *b*.

To prove part *b*, assume S is a utility-maximizing group at prices p and such that $v(p) > u((0, \dots, 0), 0)$. Let x be a utility-maximizing allocation in $F(S, p)$. Then, $v(p) = u(x, p(S))$. Given that the problem (u, F) is monotonic, for $n \in S$, and prices p' , such that $p'_n < p_n$ and $p_{-n} = p'_{-n}$, there exists $y \in F(S, p')$, such that $u(y, p'(S)) > u(x, p(S))$, so $v(p') \geq u(y, p'(S)) > u(x, p(S)) = v(p)$.

To prove part *c*, at prices p , S and x are a utility-maximizing group and a utility-maximizing allocation, respectively, so $u^*(S, p) = u(x, p(S))$. (u, F) is monotonic for each $n \in S$ and prices p' , s.t. $p'_n < p_n$ and $p_{-n} = p'_{-n}$, there exists $y \in F(S, p')$ s.t. $u(y, p'(S)) > u(x, p(S))$, so $u^*(S, p') \geq u(y, p'(S)) > u(x, p(S)) = u^*(S, p)$. ■

Lemma 4 (Conditions for an SPNE)

Consider prices and allocation (p, b, x) that is an SPNE. Assume that the utility-maximizing groups at prices p are $S_1(p), \dots, S_J(p)$, in which case:

- a. $\bigcap_{j=1}^J S_j(p) = \emptyset$.
- b. If the problem (u, F) is monotonic in prices, there exists a utility-maximizing group, and without loss of generality, assume it is $S_1(p)$, s.t. $\forall n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$, $p_n = 0$.
- c. $\forall n \in \mathcal{N} \setminus \bigcup_{j=1}^J S_j(p)$, it would not increase the planner's utility even if its price decreased to 0. That is, for $p'_n = 0$ and $p' = (p'_n, p_{-n})$, $v(p) = v(p')$.

Conversely, if there exists a vector of prices p that satisfies conditions (a)-(c), then p is supported by an SPNE (with $b(p) = S_1(p)$).

Proof. a. Suppose $\bigcap_{j=1}^J S_j(p) \neq \emptyset$, then there exists intermediary $n \in \bigcap_{j=1}^J S_j(p)$. The indirect utility function without using intermediary n is $v_{-n}(p_{-n})$. Since intermediary n is

in every utility-maximizing group, the utility-maximizing group in $\mathcal{N} \setminus \{n\}$ would achieve lower utility at prices p_{-n} , $v_{-n}(p_{-n}) < v(p)$. By Lemma 2, the indirect utility function is continuous, and there exists ϵ small enough, such that intermediary n posts price $p'_n = p_n + \epsilon$, $p' = (p'_n, p_{-n})$, in which case there is $v(p) \geq v(p') > v_{-n}(p_{-n})$. Intermediary n would still be paid by the planner with price p'_n , so intermediary n will deviate, and p cannot be an SPNE. Hence, $\bigcap_{j=1}^J S_j(p) = \emptyset$.

b. Suppose that we cannot find such a utility-maximizing group $S(p)$ at prices p that includes all intermediaries with a positive price in $\bigcup_{j=1}^J S_j(p)$. Assume the planner allocates a resource with intermediaries in $S_1(p)$, and intermediary $n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$ posts $p_n > 0$ is not selected, so intermediary n receives 0 utility. Without loss of generality, assume that $n \in S_2(p)$. Consider the price vector $p' = (p_n - \epsilon, p_{-n})$ for some $\epsilon > 0$, such that $p_n > \epsilon$. Since the problem (u, F) is monotonic, by Lemma 3, $v(p') \geq u^*(S_2(p), p') > u^*(S_2(p), p) = v(p)$. Since $v(p) = v_{-n}(p_{-n}) = v_{-n}(p'_{-n})$, then intermediary n will be in the utility-maximizing group at prices p and be selected by the planner at prices p' . Thus, the pay-off for intermediary n increases from 0 to p'_n , which is a contradiction.

c. For intermediary n not in any utility-maximizing group, $S_j(p)$ at prices p . If it lowers its price and improves the maximal utility achieved by the planner, it has incentive to lower its price and get paid. In an SPNE, there is no incentive for the intermediary to decrease the price. The maximal utility of the planner will not increase if $p'_n = 0$. Thus, $v(p) = v(p')$ for $p' = (p'_n, p_{-n})$.

Consider prices p satisfying conditions (a), (b), and (c), assuming that the planner chooses utility-maximizing group $S_1(p)$, including all intermediaries in $\bigcup_{j=1}^J S_j(p)$ with a positive price. For intermediary $n \in \mathcal{N} \setminus \bigcup_{j=1}^J S_j(p)$, it has no incentive to increase the price based on the monotonicity of problem, in which case the utility from groups of intermediaries, including n , will not be maximal for the planner. Condition (c) guarantees that n has no incentive to decrease its price. For intermediary $n \in \bigcup_{j=1}^J S_j(p)$, it has no incentive to increase its price in line with condition (a), and since it is paid in condition (b), there is no incentive to decrease its price. Thus, any price p satisfying conditions (a),(b), or (c) is an SPNE. ■

Proof of Lemma 1

Proof. Consider problem (u, F) and let $(b(p), x(p))$ be a strategy of the planner that maximizes utility for prices p . Let $S_1(p), \dots, S_J(p)$ be the utility-maximizing groups at prices p . We assume the strategy of planner $(b(p), x(p))$ satisfies the following two tie-breaking rules: (1) Assume $\underline{p}_n = \min\{q_n \in [0, P_n] | v(q_n, p_{-n}) \leq u((0, \dots, 0), 0)\}$ is the minimal price of intermediary n at which the empty group maximizes utility. The indirect utility

function is continuous, so price \underline{p}_n exists. If $p_n > \underline{p}_n$, then $n \notin b(p)$. (2) If \emptyset is not a utility-maximizing group at prices p and there exists a utility-maximizing group $S_1(p)$ s.t. $p_n = 0, \forall n \in \bigcup_{j=1}^J S_j(p) \setminus S_1(p)$, then $b(p) = S_1(p)$. If there are multiple groups satisfying this condition, then the planner chooses one of them. These tie-breaking rules guarantee that inaction prevails if the intermediary posts a price too high. When there exist one or more utility-maximizing groups containing all intermediaries who post positive prices, then the planner chooses one of these groups.

Given $(b(p), x(p))$ satisfying conditions above, let \mathcal{G} be the simultaneous move game of the intermediaries, where the strategy of intermediary n is p_n and the pay-off to intermediary n is $V^n(p) = V^n(p, b(p), x(p)), \forall n \in \mathcal{N}$. If p is a Nash equilibrium price in game \mathcal{G} , then strategies $p, (b(p), x(p))$ are an SPNE of the resource transmission game generated by (u, F) . The strategy space $[0, P_n]$ of intermediary n satisfies $P_n < +\infty$.

In order to prove that the game \mathcal{G} has a Nash equilibrium, we verify the conditions of Theorem 3.1 in Reny[29]. Indeed, the strategy set at $[0, P_n]$ is a non-empty, compact, convex subset. We show that the utility function $V^n(p)$ is quasi-concave in p_n , and the game \mathcal{G} is better reply secure.

Step 1. The pay-off function $V^n(p)$ of intermediary n is quasi-concave in p_n .

First show that the maximal utility of the group of intermediaries S at prices p (denoted by $u^*(S, p)$): (i) is decreasing in p_n for $n \in S$ and (ii) is independent of p_n for $n \notin S$. Consider prices p and p' , with $p_{-n} = p'_{-n}$, $p_n < p'_n$ and $n \notin S$. The problem (u, F) is monotonic, from Lemma 3, $u^*(S, p') > u((0, \dots, 0), 0)$, $u^*(S, p') < u^*(S, p)$, for $n \in S$. For $n \notin S$, $F(S, p) = F(S, p')$ from assumption (d) of OPF in definition 1. Furthermore, $u(x, p(S)) = u(x, p'(S))$, for $p(S) = p'(S)$, $\forall x \in F(S, p)$. Hence, $u^*(S, p) = u^*(S, p')$.

The problem (u, F) is price-satiated, $u(x, P_n)$ is decreasing in price, without loss of generality, and so assume $u^*(S, (P_n, \mathbf{0}_{-n})) < u((0, \dots, 0), 0)$ for any $n \in S$, which means p_n will not be paid if the price is too high.

Given the strategy of planner $(b(p), x(p))$, a utility-maximizing group will be chosen. For any group of intermediaries S and T , $n \in S$ and $n \notin T$. As price p_n increases, $u^*(S, p)$ decreases, while $u^*(T, p)$ does not change as p_n changes. Given p_{-n} , define \bar{p}_n by satisfying $\max_S u^*(S, (\bar{p}_n, p_{-n})) = \max_T u^*(T, p)$; otherwise, $\bar{p}_n = 0$. If $p_n < \bar{p}_n$, $u^*(S, p) > u^*(T, p)$, for some S and $\forall T$. If $p_n > \bar{p}_n$, $u^*(S, p) \leq u^*(T, p)$ for some T and $\forall S$. Thus, the pay-off function $V^n(p) = p_n$ when $p_n < \bar{p}_n$, and $V^n(p) = 0$ when $p_n > \bar{p}_n$. If $p_n = \bar{p}_n$, the planner is indifferent to choosing intermediary n or not.

In order to show the quasi-concavity of $V^n(p)$, for any constant $c \geq 0$. If $n \in b(\bar{p}_n, p_{-n})$, $V^n(\bar{p}_n, p_{-n}) = \bar{p}_n$. The upper contour set $\{p_n | V^n(p) \geq c\}$ equals $[c, \bar{p}_n]$ when $c \leq \bar{p}_n$, and

$\{p_n | V^n(p) \geq c\} = \emptyset$ when $c > \bar{p}_n$. If $n \notin b(\bar{p}_n, p_{-n})$, $V^n(\bar{p}_n, p_{-n}) = 0$. The upper contour set $\{p_n | V^n(p) \geq c\}$ equals $[c, \bar{p}_n]$ when $c < \bar{p}_n$, and $\{p_n | V^n(p) \geq c\} = \emptyset$ when $c \geq \bar{p}_n$. Thus, the upper contour set $\{p_n | V^n(p) \geq c\}$ is convex for any constant c .

Step 2. The game \mathcal{G} for intermediaries is better reply secure. That is, for any price p , which is not a Nash equilibrium, there exists intermediary n who can secure a pay-off strictly above $V^n(p)$ at p .

Assume the utility-maximizing group of intermediaries at prices p are $S_1(p), \dots, S_J(p)$. From Lemma 2, the maximal utility $u^*(S, p)$ is continuous in p . The problem (u, F) is monotonic, from Lemma 4, and the price vector p satisfies conditions (a) to (c) if it is a Nash equilibrium. So, if p is not equilibrium prices, at least one condition is not satisfied.

If condition (a) is not satisfied, there exists intermediary $n \in \bigcap_j S_j(p)$. This implies that $u^*(S_j(p), p) > u^*(T, p), \forall T \neq S_j(p), \forall j$. Since $u^*(S, p)$ is continuous in p , for p'_{-n} close to p_{-n} , $u^*(S_j(p), (p_n, p'_{-n})) > u^*(T, (p_n, p'_{-n})), \forall T$. Thus, there exists $p'_n > p_n$, s.t. $u^*(S_j(p), p') > u^*(T, p')$ for some $S_j(p)$ and prices $p' = (p'_n, p'_{-n})$. Thus, group of intermediaries $S_j(p)$, with $n \in S_j(p)$ being the utility-maximizing group for prices p' , intermediary n will secure pay-off $V^n(p') = p'_n > p_n = V^n(p)$.

If condition (b) is not satisfied, there exists intermediary $n \in \bigcup S_j(p)$, who is not used by the planner, and $p_n > 0$. From Step 1, the problem (u, F) is monotonic in prices, so there is $0 < p'_n < p_n$, s.t. for group of intermediaries $S_j(p)$, where $n \in S_j(p)$, $u^*(S_j(p), (p'_n, p_{-n})) > u^*(S_j(p), p) \geq u^*(T, p) = u^*(T, (p'_n, p_{-n})), \forall T, n \notin T$. For any utility-maximizing group S at prices (p'_n, p_{-n}) with $n \in S$, $v((p'_n, p_{-n})) = u^*(S, (p'_n, p_{-n})) > u^*(T, (p'_n, p_{-n}))$. Thus, for p'_{-n} close enough to p_{-n} , we have $u^*(S, p') > u^*(T, p'), \forall T$, and $n \notin T$. For the neighborhood of p_{-n} , intermediary n chooses price p'_n to secure pay-off $V^n(p') = p'_n > 0 = V^n(p)$.

If condition (c) is not satisfied, there exists intermediary n , who could decrease the price from p_n to $p'_n > 0$ to be used by the planner, so there is a utility-maximizing group S at prices (p'_n, p_{-n}) with $n \in S$, and $u^*(S, (p'_n, p_{-n})) > u^*(T, (p'_n, p_{-n})), \forall T$, and $n \notin T$. Thus, for prices p'_n in the neighborhood of p_{-n} , $u^*(S, p') > u^*(T, p')$. Moreover, intermediary n chooses prices p'_n to secure pay-off $V^n(p') = p'_n > 0 = V^n(p)$.

Thus, for all ps that are not equilibrium prices, intermediary n can secure a strictly higher pay-off. The game \mathcal{G} is better reply secure. Hence, game \mathcal{G} has at least one pure-strategy Nash equilibrium, and there exists an SPNE in the resource transmission game generated by problem (u, F) . ■

Proof of Theorem 1

Proof. \Leftarrow) Suppose that $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$. Consider intermediary n , who posts price $p_n > 0$.

Let $p = (p_n, \mathbf{0}_{-n})$. Therefore, for any group S^n , such that $n \in S^n$, $u^*(S^n, p) \leq u^*(S^n, \mathbf{0}) \leq u^*(S_j(\mathbf{0}), \mathbf{0})$ for any j . Since $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, there exists group $S_j(\mathbf{0})$, such that $n \notin S_j(\mathbf{0})$. Hence, intermediary n will not be chosen by the planner at p . Note that in the case where $u^*(S^n, p) = u^*(S_j(\mathbf{0}), \mathbf{0})$, the planner chooses a group of intermediaries that posts zero prices, even though it is indifferent to some groups of intermediaries with positive prices.

\Rightarrow) Assume p is an FIE. Let $S_1(p), \dots, S_{J'}(p)$ be the set of utility-maximizing groups at prices p . From Lemma 4 (a), there is $\bigcap_{j=1}^{J'} S_j(p) = \emptyset$. We then show that $S = S_j(p)$ is also a utility-maximizing group at prices $\mathbf{0}$. Since p is an FIE, there exists a planner-optimal allocation x , such that $x \in F(S, p)$. Furthermore, $u(x, 0) = u^*(S_k(\mathbf{0}), \mathbf{0})$ for any k . Since $x \in F(S, p)$ and $F(S, p) \subseteq F(S, \mathbf{0})$ by the monotonicity of F , then $x \in F(S, \mathbf{0})$. Thus, x is a feasible and utility-maximizing allocation in $F(S, \mathbf{0})$. Therefore, $u^*(S, \mathbf{0}) = u(x, 0) = u^*(S_k(\mathbf{0}), \mathbf{0})$. Hence, S is a utility-maximizing group at prices $\mathbf{0}$. Finally, if p is an FIE, then $\bigcap_{i=1}^{J'} S_i(p) = \emptyset$, and $\bigcap_{i=1}^J S_i(\mathbf{0}) \subseteq \bigcap_{i=1}^{J'} S_i(p)$. Hence, $\bigcap_{i=1}^J S_i(\mathbf{0}) = \emptyset$. ■

Proof of Theorem 2

Proof.

The converse has already been shown in the text. To prove the forward part, we will use Lemmas 3 and 4.

Assume the strategy of planner $(b(p), x(p))$ satisfies the tie-breaking rules in Lemma 1. Let $\bar{u} = u^*(S_1(\mathbf{0}), \mathbf{0}) = \dots = u^*(S_J(\mathbf{0}), \mathbf{0})$ be the maximal utility achieved by the planner when the prices are zero. Assume there is a robust SPNE with $p \neq \mathbf{0}$. Then, when $\forall p_n > 0$, intermediary n is used in the utility-maximizing group of intermediaries at prices p . From part (b) of Lemma 4, there exists a utility-maximizing group S_1 at prices p , such that all the intermediaries with positive prices are in the group S_1 , that is, $\forall p_n > 0$, intermediary $n \in S_1$. The maximal utility of the planner for the use of group S_1 is $u^*(S_1, p)$. Since $\bigcap_{j=1}^J S_j(\mathbf{0}) = \emptyset$, for any intermediary k with $p_k > 0$, there exists $S_j(\mathbf{0})$, such that $k \notin S_j(\mathbf{0})$. Let $S_2 = S_j(\mathbf{0})$. $p(S_1) = \sum_{p_n > 0} p_n > \sum_{n \in S_2} p_n = p(S_2)$. S_2 is the utility-maximizing group at prices $\mathbf{0}$. $\bar{u} = u^*(S_2, \mathbf{0}) \geq u^*(S_1, \mathbf{0})$.

Since intermediary $k \in S_1$ and $k \notin S_2$, there are two cases: $S_2 \subset S_1$ or $S_2 \setminus S_1 \neq \emptyset$.

If $S_2 \subset S_1$, let the prices p^1 satisfy $p_{S_2}^1 = p_{S_2}$ and $p_{N \setminus S_2}^1 = \mathbf{0}_{N \setminus S_2}$, thus $u^*(S_2, p^1) = u^*(S_2, p)$. Then, $p(S_2) = p^1(S_2) = p^1(S_1)$. From cross-monotonicity, $u^*(S_2, p^1) \geq u^*(S_1, p^1)$. At the same time, $p_{S_1}^1 \leq p_{S_1}$ and $p_{S_1}^1 \neq p_{S_1}$. If $u^*(S_1, p) > u((0, \dots, 0), 0)$, the problem (u, F) is monotonic, by Lemma 3, $u^*(S_1, p^1) > u^*(S_1, p)$. Thus, $u^*(S_2, p) > u^*(S_1, p)$, which contradicts that S_1 is a utility-maximizing group at prices p . If $v(p) = u((0, \dots, 0), 0)$, then $\forall n, p_n > 0, p_n \geq \underline{p}_n(p_{-n})$, because $\underline{p}_n(p_{-n}) = \min\{q_n \in [0, P_n] | v(q_n, p_{-n}) \leq u((0, \dots, 0), 0)\}$, and $n \in b(p)$. Given the tie-breaking rules above, if $p_n > \underline{p}_n(p_{-n})$, intermediary n will

not be paid, so $p_n = \underline{p}_n(p_{-n})$. Problem (u, F) is monotonic, $\forall 0 < p'_n < p_n$, so there is $v(p'_n, p_{-n}) > u((0, \dots, 0), 0)$. Thus, utility increases as p_n decreases, $u^*(S_1, p^1) > u^*(S_1, p)$ holds, and $u^*(S_2, p) = u^*(S_2, p^1) \geq u^*(S_1, p^1) > u^*(S_1, p)$.

If $S_2 \setminus S_1 \neq \emptyset$, assume intermediary $i \in S_2$ and $i \notin S_1$, let prices p^2 satisfy $p^2_{S_1} = p_{S_1}$ and $p^2_i = p(S_1) - p(S_2)$, and $p^2_{N \setminus (S_1 \cup \{i\})} = 0$, in which case we have $p^2(S_1) = p^2(S_2)$. By cross-monotonicity, $u^*(S_2, p^2) \geq u^*(S_1, p^2) = u^*(S_1, p)$. At the same time, $p^2_{S_2} \geq p_{S_2}$ and $p^2_{S_2} \neq p_{S_2}$, and thus $u^*(S_2, p) > u^*(S_2, p^2)$ through the monotonicity of the problem (u, F) . So $u^*(S_2, p) > u^*(S_1, p)$, which is a contradiction with S_1 being a utility-maximizing group at prices $p \neq \mathbf{0}$. Hence, $p = \mathbf{0}$ is the uniquely robust SPNE. ■

References

- [1] Joseph Bertrand. *Review of Walras's théorie mathématique de la richesse sociale and Cournot's recherches sur les principes mathématiques de la théorie des richesses in Cournot oligopoly: Characterization and applications.* edited by A. F. Daughety. Cambridge University Press.(1988), 1883.
- [2] Philippe Bich and Rida Laraki. On the existence of approximate equilibria and sharing rule solutions in discontinuous games. *Theoretical Economics*, (12):79–108, 2017.
- [3] Larry Blume, David Easley, Jon Kleinberg, and Eva Tardos. Trading networks with price-setting agents. In *Proceedings of the 8th ACM conference on Electronic commerce*, pages 143–151. ACM, 2007.
- [4] Olivier Bochet, Rahmi Ilkılıç, Hervé Moulin, and Jay Sethuraman. Balancing supply and demand under bilateral constraints. *Theoretical Economics*, 7(3):395–423, 2012.
- [5] Syngjoo Choi, Andrea Galeotti, and Sanjeev Goyal. Trading in networks: Theory and experiments. *Journal of the European Economic Association*, 2016.
- [6] Daniele Condorelli and Andrea Galeotti. Strategic models of intermediation networks. In *The Oxford Handbook of the Economics of Networks*. 2016.
- [7] Douglas M Gale and Shachar Kariv. Trading in networks: A normal form game experiment. *American Economic Journal: Microeconomics*, 1(2):114–132, 2009.
- [8] Lining Han and Ruben Juarez. Incentive compatible resource transmission with punishment. *Mimeo University of Hawaii*, 2018.

- [9] Lining Han and Ruben Juarez. Supplementary material for free intermediation in resource transmission. <https://perma.cc/RV5L-N76A>, 2018.
- [10] Jens Leth Hougaard, Juan D Moreno-Tertero, Mich Tvede, and Lars Peter Østerdal. Sharing the proceeds from a hierarchical venture. *Games and Economic Behavior*, 102:98–110, 2017.
- [11] Jens Leth Hougaard, Hervé Moulin, and Lars Peter Østerdal. Decentralized pricing in minimum cost spanning trees. *Economic Theory*, 44(2):293–306, 2010.
- [12] Jens Leth Hougaard and Mich Tvede. Truth-telling and nash equilibria in minimum cost spanning tree models. *European Journal of Operational Research*, 222(3):566–570, 2012.
- [13] Jens Leth Hougaard and Mich Tvede. Minimum cost connection networks: Truth-telling and implementation. *Journal of Economic Theory*, 157:76–99, 2015.
- [14] Jens Leth Hougaard, Mich Tvede, and Lars Peter Østerdal. Cost sharing in chains and other fixed trees. *Discussion Papers of Business and Economics*, 12, 2013.
- [15] Matthew O Jackson. *Social and Economic Networks*. Princeton University Press, 2010.
- [16] Ruben Juarez. The worst absolute surplus loss in the problem of commons: random priority versus average cost. *Economic Theory*, 34(1):69–84, 2008.
- [17] Ruben Juarez. Group strategyproof cost sharing: the role of indifferences. *Games and Economic Behavior*, 82:218–239, 2013.
- [18] Ruben Juarez and Rajnish Kumar. Implementing efficient graphs in connection networks. *Economic Theory*, 54(2):359–403, 2013.
- [19] Ruben Juarez, Kohei Nitta, and Miguel Vargas. Profit sharing and efficient time allocations. *Under Review*, 2018.
- [20] Ruben Juarez and Michael Wu. Routing proofness in congestion-prone networks. *Under review*, 2015.
- [21] Maciej H Kotowski and C Matthew Leister. Trading networks and equilibrium intermediation. 2014.
- [22] Mihai Manea. Intermediation and resale in networks. *Mimeo MIT*, 2016.

- [23] Andreu Mas-Colell, Michael Dennis Whinston, Jerry R Green, et al. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.
- [24] Nimrod Megiddo. Optimal flows in networks with multiple sources and sinks. *Mathematical Programming*, 7(1):97–107, 1974.
- [25] Hervé Moulin. Pricing traffic in a spanning network. *Games and Economic Behavior*, 86:475–490, 2014.
- [26] Herve Moulin. Entropy, desegregation, and proportional rationing. *Journal of Economic Theory*, 162:1–20, 2016.
- [27] Herve Moulin and Jay Sethuraman. The bipartite rationing problem. *Operations Research*, 61(5):1087–1100, 2013.
- [28] Hervé Moulin and Rodrigo A Velez. The price of imperfect competition for a spanning network. *Games and Economic Behavior*, 81:11–26, 2013.
- [29] Philip J Reny. On the existence of pure and mixed strategy nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056, 1999.
- [30] Jan-Peter Siedlarek. Intermediation in networks. *Mimeo*, 2015.
- [31] Leo K Simon and William R Zame. Discontinuous games and endogenous sharing rules. *Econometrica*, pages 861–872, 1990.