

Implementing efficient graphs in connection networks

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Abstract We consider the problem of sharing the cost of a network that meets the connection demands of a set of agents. The agents simultaneously choose paths in the network connecting their demand nodes. A mechanism splits the total cost of the network formed among the participants. We introduce two new properties of implementation. The first property, Pareto Nash implementation (PNI), requires that the efficient outcome always be implemented in a Nash equilibrium and that the efficient outcome Pareto dominates any other Nash equilibrium. The average cost mechanism and other asymmetric variations are the only mechanisms that meet PNI. These mechanisms are also characterized under strong Nash implementation. The second property, weakly Pareto Nash implementation (WPNI), requires that the least inefficient equilibrium Pareto dominates any other equilibrium. The egalitarian mechanism (EG) and other asymmetric variations are the only mechanisms that meet WPNI and individual rationality. EG minimizes the price of stability across all individually rational mechanisms.

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1 Introduction

1.1 Network cost-sharing problem

We consider the problem of sharing the cost of a congestion-free network that meets the connection demands of a set of agents. A network may represent the network of roads, Internet network or telecommunication network, where each link has a cost. The cost of a link may refer to the maintenance cost of a road segment, construction cost of a connection link between servers, etc. The agents simultaneously choose paths in the network to connect their unique sources to their unique sinks, and the choice of paths of all the agents leads to the network that must be constructed and/or maintained. A mechanism splits the total cost of the network formed among the participants. We focus on the case where agents only care about being connected at the minimal cost.¹ This type of problem is well studied in the literature (see [Anshelevich et al. 2004](#); [Chen et al. 2008](#); [Epstein et al. 2007, 2009](#); [Fiat et al. 2006](#) and other papers below) and arises in many contexts ranging from water distribution systems, road networks, telecommunications services, and multicast transmission to large computer networks such as the Internet.

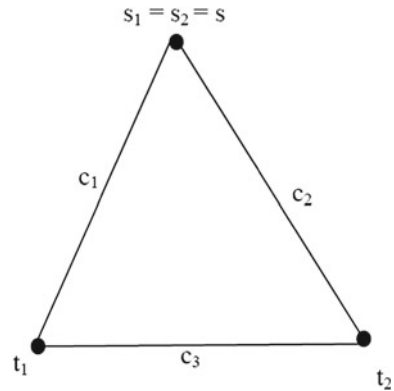
A challenge to designing mechanisms arises because a mechanism induces a game among the agents who choose their paths strategically. Therefore, the traditional objectives of the social planner may be conflicting, and thus, it may not be obvious to choose one mechanism over the other. We focus on mechanisms that are efficient, that is, the ones that minimize the cost of the network formed at the equilibrium of the game when the agents choose their paths strategically.² In other words, the planner wants to design a mechanism that implements the efficient graph.

Consider the network in [Fig. 1](#) with two agents located at the common source s and interested in going to the sinks t_1 and t_2 , respectively. The Shapley mechanism (Sh, [Chen et al. 2008](#)), which divides the cost of every edge equally across its users, may provide the wrong incentives to the players and they may end up choosing an inefficient graph at equilibrium. Indeed, if $c_3 < c_2 < c_1$, then the efficient graph is formed by the links st_2 and t_2t_1 , agent 1 chooses the path (st_2t_1) and agent 2 chooses the path (st_2) . However, at equilibrium, agent 1 does not choose the efficient path whenever $\frac{c_2}{2} + c_3 > c_1$. In general networks, even the best equilibrium of the Sh can be as costly as $H(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ times the cost of the optimal graph, where

¹ This framework can be considered as a benchmark to more general problems where agents get different utility from the path choice, for instance, links facing congestion and idiosyncratic intensities of use of a link/path by agents, among others.

² Here efficient outcome is the outcome that maximizes the sum of utilities of the agents. In our framework, this is equivalent to the outcome that minimizes the sum of costs, that is, the cost of the network fulfilling all the connection demands. This notion of efficiency is different from other optimality notions studied in the literature, such as [Myerson \(1981\)](#), [Ulku \(2012\)](#).

Fig. 1 Network with two agents at the common source s and two different sinks t_1 and t_2



k is the number of users (Anshelevich et al. 2004). Therefore, there is a need to find mechanisms that implement the efficient graph.

1.2 Robust efficient implementation

The celebrated literature on full implementation of the efficient outcome has more often than not hit impossibilities (see Maskin and Sjoström 2002 for a comprehensive survey). In the growing literature in computer science (and more recently in economics³), two measures of efficiency loss have been very fruitfully studied. On the one hand, there is the traditional price of anarchy (PoA, Koutsoupias and Papadimitriou 1999), which computes the ratio of the worst equilibrium over the efficient outcome. On the other hand, there is the price of stability (PoS, Anshelevich et al. 2004), which computes the ratio of the best equilibrium over the efficient outcome. Both of these measures have been very effective in selecting second-best mechanisms. However, these approaches lack economic justification and thus seem to be quite arbitrary in the absence of a compelling equilibrium selection rule. The following types of natural questions automatically arise. Why should we study the worst case performance of a mechanism? Why not the best case scenario?

In this paper, we fill this gap by providing new equilibrium selection rules. We introduce two new properties of implementation.⁴ The first property, Pareto Nash implementation (PNI), requires that the efficient outcome always be implemented in a Nash equilibrium (NE) and that the efficient outcome Pareto dominates any other NE. Contrary to the traditional literature on full implementation, PNI may implement multiple

³ See for instance Moulin (2008) for a comparison of three cost-sharing mechanisms using the price of anarchy. See Juárez (2008) for a comparison of two mechanisms in the problem of commons using the worst-absolute surplus loss.

⁴ Notice that the problem of implementation we consider here differs from the one considered in the traditional literature since there is no private information on the part of agents. However, it is the same problem in the sense that the planner has an objective function (here efficiency) and the cost-sharing mechanism induces a game whose equilibrium is the outcome obtained.

inefficient equilibria; however, the efficient equilibrium is always implemented and Pareto dominates any other equilibrium.

The second property, weakly Pareto Nash implementation (WPNI), requires that the least inefficient equilibrium Pareto dominates any other equilibrium. That is, WPNI might implement several equilibria (and all of them might be inefficient), but the least inefficient equilibrium should be preferred by all the agents to any other equilibrium. Thus, if the NE is a good predictor of the outcome implemented by the mechanism, the least inefficient equilibrium stands out as a reasonable selection.⁵

PNI and WPNI suggest implementation with equilibrium selection rather than arbitrarily choosing the best or the worst equilibria as the benchmark to measure the performance of a mechanism. Either the efficient equilibrium is implemented under PNI, or the least inefficient is implemented under WPNI. Therefore, PNI and WPNI point out a class of environments where the PoS, or any other measure of inefficiency that uses the best equilibrium as a benchmark, can be justified.⁶

1.3 Minimal information setting

We study the problem of designing mechanisms when the information available to the designer is minimal. Specifically, we focus on the case where the mechanism splits the total cost of the network formed by using only the *costs of the paths demanded by the agents* and the *total cost of the network formed*.

This setting has multiple applications. For instance, consider the network of roads in a state, district, or country to be financed by the users of the roads. The procurement of information on the exact paths used by drivers requires the compulsory installment of GPS (global positioning system) in all vehicles and the data to be stored and updated by a central taxing authority. Because of privacy issues, this may not be possible politically (see, for example, [Ryan 2012](#)). However, a tax based on the number of miles driven can be implemented without raising such privacy concerns. Road maintenance taxes based on the miles driven by every user have been used in pilot programs in Oregon since January 2009, and other states such as Ohio, Pennsylvania, Colorado, Florida, Rhode Island, Minnesota, and Texas are considering them (see [AP 2009a,b](#); [Patterson 2011](#); [Galbraith 2009](#)). In this setting, all the information that the designer (government) has is (a) the roads that are used more frequently (from the traffic data or the data on maintenance requirements) and (b) the miles driven by each vehicle. The maintenance of these roads must be financed by the tax collected from the drivers. This kind of

⁵ Apart from the property of being immune to unilateral deviation, the Pareto optimal NE is also immune to deviation by the grand coalition. Also, pre-play communication leads to the payoff-dominant Pareto optimal NE in many games. See for instance [Calcagno and Lovo \(2010\)](#), [Cooper et al. \(1992\)](#), [Kim \(1996\)](#).

⁶ Note that contrary to PNI, WPNI may not implement the best possible outcome. The definition of WPNI does not rule out the existence of another mechanism whose equilibria (possibly not Pareto ranked) are less inefficient than the equilibria of a WPNI mechanism. However, in our model, as we will see in the results, the mechanisms characterized under WPNI are less inefficient than any possible mechanism that satisfies individual rationality.

environment requires mechanisms where the input is the total cost of the paths⁷ used by the agents rather than the paths themselves.

Moreover, in spite of the information on the paths being available, it may sometimes be desirable to use just the total costs of the paths rather than the paths themselves. Consider, for instance, a big or highly dynamic network structure, where agents join and leave the network continuously. It may be impractical to change the formulae of our mechanism every time the network changes. One such example is sharing the cost of a telephone network or the Internet where the agreement is generally monthly but there are agents entering and leaving the network continuously. Notice that charging the same amount for long distance calls makes sense irrespective of the number of users who share the edges.⁸ Alternative examples include the fare charged by a taxi (which usually depends only on the distance driven) or the division of a joint electricity bill in condominiums. There are normative concerns too for penalizing agents who may not be responsible for the fact that their links are not shared by a lot of users. Examples include electricity/water supply or postal service to remote villages. There is a reasonable case against charging higher prices for these services to the poor villagers living in a small village on the top of a mountain.

This setup has a natural resemblance to the classic rationing problem (also referred to in the literature as a bankruptcy, taxation, or claims problem), where a given amount of a resource (e.g., money) must be divided among beneficiaries with unequal claims on the resource (see [Moulin 2002](#); [Thomson 2003](#) and below for related work).

1.4 Overview of the results

Theorem 1 characterizes the class of mechanisms that satisfy PNI. The mechanisms are monotonic in the total cost and do not depend on the demands of the agents. The average cost mechanism (AC) ([Moulin and Shenker 2001](#); [Juarez 2008](#), which divides the total cost of the network equally among its participants (Theorem 2), is the only symmetric mechanism in this class. These mechanisms are also characterized under strong Nash implementation, which requires the efficient equilibrium to be a strong NE.

The main downside of AC and the above variations is that they do not meet individual rationality (IR, also referred to in the literature as voluntary participation): agents demanding cheap links may pay more than the cost of their demands; thus, they may subsidize agents who demand expensive links. We provide a class of mechanisms that meet both IR and WPNI. The egalitarian mechanism (EG, [Sprumont 1982](#)), a symmetric mechanism reminiscent of AC that meets IR, also satisfies WPNI. Theorem 3 introduces a new class of mechanisms that are non-symmetric variations of EG. Such mechanisms are the only mechanisms that meet WPNI.

⁷ In this example, the cost of a link in the network is proportional to the distance between the nodes that this link connects.

⁸ The choice of path is not a strategy for the telephone user, and thus, the setting is not exactly the same. But, the cost-sharing mechanism has a similar motivation; namely, it is simpler than charging every caller differently based on the path used.

We show that EG has a PoS equal to $H(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k}$, where k is the number of agents in the network. EG is also an optimum across all mechanisms meeting IR under the PoS measure.⁹

1.5 Related literature

The literature on connection networks has mainly focused on the performance of the Sh. For example, [Chen et al. \(2008\)](#) studies the equilibrium behavior of separable mechanisms. The PoS of separable mechanisms with a linear cost-sharing function is at least $H(k)$ (which is $O(\log k)$), where k is the number of agents ([Chen et al. 2008](#)). The number $H(k)$ is also the upper bound on PoS(Sh) in general graphs ([Anshelevich et al. 2004](#)). This upper bound is achieved in directed graphs. If the graph is undirected, PoS(Sh) is lower than $H(k)$. [Fiat et al. \(2006\)](#) finds a new upper bound of $O(\log \log k)$ when the graph is single source and there are no Steiner nodes. [Li \(2009\)](#) finds a new upper bound of $O(\log k / \log \log k)$ for single source networks when Steiner nodes are allowed. [Chen et al. \(2008\)](#) shows that the upper bound in a two player case with a single source is $\frac{4}{3}$. [Kumar \(2010\)](#) finds that $\frac{4}{3}$ is also the upper bound in a general multi-commodity case. [Epstein et al. \(2009\)](#) investigates the conditions on network topologies that admit a strong equilibrium under Sh and finds the upper bound on strong PoA ([Andelman et al. 2009](#)) under Sh to be $H(k)$. [Hougaard and Tvede \(2012\)](#) considers a problem similar to ours where the designer's objective is to implement the minimum cost spanning tree but the private information about the link costs is not known to the designer. They characterize the set of cost-sharing mechanisms under which true revelation of link costs is a NE.

The paper also connects to the literature on the rationing problem (also referred to as bankruptcy, taxation, or claims problem). This literature was started by [O'Neill \(1982\)](#), [Aumann and Maschler \(1985\)](#) and nicely surveyed by [Moulin \(2002\)](#), [Thomson \(2003\)](#). The class of asymmetric parametric rules plays a key role in [Theorem 3](#). [Young \(1987\)](#) characterizes the class of symmetric parametric rules by consistency in the population of agents, continuity and symmetry. This class is extended to richer settings by [Kaminski \(2006\)](#). The axiom of consistency, which is the key component of the parametric rules, has been extensively explored in the literature. See [Dagan and Volij \(1997\)](#), [Kaminski \(2000\)](#), [Thomson \(2007\)](#) for characterizations of the rules that are consistent. [Moulin \(2000\)](#), [Chambers \(2006\)](#), [Thomson \(2003\)](#), [Moulin \(2002\)](#), [Young \(1988\)](#) provide important collections of consistent rules.

2 The model

We denote the set of agents by $\bar{K} = \{1, 2, \dots, k\}$. A network cost-sharing problem is a tuple $N = \langle G, K \rangle$, where $G = (V, E)$ is a network that is directed or undirected such that each edge $e \in E$ has a nonnegative cost c_e . The set

⁹ The Shapley mechanism even though it looks like a natural mechanism in this setting, fails basic tests such as efficiency, symmetry at equilibrium and continuity. It also does not satisfy minimal information, since the cost-share of an agent depends on the number of users of his demanded links.

$K = ((s_1, t_1), (s_2, t_2), \dots, (s_k, t_k))$, where $(s_i, t_i) \in V \times V$ for all $i \in \bar{K}$, represents the sources and sinks that agents want to connect. When there is no confusion, we also denote $K = \bar{K}$ as the set of agents. Let the set of all graphs be \mathbf{G} , and the set of all network cost-sharing problems be denoted by \mathbf{N} .

Given a problem $N \in \mathbf{N}$, a strategy¹⁰ for agent i is a path $P_i \subseteq E$ that connects s_i to t_i . Let the set of paths connecting s_i to t_i be $\Pi_i(N)$. Let $\Pi(N) \equiv \times_{i \in \bar{K}} \Pi_i(N)$ be the set of strategy profiles of all agents in network N . The vector $P = (P_1, \dots, P_k) \in \Pi(N)$ will be used to denote a strategy profile of the agents. When there is no confusion we denote $\Pi_i(N)$ and $\Pi(N)$ simply as Π_i and Π , respectively. Let $G_P = (V, \bigcup_{i \in \bar{K}} P_i)$ be the network formed by the choice of paths by different agents. Let $C(P) = \sum_{e \in G_P} c_e$ be the cost of the graph formed by strategies P .

Let $\mathcal{N} = \{(P, N) | P \in \Pi(N), N \in \mathbf{N}\}$ be the union of all problems with their respective strategies.

Definition 1 A cost-sharing mechanism is a continuous¹¹ mapping $\varphi : \mathcal{N} \rightarrow \mathbb{R}_+^k$ such that

$$\sum_{i \in \bar{K}} \varphi_i(P, N) = C(P) \text{ for all } (P, N) \in \mathcal{N}.$$

A cost-sharing mechanism assigns nonnegative cost-shares to the users of the network based on their demands such that the total cost of the network formed is exactly collected.

Continuity in the mechanism, which at first looks harmless, plays a key role in the proofs.¹² Continuity captures the fact that small perturbations on the demand or cost of the network should not change the total allocation of the cost.

Example 1 – The Sh, divides the cost of every link equally across its users. That is, $Sh_i(P, N) = \sum_{e \in P_i} \frac{c_e}{U(e, P)}$ for all $i \in \bar{K}$, where $U(e, P)$ is the number of users of link e in the strategy profile P .

- The proportional to the stand-alone mechanism, η^{pr} , divides the cost of the network in proportion to every user’s stand-alone cost. That is, $\eta_i^{pr}(P, N) = \frac{SA_i(N)}{SA_1(N) + \dots + SA_k(N)} C(P)$ for all $i \in \bar{K}$, where $SA_i(N) = \min_{P_i \in \Pi_i(N)} C(P_i)$ is the stand-alone cost of agent i in network N .
- The AC, divides the cost of the network formed equally across all users. That is, $AC_i(P, N) = \frac{C(P)}{k}$ for all $i \in \bar{K}$.

The Sh is a separable mechanism, that is, it divides the cost of every link only across its users and adds those costs for all links in the network formed. Alternative separable mechanisms can be constructed by considering different cost-sharing rules for the links, for instance, by giving priority across users. Nevertheless, Sh is the optimal

¹⁰ We use the word strategy to be consistent with the game that will be defined in Sect. 2.1.

¹¹ Continuous with the Euclidean distance as a function of the costs in the network.

¹² It is particularly crucial in the proofs of the key Separability Lemma (Lemma 4) and the proofs of Theorems 1 and 2.

mechanism (using the PoS measure; see below) across all separable mechanisms (Chen et al. 2008). Sh can be computed in polynomial time.

On the other hand, η^{pr} divides the cost of the network in proportion to the stand-alone cost of the agents. Since the stand-alone cost of every agent has to be computed for every network, this mechanism uses the full information of the network.

AC divides the cost of the network formed equally across the users of the network. It is the most EG, reminiscent of the classic head tax rule, where the size of agents' demands is not relevant; only the size of the total cost of the network formed is relevant. AC uses less information than Sh or η^{pr} , since only the total cost of the network formed and the number of agents are needed to compute the cost-sharing allocation. There is no need to know the stand-alone cost of the agents or the users of certain links. As a result, its computation complexity is minimal.

To contrast the allocation of the three mechanisms, consider the network in Fig. 1, where $c_1 = 2, c_2 = 1$ and $c_3 = 1$. Assume that the demand of agent 1 is st_1 and the demand of agent 2 is st_2 . The Sh splits the cost of link st_2 equally among agents; therefore, it allocates payments $Sh_1 = \frac{1}{2} + 1$ and $Sh_2 = \frac{1}{2}$. The stand-alone cost of agent 1 equals 2, and the stand-alone cost of agent 2 equals 1. Therefore, the proportional to the stand-alone mechanism allocates payments $\eta_1^{pr} = \frac{2}{3}(2) = \frac{4}{3}$ and $\eta_2^{pr} = \frac{1}{3}(2) = \frac{2}{3}$. Finally, the AC splits the cost equally; therefore, $AC_1 = AC_2 = 1$.

Definition 2 A cost-sharing mechanism φ uses *minimal information* if for any two problems $N = (G, K)$ and $N' = (G', K')$ and strategies $P \in \Pi(N)$ and $P' \in \Pi(N')$ such that $C(P_i) = C(P'_i)$ for all $i \in \bar{K}$ and $C(P) = C(P')$: $\varphi(P, N) = \varphi(P', N')$.

Minimal information captures the mechanisms that depend only on the cost of the network formed and the cost of the agents' demands. For instance, in Fig. 2, we represent three different networks formed by the demands of three agents. The first coordinate represents the cost of the network formed. The second, third, and fourth coordinates represent the cost of the demands of agents 1, 2, and 3, respectively. A cost-sharing mechanism that uses minimal information would allocate the same payments to the agents in all three problems if the following four conditions hold: (i) the total cost of the network formed does not vary across problems ($w + x + y + z = p + q + r + s = a + b + c + d + e$); (ii) the cost of the demand of agent 1 does not

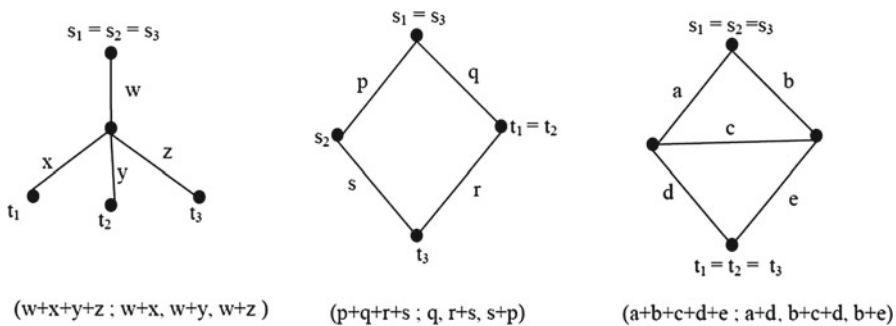


Fig. 2 Equivalence in cost-shares under minimal information

vary across problems ($w + x = q = a + d$); (iii) the cost of the demand of agent 2 does not vary across problems ($w + y = r + s = b + c + d$); and (iv) the cost of the demand of agent 3 does not vary across problems ($w + z = s + p = b + e$).

Neither Sh nor η^{pr} uses minimal information. On the other hand, AC uses only the total cost of the network formed and the number of users; thus, it uses minimal information. More complex mechanisms that use minimal information are discussed in the following sections.

All the mechanisms studied in this paper use minimal information. Minimal information is always assumed and we do not refer to it when there is no confusion.

Let $\mathcal{S}^k = \{(c; y) \in \mathbb{R}_+ \times \mathbb{R}_+^k \mid \max_i y_i \leq c \leq \sum_i y_i\}$.

Lemma 1 *A cost-sharing mechanism φ uses minimal information if and only if there is a continuous function $\xi : \mathcal{S}^k \rightarrow \mathbb{R}_+^k$ such that $\sum_i \xi_i(c; y) = c$ for all $(c; y) \in \mathcal{S}^k$, and*

$$\varphi(P, N) = \xi(C(P); C(P_1), \dots, C(P_k))$$

for all problems $(P, N) \in \mathcal{N}$.

Proof The sufficiency part is obvious. We prove the necessity only.

First, for any $(c; y) \in \mathcal{S}^k$, we construct the network $\tilde{N}(c; y)$ as follows. Assume without loss of generality that $y_1 \geq y_2 \geq \dots \geq y_k$. Choose i , where $i \in \{1, \dots, k\}$, such that

$$y_1 + y_2 + \dots + y_i \leq c < y_1 + y_2 + \dots + y_{i+1}.$$

Let $\tilde{N}(c; y)$ be a linear network such that every agent has a unique strategy (see Fig. 3). In this Figure, the source node of agent 1 is also the source node of agent $i + 2$, agent $i + 3$, ..., agent k (the first node on the left). The sink node of agent m is also the source node of agent $m + 1$ for all $m = 1, 2, \dots, i - 1$. Agents 1 to i have demands that do not overlap. Agent $i + 1$ has demand y_{i+1} such that a segment of length $c - (y_1 + y_2 + \dots + y_i)$ does not intersect the demand of other agents, and

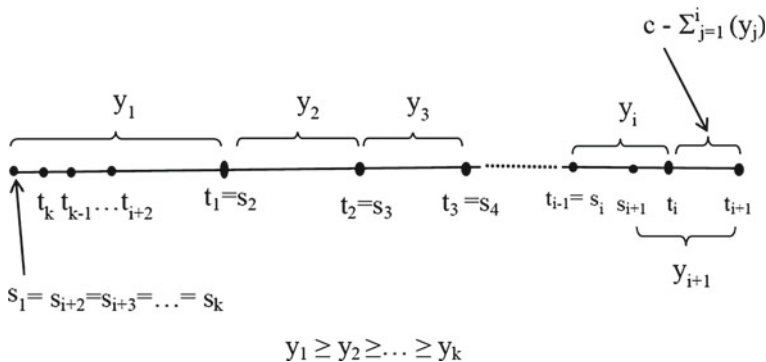


Fig. 3 Linear network where all the agents have exactly one strategy and the total cost of the network equals c

$y_1 + y_2 + \dots + y_{i+1} - c$ intersects the demand of agent i . Agent j , such that $j > i + 1$, has demand y_j that is contained in the demand of agent 1.

Clearly, the unique strategy of agent k in $\tilde{N}(c; y)$ is y_k , and the network formed by all strategies has cost c . Define $\xi : \mathcal{S}^k \rightarrow \mathbb{R}_+^k$ as $\xi(c; y) = \varphi(\tilde{N}(c; y))$.

Second, consider any arbitrary network $N = \langle G, K \rangle$ and a set of demands P . On the one hand, notice that $C(P) \geq C(P_i)$ for every agent i , since $P_i \subseteq P$. On the other hand, notice that $C(P) \leq C(P_1) + \dots + C(P_k)$, since $P \subseteq P_1 \cup P_2 \cup \dots \cup P_k$.

Let $y_i = C(P_i)$ and $c = C(P)$. Then, $(c; y) \in \mathcal{S}^k$. By minimal information $\varphi(P, N) = \varphi(\tilde{N}(c; y)) = \xi(c; y)$.

Finally, the continuity of ξ follows from the continuity of φ . □

Notice that a mechanism that uses minimal information can be expressed as a function ξ that is similar to a rationing rule (Thomson 2003; Moulin 2002). Since we work only with mechanisms that use minimal information, we often refer without loss of generality to the rule ξ as a mechanism. When there is no confusion, the total cost of a path demanded by an agent will be referred to as his demand.

Example 2 In this example, we see that the network plays a very critical role in the implementation problem of mechanisms that use minimal information. In particular, we see that for the same network, the same total cost and demand may correspond to different equilibria.

Consider the networks in Fig. 2 (left and right). Assume that the following four conditions are satisfied (i) $w + x + y + z = a + b + c + d + e$, (ii) $w + x = a + d$, (iii) $w + y = b + c + d$, and (iv) $w + z = b + e$. Thus, any mechanism that uses minimal information allocates the same payments at the given demands.

The demand profile in Fig. 2 (left) will always be an equilibrium in any minimal information mechanism. The reason is that there are no other alternatives to the players. However, when we consider the third network in Fig. 2 (right), the same demand profile will not be an equilibrium for the AC mechanism for any positive c . Players 1 and 2 have profitable deviations to path be from paths ad and bcd , respectively.

Therefore, the equilibria do depend on the network chosen.

2.1 Efficiency and other desirable properties

Since we focus only on minimal information mechanisms, all the desirable properties that we require are imposed in the function ξ given by Lemma 1.

Given a problem $N = \langle G, K \rangle$, we say P^* is an *efficient* graph if $P^* \in \arg \min_{P \in \Pi(N)} C(P)$. That is, P^* is a graph that connects all the agents at a minimal cost. Let $\text{Eff}(N)$ be the set of efficient graphs in the problem N .

Given the problem $N = \langle G, K \rangle$, the mechanism ξ induces the following non-cooperative game $\Gamma^\xi(N) \equiv (\bar{K}, \{\Pi_i(N)\}_{i \in \bar{K}}, \{\xi_i\}_{i \in \bar{K}})$, where the representation of the game is the standard representation of the game in normal form. Namely, $\bar{K} = \{1, \dots, k\}$ is the set of players, $\Pi_i(N)$ is the strategy space of player i , and ξ_i is the (negative of) payoff function of player i that maps a strategy profile to real numbers.

$$P \text{ is a NE of } \Gamma^\xi(N) \text{ if } P_i \in \arg \min_{\hat{P}_i \in \Pi_i(N)} \xi_i(\hat{P}_i, P_{-i}) \text{ for all } i.$$

Let $NE(\Gamma^\xi(N)) \equiv \{P \in \Pi(N) | P \text{ is a NE of } \Gamma^\xi(N)\}$ be the set of Nash equilibria of the game $\Gamma^\xi(N)$.

Since every agent in the game $\Gamma^\xi(N)$ has a finite number of strategies, the game has a finite number of equilibria or there is no equilibrium.

We say that ξ (weakly) implements P if $P \in NE(\Gamma^\xi(N))$.

Definition 3 The mechanism ξ is efficient (EFF) if it implements an efficient graph for any problem N , that is, $P^* \in NE(\Gamma^\xi(N))$ for some efficient graph P^* .¹³

The definition of efficiency just requires an efficient graph to be a NE. This does not preclude the existence of other inefficient equilibria.

Notice that AC is efficient. Indeed, at any efficient strategy profile P^* , every agent is paying $\frac{C(P^*)}{k}$. If an agent i deviates from P^* by reporting P_i , then he will pay $\frac{C(P_i, P_{-i}^*)}{k}$. Clearly, $\frac{C(P_i, P_{-i}^*)}{k} \geq \frac{C(P^*)}{k}$ by the optimality of P^* .

Section 4 discusses a variety of mechanisms that are not efficient.

For the vectors $z, \tilde{z} \in \mathbb{R}^m$, we say $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ for all i .

Definition 4 The mechanism ξ Pareto Nash implements (PNI) an efficient graph if for any problem N , it implements an efficient graph and that graph Pareto dominates any other equilibrium, that is, for any problem N

- there is an efficient graph P^* such that $P^* \in NE(\Gamma^\xi(N))$, and
- $\xi(P^*) \leq \xi(P)$ for any other $P \in NE(\Gamma^\xi(N))$.

PNI is a very robust property that provides agents with the incentives to select the efficient allocation even when a multiplicity of equilibria arise. In the case of a multiplicity of equilibria, PNI requires that all agents would prefer the efficient graph to any other equilibrium. Hence, under a multiplicity of equilibria, it serves as a selection rule.

In particular, this implies that whenever there is a multiplicity of equilibria such that agent i prefers equilibrium P^i to P^j and agent j prefers equilibrium P^j to P^i , there should exist another equilibrium P^* (the efficient equilibrium) such that agent i prefers equilibrium P^* to P^i and agent j also prefers equilibrium P^* to P^j .

The AC mechanism PNI the efficient graph. Indeed, at the efficient graph P^* , this equilibrium would Pareto dominate any other equilibrium \tilde{P} since $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P})}{k}$.¹⁴

Another point in favor of AC (and its asymmetric variations discussed below) is that it generates an ordinal potential game on the set of players where the potential function is equal to the total cost. Since the vast family of decentralized learning/tatonnement mechanisms converge to a NE in a potential game,¹⁵ they will also do so in the AC

¹³ The reader should not confuse this definition with other definitions of efficiency irrespective of incentive-compatibility considerations.

¹⁴ Note that this property and some others discussed in this section are true not just for the network cost-sharing problems but also for other cost-sharing problems. However, as we will see soon, in the network cost-sharing framework, this property together with symmetry characterizes the AC mechanism. Indeed, all our major characterization results use networks in an essential sense.

¹⁵ See [Monderer and Shapley \(1996a,b\)](#) for convergence of fictitious play and best response (br) dynamics. See [Sandholm \(2001\)](#) for more general dynamics.

mechanism. Moreover, with the presence of a non-binding coordinator (who knows the optimal path in advance), the agents can easily converge to the best NE.

Definition 5 The mechanism ξ strongly Nash implements (SNI) an efficient graph if for any problem N it implements an efficient graph in a strong NE, that is, for any problem N

- there is an efficient graph P^* such that $P^* \in NE(\Gamma^\xi(N))$, and
- for any group of agents $S \subset \{1, \dots, k\}$, and $P \in \Pi(N)$ such that $P_{-S} = P^*_{-S}$, if $\xi_i(P) < \xi_i(P^*)$ for some $i \in S$, then $\xi_j(P) > \xi_j(P^*)$ for some $j \in S$.

Under SNI, there is no group of agents that can coordinate paths and weakly improve all of them, and at least one agent in the group strictly improves. This is related to the strong NE¹⁶ and to the literature on group strategyproofness (Juárez 2012; Moulin 1999).

The AC mechanism SNI the efficient graph. Indeed, at any deviation \tilde{P}_S of the group of agent S from the efficient graph P^* , it should be the case that $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P}_S, P^*_{N \setminus S})}{k}$ for all $i \in S$. Hence, no agent in S would strictly improve by deviating.

Definition 6 – The mechanism ξ that uses minimal information is demand monotonic (DM) if for all feasible problems $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ such that $y_{-i} = \tilde{y}_{-i}$ and $y_i < \tilde{y}_i : \xi_i(c; y) \leq \xi_i(c; \tilde{y})$.

- The mechanism ξ that uses minimal information is strongly demand monotonic (SDM) if for all feasible problems $(c; y), (c; \tilde{y}) \in \mathcal{S}^k$ such that $y_{-i} = \tilde{y}_{-i}$ and $y_i < \tilde{y}_i : \xi_{-i}(c; y) \geq \xi_{-i}(c; \tilde{y})$.

Demand monotonicity is a weak property that requires that whenever the demand of an agent increases, everything else fixed, his payment should not decrease. Notice that does not preclude the possibility that the payment of other agents would change. Under SDM, the increase in the demand of one agent does not increase the payment of other agents. In particular, notice that SDM implies DM since all of the agents' payments have to add up to a constant.

AC is clearly SDM since $AC(c; y) = AC(c; \tilde{y})$. Thus an increase in the demand of one agent does not change the payments of the other agents.

3 Implementing the efficient equilibrium

We now turn to the first main result of the paper. We characterize the mechanisms that meet the efficiency properties discussed above.

Theorem 1 Assume that there are three or more agents. Then, the following four statements are equivalent for a mechanism ξ that uses minimal information:

1. the mechanism ξ is EFF and SDM;
2. the mechanism ξ PNI the efficient graph;

¹⁶ SNI is more demanding than the original strong NE (Aumann 1959) that requires strict improvements by all agents involved in the deviation.

- 3. the mechanism ξ SNI the efficient graph;
- 4. there is a monotonic function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^k$ such that $\sum_i f_i(c) = c$ and $\xi(c; y) = f(c)$ for any feasible problem $(c; y)$.

The mechanisms characterized by Theorem 1 are demand independent, that is, the cost-share of every agent does not depend on whether the agents are demanding cheap or expensive links. Instead, they depend only on the total cost of the network formed. The AC, generated by $f(c) = (\frac{c}{k}, \dots, \frac{c}{k})$, is the only mechanism in this class that treats equal agents equally.

Notice that efficiency alone is not sufficient to characterize the above mechanisms. Indeed, consider the mechanism

$$\tilde{\xi}(c; y) = \left(\min \left\{ y_3, \frac{c}{k} \right\}, \frac{2c}{k} - \min \left\{ y_3, \frac{c}{k} \right\}, \frac{c}{k}, \dots, \frac{c}{k} \right).$$

First, notice that $\tilde{\xi}$ implements the efficient graph because at the efficient graph agents $\{3, \dots, k\}$ do not have the incentive to deviate since by doing so their payment is going to increase. On the other hand, agents $\{1, 2\}$ do not have any incentive to deviate from the efficient equilibrium since the functions $\min\{y_3, \frac{c}{k}\}$ and $\frac{2c}{k} - \min\{y_3, \frac{c}{k}\}$ are weakly monotonic in the total cost of the network and do not depend on their report.

The mechanism $\tilde{\xi}$ is also an example of a mechanism that is not SNI, but agents cannot *strictly* improve by coordinating. To see that $\tilde{\xi}$ does not SNI the efficient graph, notice that in the problems where agent 3 has multiple strategies leading to the same efficient cost c , he can help either agent 1 or agent 2 without having his own payoff changed.

Minimal information is crucial to get this result. The proportional to the stand-alone mechanism η^{pr} PNI and SNI the efficient graph. However, η^{pr} does not use minimal information.

3.1 Efficient mechanisms for two agents

The example above shows that for three or more agents, EFF is not enough to characterize the demand-independent mechanisms. On the other hand, this property is enough when there are two agents. The property is an immediate consequence of a Separability Lemma described below.

Proposition 1 *Assume that there are two agents. A mechanism that uses minimal information is efficient if and only if there is a monotonic function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ such that $f_1(c) + f_2(c) = c$ and $\xi(c; y) = f(c)$ for any feasible problem $(c; y)$.*

3.2 Equal treatment of equals

Definition 7 The mechanism ξ that uses minimal information satisfies equal treatment of equals (ETE) if $\xi_i(c; y) = \xi_j(c; y)$ for any agents i and j , and any feasible problem $(c; y)$ such that $y_i = y_j$.

ETE is the standard property of equal responsibility for the cost of the good. Equal agents with the same demand should be allocated the same cost. There is a large class of solutions that meet ETE. In Sect. 4, we describe alternative mechanisms that meet ETE, such as the proportional and the egalitarian solution.

Theorem 2 *A mechanism that uses minimal information is EFF and ETE if and only if it is AC.*

Notice that this statement is not directly implied by Theorem 1, since we do not need SDM.

4 Individually rational mechanisms

Definition 8 A mechanism ξ that uses minimal information is individually rational (IR) if $\xi_i(c; y) \leq y_i$ for any feasible problem $(c; y)$ and any agent i .

Individually rational mechanisms rule out cross-subsidies, that is, no agent pays more than the cost of his demand.

Notice that neither AC nor any mechanism discussed in Theorem 1 meets individual rationality. Therefore, the traditional incompatibility of strategyproofness, efficiency, budget balance, and individual rationality (Green and Laffont 1979) also holds in this problem.

This incompatibility holds only because we consider mechanisms that use minimal information. If we remove this constraint, there is a large class of mechanisms that always implement the efficient network and at the same time meet individual rationality. For instance, consider the proportional to the stand-alone mechanism η^{pr} discussed above. η^{pr} is IR because no agent pays more than his stand-alone cost, which in turn is less than his demand. On the other hand, η^{pr} implements the efficient allocation because the cost-share of every agent is in proportion to the cost of the network; therefore, any deviation from the efficient graph that increases the total cost of the network formed would increase the cost-share of all agents.

On the other hand, there is a large class of IR mechanisms that use minimal information: most of the mechanisms discussed in the rationing/bankruptcy literature meet IR; see, for instance, Thomson (2003), Moulin (2002).

A class of rationing mechanisms that is especially compelling is derived from the class of asymmetric parametric rules.

Definition 9 For every agent i , consider $F_i : [0, \Lambda] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous in both variables, non-decreasing in the first variable and such that $F_i(0, z) = 0$ and $F_i(\Lambda, z) = z$ for all z . A parametric rationing mechanism is defined as.¹⁷

$$\varphi_i(c; y) = F_i(\lambda^*, y_i) \text{ where } \lambda^* \text{ solves } \sum_{i \in \bar{K}} F_i(\lambda^*, y_i) = c$$

¹⁷ Notice that there could be more than one solution λ^* to the system; however, they will give the same cost-shares to the agents.

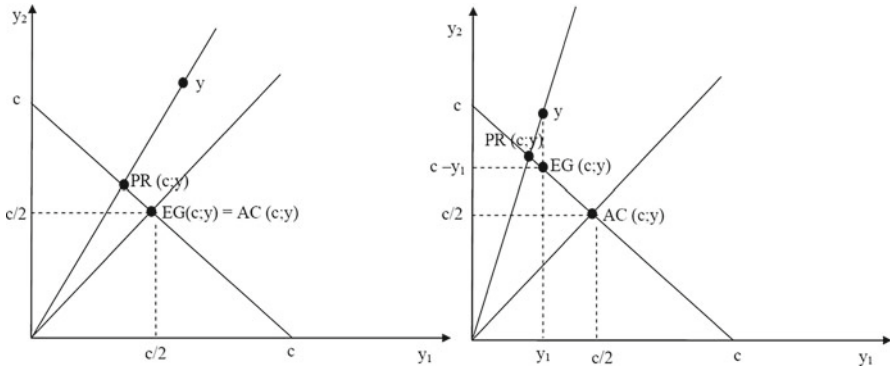


Fig. 4 Contrast between AC, EG and PR for two agents

The class of parametric rationing mechanisms is very rich since it contains almost any rationing rule discussed in the literature.¹⁸ In particular, it contains two basic rationing mechanisms: proportional and egalitarian. The proportional mechanism (PR) divides the cost of the agents in proportion to their demands, that is,

$$PR_i(c; y) = \frac{y_i}{y_1 + \dots + y_k} c.$$

On the other hand, the EG divides the cost equally across the agents subject to no agent paying more than his demand, that is,

$$EG_i(c; y) = \min\{y_i, \lambda^*\} \text{ where } \lambda^* \text{ solves } \sum_i \min\{y_i, \lambda^*\} = c.$$

The parametric description of these two mechanisms is given by

- Proportional** : $F_i(\lambda, z) = \lambda z, \Lambda = 1$;
- Egalitarian** : $F_i(\lambda, z) = \min\{\lambda, z\}, \Lambda = \infty$.

Figure 4 illustrates the allocation of payments for AC and EG at the problem $(c; y_1, y_2)$. In both figures, AC would allocate a payment equal to $\frac{c}{2}$ to every agent. In Fig. 4 left, both agents have a demand above the average cost $\frac{c}{2}$. Therefore, EG coincides with AC and allocates a payment of $\frac{c}{2}$ to every agent. On the other hand, in Fig. 4 right, agent 1 demands less than the average cost $\frac{c}{2}$. Therefore, his payment under EG would be equal to his demand y_1 , whereas agent 2 would pay the difference to cover the cost $c - y_1$. The PR allocates the point of intersection of the simplex with the line joining the origin and the demand vector.

We now introduce a class of mechanisms that generalize the EG introduced above. These mechanisms, which resemble a fixed path rule, are briefly introduced and dis-

¹⁸ See Young (1987) for a characterization of the symmetric parametric rules; see Moulin (2002) for a more detailed description of the rules.

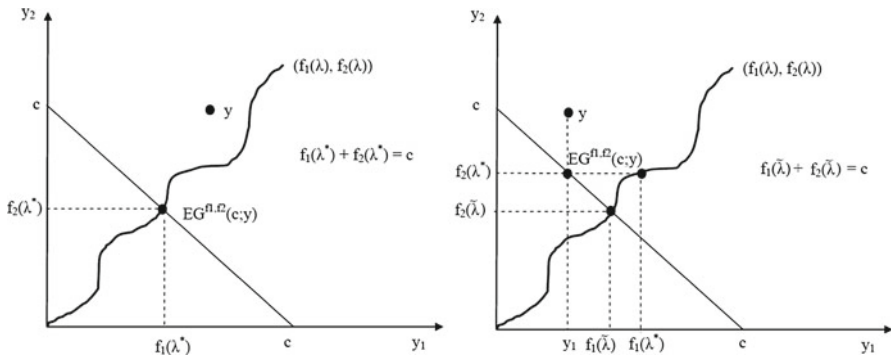


Fig. 5 The asymmetric egalitarian mechanisms for two agents

cussed in section 1.8 of [Moulin \(2002\)](#). To illustrate the class of mechanisms, consider a non-decreasing function $f_i : [0, \Lambda] \rightarrow \mathbb{R}_+$ such that $f_i(0) = 0$ and $f_i(\Lambda) = \infty$, for every agent $i \in \{1, \dots, k\}$. Given the demands of the agents (y_1, \dots, y_k) and a cost of the network c , the cost-share of agent i is given by

$$EG_i^{f_1, f_2, \dots, f_k}(c; y_1, y_2, \dots, y_k) = \min\{f_i(\lambda^*), y_i\},$$

where λ^* solves $\sum_{i=1}^k \min\{f_i(\lambda^*), y_i\} = c$.

Notice that the mechanism $EG_i^{f_1, f_2, \dots, f_k}$ clearly meets IR since

$$EG_i^{f_1, f_2, \dots, f_k}(c; y_1, y_2, \dots, y_k) \leq y_i.$$

The mechanism $EG_i^{f_1, f_2, \dots, f_k}$ will be called an asymmetric egalitarian mechanism (AEM).

Figure 5 illustrates the allocation of payments for EG^{f_1, f_2} at the problem $(c; y_1, y_2)$. The path used to compute the payments, $\{(f_1(\lambda), f_2(\lambda)) | \lambda \geq 0\}$, is generated by the functions $f_1(\lambda)$ and $f_2(\lambda)$. In Fig. 5 left, the mechanism would allocate payments equal to $(f_1(\lambda^*), f_2(\lambda^*))$ because $y_1 \geq f_1(\lambda^*)$ and $y_2 \geq f_2(\lambda^*)$. On the other hand, in Fig. 5 right, the mechanism would allocate payments equal to $(y_1, c - y_1)$ because $y_1 < f_1(\bar{\lambda})$.

The EG is constructed by picking functions $f_1 = f_2 = \dots = f_k$. The path generated by these functions is the identity line

$$\{(\lambda, \lambda, \dots, \lambda) \in \mathbb{R}^k | \lambda \geq 0\}.$$

Alternatively, the weighted egalitarian mechanisms are constructed when $f_i(\lambda) = w_i \lambda$ for all i , for a given set of constants w_1, \dots, w_k . The path generated by such functions is the ray $\{\lambda(w_1, w_2, \dots, w_k) | \lambda \geq 0\}$

Contrary to the traditional analysis of this problem, the games induced by the AEMs are not potential games; see Sect. 9.2 for an example illustrating that. Therefore, the previous potential techniques used in the analysis of this problem do not

work anymore. In general, a mechanism induced by a rationing rule could not have a pure strategy NE. Nevertheless, we show below that AEM and PRs always have pure strategy Nash equilibria and provide algorithms to compute them.¹⁹

Lemma 2 *The proportional and the asymmetric egalitarian mechanisms always have a pure strategy NE.*

Proof We prove the Lemma for an AEM. The proof for the PR is written in Sect. 6.

Consider the asymmetric EG φ generated by the functions f^1, \dots, f^k .

Let $s^i = \min_{P_i \in \Pi_i(N)} C(P_i)$ be the stand-alone cost of agent i , and let

$$S^i \in \arg \min_{P_i \in \Pi_i(N)} C(P_i)$$

be his stand-alone path.

Let λ^i be the largest number such that $f^i(\lambda^i) = s^i$. Without loss of generality, assume that $\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^k$.

Consider a strategy profile $T = (T^1, T^2, \dots, T^k)$ and let $t^i = C(T^i)$ be the cost of strategy T^i . Let λ^* be a solution to the system $\sum_i \min\{t^i, f^i(\lambda^*)\} = C(T)$.

Given the strategy profile T , a br \tilde{T}^i of agent i is

$$\tilde{T}^i \in \arg \min_{\tilde{T}^i \in \Pi_i(N)} \varphi_i(\tilde{T}^i, T^{-i}).$$

We say that the strategy profile T is *Nash-convergent* if for every agent i : (a) $\varphi_i(T) = s^i$, or (b) $\varphi_i(T) = f^i(\lambda^*)$ and $\lambda^* \leq \lambda^i$.

Step 1. Consider a Nash-convergent profile T . Suppose that the br of agent j , with path Y^j and cost $C(Y^j) = y^j$, leads to a decrease in his cost-share, that is, $\varphi_j(T) > \varphi_j(Y^j, T^{-j})$. Then, the profile (Y^j, T^{-j}) is also a Nash-convergent profile with smaller λ^* .

Let $\tilde{\lambda}$ be the smallest solution to $\min\{y^j, f^j(\tilde{\lambda})\} + \sum_{i \neq j} \min\{t^i, f^i(\tilde{\lambda})\} = C(Y^j, T^{-j})$.

The relation $\tilde{\lambda} < \lambda^$ holds*

Since T is Nash-convergent, then $\varphi_j(T) = s^j$, or $\varphi_j(T) = f^j(\lambda^*)$ and $\lambda^* \leq \lambda^j$.

Case 1 $\varphi_j(T) = s^j$.

Note that

$$\min\{t^j, f^j(\lambda^*)\} = s^j > \min\{y^j, f^j(\tilde{\lambda})\}.$$

Since s^j is the stand-alone cost of agent j , then $y^j \geq s^j$. Therefore, $\min\{y^j, f^j(\tilde{\lambda})\} = f^j(\tilde{\lambda})$. Thus,

$$\min\{t^j, f^j(\lambda^*)\} > f^j(\tilde{\lambda}).$$

¹⁹ Surprisingly, in both cases, the traditional br tatonnement, where at every step an agent picks the path that minimizes his cost-share, starting from some profiles will converge to a NE.

This implies, $f^j(\lambda^*) > f^j(\tilde{\lambda})$. By the monotonicity of f^j , we have $\lambda^* > \tilde{\lambda}$.

Case 2 $\varphi_j(T) = f^j(\lambda^*)$ and $\lambda^* \leq \lambda^i$.

Note that

$$\min\{t^j, f^j(\lambda^*)\} = f^j(\lambda^*) > \min\{y^j, f^j(\tilde{\lambda})\}. \tag{1}$$

We show that $y^j \geq f^j(\tilde{\lambda})$. Indeed, assume the contrary, that is $f^j(\tilde{\lambda}) > y^j$. Then, $\min\{y^j, f^j(\tilde{\lambda})\} = y^j$. Thus, from Eq. 1,

$$f^j(\lambda^*) > y^j \geq s^j. \tag{2}$$

Since $\lambda^j \geq \lambda^*$, then $s^j = f^j(\lambda^j) \geq f^j(\lambda^*)$. This contradicts equation 2. Therefore, $y^j \geq f^j(\tilde{\lambda})$.

Now, we show that $\lambda^* > \tilde{\lambda}$. Since $y^j \geq f^j(\tilde{\lambda})$, then from Eq. 1, $f^j(\lambda^*) > f^j(\tilde{\lambda})$. Thus, $\lambda^* > \tilde{\lambda}$ by the monotonicity of f^j .

The profile (Y^j, T^{-j}) is Nash-convergent

- First, we consider an agent i , where $i \neq j$, that satisfies case (a). That is, $\varphi_i(T) = \min\{t^i, f^i(\lambda^*)\} = s^i$.
Since $\lambda^* > \tilde{\lambda}$, then

$$s^i = \min\{t^i, f^i(\lambda^*)\} \geq \min\{t^i, f^i(\tilde{\lambda})\}. \tag{3}$$

If $\varphi_i(Y^j, T^{-j}) = t^i$, then $\min\{t^i, f^i(\tilde{\lambda})\} = t^i$. Therefore, by Eq. 3, $s^i = t^i$. Hence, $\varphi_i(Y^j, T^{-j}) = s^i$.

On the other hand, if $\varphi_i(Y^j, T^{-j}) = f^i(\tilde{\lambda})$, then $\min\{t^i, f^i(\tilde{\lambda})\} = f^i(\tilde{\lambda})$. Therefore, by Eq. 3, $s^i \geq f^i(\tilde{\lambda})$. Since $s^i = f^i(\lambda^i)$, then $f^i(\lambda^i) \geq f^i(\tilde{\lambda})$. Hence, $\lambda^i \geq \tilde{\lambda}$ by the monotonicity of f^i and because λ^i is the maximal value that satisfies $f^i(\lambda^i) = s^i$.

- Second, we consider an agent i , where $i \neq j$, that satisfies case (b) $\varphi_i(T) = f^i(\lambda^*)$ and $\lambda^* \leq \lambda^i$.
Since $\varphi_i(T) = f^i(\lambda^*)$, then $\min\{t^i, f^i(\lambda^*)\} = f^i(\lambda^*)$. Thus, $t^i \geq f^i(\lambda^*)$. Also, note that $f^i(\lambda^*) \geq f^i(\tilde{\lambda})$ because $\lambda^* > \tilde{\lambda}$. Hence, $t^i \geq f^i(\tilde{\lambda})$. Thus, $\varphi_i(Y^j, T^{-j}) = \min\{t^i, f^i(\tilde{\lambda})\} = f^i(\tilde{\lambda})$.
Since $\lambda^* \leq \lambda^i$ and $\tilde{\lambda} \leq \lambda^*$, then $\tilde{\lambda} \leq \lambda^i$ as desired.
- Third, we consider the agent j who changed his strategy. Since agent j is using his br, then

$$\varphi_j(Y^j, T^{-j}) \leq \varphi_j(S^j, T^{-j}) \leq s^j,$$

where the last inequality comes from individual rationality.

If $\varphi_j(Y^j, T^{-j}) = \min\{y^j, f^j(\tilde{\lambda})\} = y^j$, then $y^j \leq s^j$. Since s^j is the stand-alone cost then $s^j \leq y^j$. Therefore, $y^j = s^j$ and case (a) is satisfied.

On the other hand, if $\varphi_j(Y^j, T^{-j}) = \min\{y^j, f^j(\tilde{\lambda})\} = f^j(\tilde{\lambda})$, then $f^j(\tilde{\lambda}) \leq s^j = f^j(\lambda^j)$. Therefore $\tilde{\lambda} \leq \lambda^j$ by the monotonicity of f^j and because λ^j is the maximal value that satisfies $f^j(\lambda^j) = s^j$. Thus, case (b) is satisfied.

Step 2. The profile (S^1, \dots, S^k) is a Nash-convergent profile.

Consider a solution λ^{**} to the equation

$$\sum_i \min\{s^i, f^i(\lambda^{**})\} = C(S^1, \dots, S^k).$$

If $\varphi_i(S^1, \dots, S^k) = s^i$, then case (a) holds.

If $\varphi_i(S^1, \dots, S^k) = f^i(\lambda^{**})$, then $f^i(\lambda^{**}) \leq s^i$ holds by individual rationality. Therefore, $f^i(\lambda^{**}) \leq f^i(\lambda^i)$. Hence, $\lambda^{**} \leq \lambda^i$ by the monotonicity of f^i and because λ^i is the maximal value that satisfies $f^i(\lambda^i) = s^i$. Thus, case (b) holds.

Step 3. Finally, we show the existence of equilibrium. The br tatonnement, where at every step an agent picks a path that minimizes his cost-share, starting at profile (S^1, \dots, S^k) , converges to a strategy profile in a finite number of iterations because (S^1, \dots, S^k) is Nash-convergent; then at every step, the value λ^* decreases and there are a finite number of strategies. The limit profile is a NE. □

4.1 Weakly Pareto Nash implementation

Since the mechanisms characterized by Theorem 1 are demand independent, there are no IR mechanisms that PNI the efficient graph. WPNI requires that the least inefficient equilibrium Pareto dominate any other equilibrium. That is, WPNI might implement several equilibria (and all of them might be inefficient), but the least inefficient equilibrium should be preferred by all agents to any other equilibrium. Clearly if a mechanism satisfies PNI, then it satisfies WPNI.

Definition 10 A mechanisms ξ satisfies WPNI if

- for any problem N , the mechanism ξ has at least one NE,
- let P^* be the equilibrium profile with the minimal cost; that is, $C(P^*) \leq C(\tilde{P})$ for any other equilibrium \tilde{P} . Then, $\xi(P^*) \leq \xi(\tilde{P})$.

WPNI serves an equilibrium selection rule. If the NE is a good predictor of the outcome implemented by the mechanism, then the least inefficient equilibrium stands out as a selection since all agents prefer it.

Theorem 3 *An asymmetric parametric mechanism meets WPNI if and only if it is an asymmetric egalitarian mechanisms.*

All the mechanisms discussed in this section are inefficient. The measure below will serve as a selection criterion for different mechanisms.

Definition 11 The PoS of the mechanism ξ equals

$$\max_{N \in \mathcal{N}, P^* \in Eff(N)} \left\{ \frac{\min_{P \in NE(\Gamma^\xi(N))} C(P)}{C(P^*)} \right\}$$

The PoS, which computes the ratio between the cost of the best equilibrium and the cost of the efficient graph, is a compelling measure of the inefficiency generated by WPNI mechanisms since the agents' incentives are aligned to pick the best NE.

Notice that PoS is always greater than or equal to 1. A mechanism is efficient if it has a PoS equal to 1. The smaller the PoS, the more efficient the mechanism is.

Corollary 1 *i. EG has the smallest PoS across all asymmetric parametric mechanisms meeting WPNI. It has a PoS equal to $H(k) = 1 + \frac{1}{2} + \dots + \frac{1}{k}$.*
ii. EG minimizes the PoS across all IR mechanisms.
iii. The PoS of PR is of order k .

Since the Sh has a PoS equal to $H(k)$, EG is no more inefficient than the Sh. No other IR mechanism can be more efficient than EG and Shapley. On the other hand, the traditional PR is extremely inefficient; since its PoS is bounded by k , its maximal loss approaches that in the limit.

The definition of WPNI does not rule out the existence of another mechanism whose equilibria (possibly not Pareto ranked) are less inefficient than the equilibria of a WPNI mechanism. However, from part *ii*, we can see that EG is not more inefficient than any other mechanism that satisfies individual rationality. Therefore, EG stands out as a more efficient selection even among mechanisms that do not Pareto rank the equilibria.

4.2 Strong Nash implementation of the minimal cost

Definition 12 A mechanism ξ strongly Nash implements the equilibrium with the minimal cost (SNIMC) if

- for any problem N , the mechanism ξ has at least one NE, and
- let P^* be the equilibrium profile with the minimal cost; that is, $C(P^*) \leq C(\tilde{P})$ for any other equilibrium \tilde{P} . Then, P^* is a strong NE, that is, if $\xi_i(P_S, P_{N \setminus S}^*) > \xi_i(P^*)$ for some demands P_S of the agents in S and $i \in S$, then $\xi_j(P_S, P_{N \setminus S}^*) < \xi_j(P^*)$ for some agent $j \in S$.

If a mechanism SNI the efficient graph (SNI), then it SNIMC. The converse is not true, since the equilibrium with the minimal cost might not be efficient. The AEMs are such examples.

Proposition 2 *The asymmetric egalitarian mechanisms SNIMC.*

Notice that together with the WPNI property of AEM, this proposition implies that there is one and only one strong NE under AEM. This comes from the fact that the Nash equilibria are Pareto ranked, and thus, under any NE other than the cheapest, the grand coalition has a profitable deviation. This means that the strong price of anarchy (SPoA) of the EG equals the strong PoS of EG, and they equal $H(k)$. This earmarks another advantage of EG over Sh, since we know that Sh does not always admit strong NE [Anshelevich et al. \(2004\)](#), [Chen et al. \(2008\)](#), and therefore, SPoA does not exist for Sh.

We conjecture that the only mechanisms that SNIMC are the AEMs.

5 Proof of Theorems 1 and 2

5.1 Preliminary Lemmas

Definition 13 The mechanism ξ that uses minimal information is monotonic in cost if for all feasible problems $(c; y), (c'; y) \in \mathcal{N}^K$ such that $c < c'$: $\xi(c; y) \leq \xi(c'; y)$.

Lemma 3 If the mechanism ξ that uses minimal information is efficient then, it is monotonic in total cost.

Proof Consider two feasible problems $(c; y)$ and $(c'; y)$, where $c' > c$ and $(c' - c) < \min_{i \in K} \{y_i\}$. Suppose that there exists an agent i and an efficient ξ such that $\xi_i(c'; y) < \xi_i(c; y)$.

We construct a network that has two potential profiles $(c; y)$ and $(c'; y)$.

Indeed, consider a network where agents $j \neq i$ have just one strategy each, P_j , which costs y_j . Agent i has two strategies, P_i and P'_i , both of which cost y_i , but P_i makes the total cost of the network c , and P'_i makes the total cost go up to c' .

Case 1: $c \leq \sum_{j \neq i} y_j$.

In this case, we can have a configuration as shown in Fig. 6. Here, the demands of agents in $K \setminus \{i\}$ are contained in the interval $a \rightarrow b$, which costs c . This is possible since when $c = \sum_{j \neq i} y_j$, we can have $a \rightarrow b$ as the concatenation of the demand links of the agents $j \neq i$. When $c < \sum_{j \neq i} y_j$, we can have the demand links overlapping, for example, when $\max_{j \neq i} \{y_j\} = c$, then $a \rightarrow b$ is the demand link of the biggest demander and all other demands overlap with his. $P_i = s_i \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow t_i$ and $P'_i = s_i \rightarrow v_2 \rightarrow v_3 \rightarrow t_i$. All the costly links of P_i are contained in $\{\cup_{j \neq i} P_j\}$, whereas there are links of cost $c' - c$ that are not contained in $\{\cup_{j \neq i} P_j\}$ under P'_i . Again, this is possible since c' and c are close enough to guarantee that for all i we can have such paths.

Case 2: $\sum_{j \in K} y_j > c > \sum_{j \neq i} y_j$.

In this case, we can have a configuration as shown in Fig. 7. Here, the interval $a \rightarrow b$ is the concatenation of the demand links of agents in $K \setminus \{i\}$. Thus, $|a \rightarrow b| = \sum_{j \neq i} y_j, |s_i \rightarrow a| = c - \sum_{j \neq i} y_j, |a \rightarrow d| = c' - c, |s_i \rightarrow a \rightarrow d| = |s_i \rightarrow a' \rightarrow$

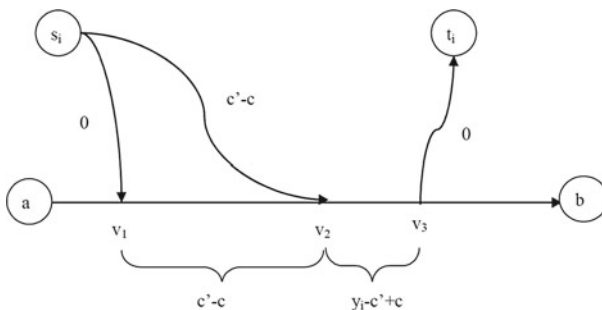


Fig. 6 EFF implies cost monotonicity (case 1)

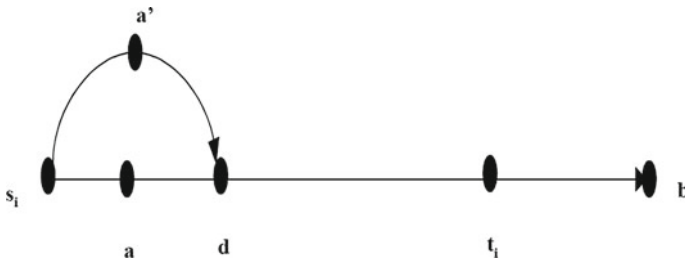


Fig. 7 EFF implies cost monotonicity (case 2)

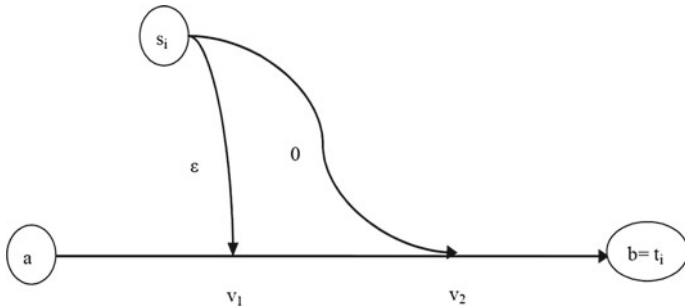


Fig. 8 EFF implies separability

$d| = c' - \sum_{j \neq i} y_j$. $P_i = s_i \rightarrow a \rightarrow d \rightarrow t_i$ and $P'_i = s_i \rightarrow a' \rightarrow d \rightarrow t_i$. Notice that it may be the case that $t_i = b$.

Now clearly in both cases, i will have a profitable deviation from the efficient graph of cost c , thus contradicting the efficiency of ξ . Thus, we have shown that efficient ξ must be monotonic in total cost in some open neighborhood of c , for all c . Therefore, we can extend the argument to conclude that ξ must be monotonic in total cost in general. \square

Lemma 4 (Separability Lemma) *If the mechanism ξ that uses minimal information is efficient then $\xi(C; y) = (\xi_1(C; y_{-1}), \xi_2(C; y_{-2}), \dots, \xi_k(C; y_{-k}))$. That is, any efficient mechanism is separable and assigns the costs-shares to the agents independently of their demand.*

Proof If we prove that for any feasible problems $(c; y)$ and $(c; \tilde{y}_i, y_{-i})$, any continuous and efficient ξ must have $\xi_i(c; y) = \xi_i(c; \tilde{y}_i, y_{-i})$, then we are done. Consider a feasible problem $(c; y)$. Consider a graph as shown in Fig. 8, which generates this problem. The sources and sinks of agents $j \neq i$ lie on the ray $a \rightarrow b$ according to the demand profile, that is, the agent with the highest demand covers most of the span on $a \rightarrow b$, and so on. Thus, an agent $j \neq i$ has one strategy that generates the demand y_j . Agent i has two strategies -connect $s_i - t_i$ either through v_1 or through v_2 . The demands of agent i when connecting through v_1 and v_2 are \tilde{y}_i and y_i , respectively. Now, the total costs when i uses v_1 and v_2 are, respectively, $c + \epsilon$ and c . Notice that by moving the position of v_2 and arranging the demand links of the agents $j \neq i$, we can generate all the feasible problems $(c; y_i, y_{-i})$. Also, by moving the position of v_1

and arranging the demand links of the agents $j \neq i$, we can generate all the feasible problems $(c + \epsilon; y_i, y_{-i})$. Consider an efficient ξ that is continuous. The efficiency of ξ requires the following inequality

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c + \epsilon; \tilde{y}_i, y_{-i}) \tag{4}$$

Using continuity, we get

$$\xi_i(c; y_i, y_{-i}) \leq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{5}$$

Similarly, switching the position of v_1 and v_2 and using continuity again we get

$$\xi_i(c; y_i, y_{-i}) \geq \xi_i(c; \tilde{y}_i, y_{-i}) \tag{6}$$

Thus, we conclude that $\xi_i(c; y_i, y_{-i}) = \xi_i(c; \tilde{y}_i, y_{-i})$ for all feasible problems $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$. □

5.2 Proof of Proposition 1

Proof Consider a problem $(c; y_1, y_2) \in S^2$.

By Separability Lemma $\xi_1(c; y_1, y_2) = \xi_1(c; c, y_2)$.

By budget balance $\xi_2(c; y_1, y_2) = \xi_2(c; c, y_2)$. Thus, $\xi(c; y_1, y_2) = \xi(c; c, y_2)$.

By Separability Lemma $\xi_2(c; c, y_2) = \xi_2(c; c, c)$.

By budget balance $\xi_1(c; c, y_2) = \xi_1(c; c, c)$. Thus, $\xi(c; c, y_2) = \xi(c; c, c)$.

Therefore, $\xi(c; y_1, y_2) = \xi(c; c, c)$.

Let $f(c) = \xi(c; c, c)$. Since the mechanism is monotonic in the total cost (Lemma 3), $f(c)$ is monotonic in the total cost. □

5.3 Proof of Theorem 1

5.3.1 1. \implies 4.

Consider a continuous ξ that is efficient and strongly monotonic. Consider two arbitrary feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from Lemma 3. Let $a = \frac{1}{k} \sum_{i \in K} y_i$ and $\tilde{a} = \frac{1}{k} \sum_{i \in K} \tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq \dots \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq \dots \leq \tilde{y}_k$.

Step 1: $\xi(c; y) = \xi(c; a, a, \dots, a)$ and $\xi(c; \tilde{y}) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$

Proof Consider the following problems:

$$P_0 = (c; y), P_1 = (c; a, y_2, y_3, \dots, y_k),$$

$$P_2 = (c; a, a, y_3, y_4, \dots, y_k), \dots, P_k = (c; a, a, \dots, a).$$

First, notice that the feasibility of P_0 implies the feasibility of

$$P_1, P_2, \dots, P_k.$$

This is true because all individual demands are bounded above by y_k , and the aggregate demand at any problem is at least $k(a) = \sum_{i \in K} y_i \geq c$ by the feasibility of P_0 .

Similarly, we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k$ where

$$\tilde{P}_i = (c; \tilde{a}, \tilde{a}, \dots, \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, \dots, \tilde{y}_{k-1}, \tilde{y}_k).$$

By the above argument these problems are also feasible.

Now, due to the Separability Lemma, we must have $\xi_1(P_0) = \xi_1(P_1)$. By strong monotonicity and budget balance $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument, we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k - 1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$ as desired. \square

Step 2: $\xi(c; a, a, \dots, a) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$

Proof First, notice that the feasibility of $(c; a, a, \dots, a)$ and $\xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$ implies that any problem $(c; \hat{a})$ where some of the $\hat{a}_i = a$ and other $\hat{a}_i = \tilde{a}$ is also feasible. Now, the Separability Lemma implies $\xi_1(c; a, \tilde{a}, \dots, \tilde{a}) = \xi_1(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$. Now, there can be three cases: (1) $a < \tilde{a}$ (2) $a > \tilde{a}$ or (3) $a = \tilde{a}$. In the first two cases, strong monotonicity and budget balance imply $\xi_{-1}(c; a, \tilde{a}, \dots, \tilde{a}) = \xi_{-1}(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$ and we get $\xi(c; a, \tilde{a}, \dots, \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$. The third case trivially implies

$$\xi(c; a, \tilde{a}, \dots, \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a}).$$

Similarly, we get

$$\xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a}) = \xi(c; a, \tilde{a}, \dots, \tilde{a}) = \xi(c; a, a, \tilde{a}, \dots, \tilde{a}) = \dots = \xi(c; a, a, \dots, a).$$

\square

5.3.2 2. \implies 1.

Proof We know that ξ PNI efficient graph implies that ξ is efficient. We will prove that if ξ PNI the efficient graph, then ξ is strongly monotonic. Consider a ξ that PNI the efficient graph and a feasible problem $(c; y)$ and assume without loss of generality that $y_1 < y_2 < \dots < y_k$.²⁰ Now, consider a graph as shown in Fig. 9 below.

Here every agent has two strategies -either use the path in the solid graph or use that in the dotted graph. We call the solid graph ** and the dotted graph *. Let ϵ be a small perturbation of ** as follows. The cost of the path of an agent $j \neq i$ in both graphs is y_j . The cost of the paths of agent i in ** and * are y_i and \tilde{y}_i where \tilde{y}_i is in a neighborhood of y_i and $\tilde{y}_i > y_i$ and $|\tilde{y}_i - y_i| < \min_{j,k \in K} |y_j - y_k|$. This

²⁰ The case of weak inequality follows by continuity of the mechanism.

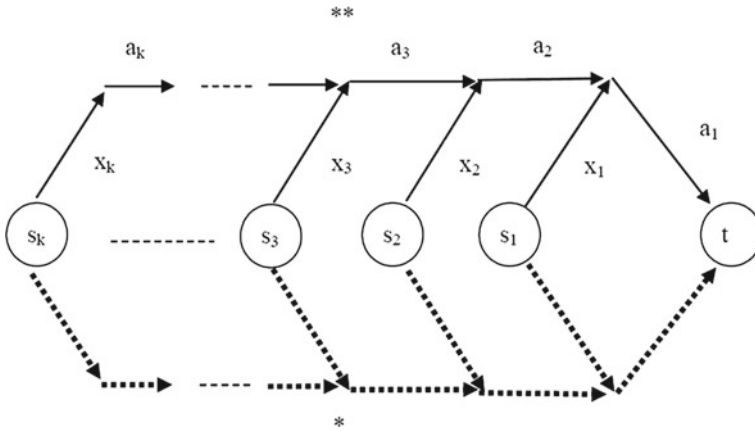


Fig. 9 PNI implies SM

restriction guarantees that the ranking will be preserved in the perturbed problem. Let the total cost of ** and * be $c - \epsilon$ and c , respectively. First, we will show that this graph generates all feasible problems $(c; y)$. This happens if and only if the following system has a solution

$$\begin{aligned}
 x_1 + a_1 &= y_1 \\
 x_2 + a_2 + a_1 &= y_2 \\
 x_3 + a_3 + a_2 + a_1 &= y_3 \\
 &\vdots \\
 &\vdots \\
 x_k + a_k + a_{k-1} + \dots + a_1 &= y_k \\
 \sum_{i=1}^k x_i + \sum_{i=1}^k a_i &= c \\
 \forall i \in K \quad x_i, a_i &\geq 0
 \end{aligned}$$

We use Farka’s Lemma to prove that this system indeed has a solution.

From Farka’s Lemma, we know that $Ax = b$ and $x \geq 0$ has a solution if and only if $A^T z \geq 0$ and $b^T z < 0$ does not have a solution.

Here, the $(k + 1) \times (2k)$ matrix A , vector x , and vector b are defined as follows

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots & 1 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots \end{bmatrix} \\
 x &= [x_1 \ x_2 \ \dots \ x_k \ a_1 \ a_2 \ \dots \ a_k]^T \\
 b &= [y_1 \ y_2 \ \dots \ y_k \ c]^T
 \end{aligned}$$

$A^T z \geq 0$ and $b^T z < 0$ gives the following $(2k + 1)$ inequalities:

$$z_1 + z_2 + \dots + z_{k+1} \geq 0 \tag{1}$$

$$z_2 + z_3 + \dots + z_{k+1} \geq 0 \tag{2}$$

$$\vdots \tag{(:)}$$

$$z_k + z_{k+1} \geq 0 \tag{k}$$

$$z_1 + z_{k+1} \geq 0 \tag{k + 1}$$

$$z_2 + z_{k+1} \geq 0 \tag{k + 2}$$

$$\vdots \tag{(:)}$$

$$z_k + z_{k+1} \geq 0 \tag{(2k)}$$

$$y_1 z_1 + y_2 z_2 + \dots + y_k z_k + c z_{k+1} < 0 \tag{(2k + 1)}$$

Now, we do the following operation on the first k inequalities:

$$y_1 \times (1) + (y_2 - y_1) \times (2) + \dots + (y_k - y_{k-1}) \times (k),$$

and the result is

$$y_1 z_1 + y_2 z_2 + \dots + y_k z_k + y_k z_{k+1} \geq 0. \tag{(2k + 2)}$$

Now, for the inequalities $(2k + 1)$ and $(2k + 2)$ to be compatible, it must be the case that $z_{k+1} < 0$. Let this be the case and let $(2k + 2)$ and $(2k + 1)$ hold. Then, $(2k + 1)$ implies

$$y_1 z_1 + y_2 z_2 + \dots + y_k z_k + \left(\sum_{i \in K} y_i \right) z_{k+1} < 0 \tag{(2k + 3)}$$

This is true because feasibility requires $\sum_{i \in K} y_i \geq c$. Now, if we do the following operation on inequalities $(k + 1)$ through $(2k)$: $y_1 \times (k + 1) + y_2 \times (k + 2) + \dots + y_n \times (2k)$, then we get

$$y_1 z_1 + y_2 z_2 + \dots + y_k z_k + \left(\sum_{i \in K} y_i \right) z_{k+1} \geq 0 \tag{2k + 4}$$

which contradicts $(2k + 3)$, to give us the desired result.

We now prove the strong monotonicity of ξ . Clearly, the efficiency of ξ implies that $**$ is a NE. Since $*$ is a perturbation of $**$, then we will have $*$ also as a NE for a perturbation small enough. Since ξ PNI the efficient graph implies that

$$\xi(c - \epsilon; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Using continuity, we get

$$\xi(c; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Now consider a perturbation where everything is exactly the same except $**$ costs $c + \epsilon$. Using the same argument of Pareto Nash implementability and continuity we get

$$\xi(c; y_i, y_{-i}) \geq \xi(c; \tilde{y}_i, y_{-i})$$

Thus, $\xi(c; y_i, y_{-i}) = \xi(c; \tilde{y}_i, y_{-i})$ for \tilde{y}_i in an open neighborhood of y_i .
By repeatedly using the open neighborhood argument, we get

$$\xi(c; y_i, y_{-i}) = \xi(c; \tilde{y}_i, y_{-i})$$

for any arbitrary y_i and \tilde{y}_i as long as $(c; y_i, y_{-i})$ and $(c; \tilde{y}_i, y_{-i})$ are both feasible problems. □

5.3.3 3. \implies 4.

Consider a continuous ξ that implements the efficient graph in strong NE. Consider two feasible problems $(c; y)$ and $(c; \tilde{y})$. We will prove that $\xi(c; y) = \xi(c; \tilde{y}) = f(c)$. The monotonicity of f comes from Lemma 3. Let $a = \frac{1}{k} \sum_{i \in K} y_i$ and $\tilde{a} = \frac{1}{k} \sum_{i \in K} \tilde{y}_i$. Assume without loss of generality that $y_1 \leq y_2 \leq y_3 \leq \dots \leq y_k$ and $\tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq \dots \leq \tilde{y}_k$.

Step 1: $\xi(c; y) = \xi(c; a, a, \dots, a)$ and $\xi(c; \tilde{y}) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$

Proof Consider the following problems:

$$P_0 = (c; y), P_1 = (c; a, y_2, y_3, \dots, y_k), P_2 = (c; a, a, y_3, y_4, \dots, y_k), \dots \\ \dots, P_k = (c; a, a, \dots, a).$$

First, notice that the feasibility of P_0 implies the feasibility of

$$P_1, P_2, \dots, P_k.$$

This is true because all individual demands are bounded above by y_k , and the aggregate demand at any problem is at least $k(a) = \sum_{i \in K} y_i \geq c$ by the feasibility of P_0 .

Similarly, we define the counterpart problems $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k$ where

$$\tilde{P}_i = (c; \tilde{a}, \tilde{a}, \dots, \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, \dots, \tilde{y}_{k-1}, \tilde{y}_k).$$

By the above argument these problems are also feasible.

Now, due to the Separability Lemma, we must have $\xi_1(P_0) = \xi_1(P_1)$. Also, strong Nash implementability implies that $\xi_{-1}(P_0) = \xi_{-1}(P_1)$. To see this, suppose that it is not the case and for some agent $j \neq 1$, we have $\xi_j(P_0) \neq \xi_j(P_1)$. Assume without loss of generality that $\xi_j(P_0) < \xi_j(P_1)$. This means that there exists $\hat{j} \in K \setminus \{1, j\}$ such that $\xi_{\hat{j}}(P_0) > \xi_{\hat{j}}(P_1)$ (by budget balance). Consider a network where all the agents $2, 3, \dots, k$ have just one strategy, which costs y_2, y_3, \dots, y_k and agent 1 has two strategies, where one of them costs y_1 and the other costs a . In both cases, the total cost of the network is c . Thus, one of the configurations generates the problem P_0 and the other P_1 . Now both the configurations of the network are efficient, and therefore, at least one of them must be a strong NE under ξ . But, clearly none of them is a strong NE. From P_1 , the group $\{1, j\}$ has a profitable deviation and from P_0 the group $\{1, \hat{j}\}$. Thus, we have $\xi(P_0) = \xi(P_1)$. Using the same argument, we have $\xi(P_i) = \xi(P_{i+1})$ and $\xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1})$ for all $0 \leq i \leq k - 1$. Thus, we have $\xi(P_0) = \xi(P_k)$ and $\xi(\tilde{P}_0) = \xi(\tilde{P}_k)$, as desired. \square

Step 2: $\xi(c; a, a, \dots, a) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$

Proof First, notice that the feasibility of problems

$$(c; a, a, \dots, a) \text{ and } \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$$

implies that any problem $(c; \hat{a})$ where $\hat{a}_i = a$ or $\hat{a}_i = \tilde{a}$ for all i is also a feasible problem. Now, the Separability Lemma implies $\xi_1(c; a, \tilde{a}, \dots, \tilde{a}) = \xi_1(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$. Again, the strong Nash implementability implies

$$\xi_{-1}(c; a, \tilde{a}, \dots, \tilde{a}) = \xi_{-1}(c; \tilde{a}, \tilde{a}, \dots, \tilde{a}).$$

The proof of this statement is analogous to the one in step 1. Thus, we have $\xi(c; a, \tilde{a}, \dots, \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a})$. Similarly, we get

$$\xi(c; \tilde{a}, \tilde{a}, \dots, \tilde{a}) = \xi(c; a, \tilde{a}, \dots, \tilde{a}) = \xi(c; a, a, \tilde{a}, \dots, \tilde{a}) = \dots = \xi(c; a, a, \dots, a).$$

\square

The results “4. \implies 1,” “4. \implies 2,” and “4. \implies 3” are straightforward and their proofs are omitted.

5.4 Proof of Theorem 2

Proof The “if” part is clear. For “only if,” consider a feasible problem $(c; y)$. Assume without loss of generality that $y_1 \geq y_2 \geq y_3 \geq \dots \geq y_k$. Let $a = \frac{1}{k} \sum_{i=1}^k y_i$. Consider a problem $(c; a, a, \dots, a)$ and suppose that ξ is continuous, efficient and satisfies ETE. Notice that the feasibility of $(c; y)$ implies the feasibility of $(c; a, a, \dots, a)$ and any other problem $(c; \hat{y})$ where $\hat{y}_i = y_i$ for all $i \in \{1, 2, \dots, l\}$ and $\hat{y}_i = a$ for all $i \in \{l, l + 1, \dots, k - 1, k\}$. Now, the ETE property of ξ implies

$$\xi(c; a, a, \dots, a) = (c/k, c/k, \dots, c/k). \tag{7}$$

Using the Separability Lemma and applying ETE again, we get

$$\xi(c; y_1, a, \dots, a) = (c/k, c/k, \dots, c/k). \tag{8}$$

Now again applying the Separability Lemma and ETE, we have

$$\xi(c; y_1, y_2, a, a, \dots, a) = (x_1, c/k, x, x, \dots, x). \tag{9}$$

But if we change the ordering of 1 and 2 while arriving at the above profile, then we have

$$\xi(c; y_1, y_2, a, a, \dots, a) = (c/k, x_2, x, x, \dots, x). \tag{10}$$

But since the ordering is immaterial, we must have $x_1, x_2, x = c/k$. And thus we have

$$\xi(c; y_1, y_2, a, a, \dots, a) = (c/k, c/k, \dots, c/k). \tag{11}$$

Repeating the same argument, we conclude that

$$\xi(c; y) = (c/k, c/k, \dots, c/k).$$

□

6 Proof of Lemma 2

We show the existence of equilibrium for PR.

Proof We prove a stronger property, which is that the br dynamics (one agent at a time) of any arbitrarily fixed ordering of agents converges to a NE, no matter where we start the br dynamics from. Suppose, on the contrary, that for some fixed ordering of the agents the br dynamics from some strategy profile s does not converge. This means that there is a cycle of a finite length $l : s(1) \rightarrow s(2) \rightarrow s(3) \rightarrow \dots \rightarrow s(l) \rightarrow s(1)$. Say, without loss of generality, that this cycle includes deviations by the set of agents $M = \{1, 2, \dots, m\} \subseteq K$. The strategy of agents in $K \setminus M$ is fixed at s^{-M} . Notice that

l is at least as big as $2m$. This is so because after the l brs, we arrive at the original strategy profile $s(1)$. Every agent in M is a part of the cycle, which in turn means that they change their strategy at least once. Therefore, it must be the case that every agent in M takes its turn at least twice so that they reach the original profile $s(1)$. We assume that agent $i \in M$ takes its turn in the br dynamics n_i number of times, where $n_i > 1$, so that $\sum_{i \in M} n_i = l$. Let the strategies played by agent i in the cycle be $s^{i;1}, s^{i;2}, \dots, s^{i;n_i}, s^{i;1}$, and so on. We call the agent who takes his turn of br in the movement from s_t to s_{t+1} agent a_t . Therefore, $s(1) = (s^{1;1}, s^{2;1}, \dots, s^{m;1}, s^{-M})$, $s(2) = (s^{a_1;2}, s_{-a_1}(1))$, $s(3) = (s^{a_2;2}, s_{-a_2}(2)), \dots, s(l-1) = (s^{a_{l-1};n_{a_{l-1}}}, s_{-a_{l-1}}(l-2))$, $s(l) = (s^{a_l;n_{a_l}}, s_{-a_l}(l-1))$. Here, we use the standard notation where $s_{-i}(t)$ represents the strategy profile of $K \setminus \{i\}$ fixed at that in $s(t)$. We abuse the notation and say that the cost of $s^{p;i}$ is equal to $s^{p;i}$. Here the cost of the network formed by the strategy profile $s(i) = C(G_{s(i)})$. Now, $PR(C(G_{s(i)}); s(i)) = s^{j;p} A_i$ where A_i is fixed for any particular $s(i)$ and $s^{j;p}$ represents the strategy of agent j in $s(i)$. The fixed A_i for any $s(i)$ is the ratio of $C(G_{s(i)})$ to the sum of the costs of individual paths in $s(i)$.

Now every step of the cycle corresponds to an inequality, which we will present as follows:

Step 1: $s(1) \rightarrow s(2) \implies$

$$s^{a_1;2} \times A_2 < s^{a_1;1} \times A_1 \tag{1}$$

Step 2: $s(2) \rightarrow s(3) \implies$

$$s^{a_2;2} \times A_3 < s^{a_2;1} \times A_2 \tag{2}$$

Step 3: $s(3) \rightarrow s(4) \implies$

$$s^{a_3;t} \times A_4 < s^{a_3;t-1} \times A_3 \quad \text{where } t = \begin{cases} 3 & \text{if } a_3 = a_1 \\ 2 & \text{otherwise} \end{cases} \tag{3}$$

Step p : $s(p) \rightarrow s(p+1) \implies$

$$s^{a_p;t} \times A_{p+1} < s^{a_p;t-1} \times A_p \quad \text{where } t \in \{1, 2, \dots, n_{a_p}\} \tag{p}$$

Step l : $s_l \rightarrow s_1 \implies$

$$s^{a_l;n_{a_l}} \times A_1 < s^{a_l;n_{a_l}-1} \times A_l \tag{l}$$

If we multiply the systems (2), (3), \dots , (l) together,²¹ then everything else cancels out and we are left with $s^{a_1;2} \times A_2 > s^{a_1;1} \times A_1$, which contradicts the inequality (1). Therefore, we conclude that there cannot be any cycle regardless of the ordering of agents and regardless of at which initial point we follow the br dynamics.

□

²¹ Notice that we can do that since all expressions are positive.

7 Proof of Theorem 3

7.1 Any AEM meets WPNI

We start from an AEM φ determined by the functions f^1, \dots, f^k .

Let s^i be the stand-alone cost of agent i and let S^i be a stand-alone path of agent i . That is, S^i is a demand path of agent i with cost s^i .

Let λ^i be the largest number such that $f^i(\lambda^i) = s^i$. Without loss of generality, assume that $\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^k$.

Let λ^* be such that

$$\sum_i \min\{C(X^i), f^i(\lambda^*)\} = C(X^1, \dots, X^k).$$

Step 1. For any NE $X = (X^1, \dots, X^k)$, there is an index m such that:

- i. $\varphi_i(X) = s^i$ for $i = 1, \dots, m$,
- ii. $\varphi_h(X) = f^h(\lambda^*)$ for $h > m$, and
- iii. $\lambda^m < \lambda^* \leq \lambda^{m+1}$.

Proof When there is no confusion, we denote $\varphi_j(X)$ simply as φ_j for any agent j .

Consider the set $M = \{i \in \bar{K} \mid \varphi_i < f^i(\lambda^*)\}$.

First, note that if $j \in M$, then $\varphi_j = s^j$ and $C(X_j) = s^j$. Since

$$\varphi_j = \min\{C(X^j), f^j(\lambda^*)\} < f^j(\lambda^*),$$

then $\varphi_j = C(X^j)$. Note that $C(X^j) \geq s^j$ because s^j is the stand-alone cost of agent j . If $C(X^j) > s^j$ then

$$\varphi_j(X) = C(X^j) > s^j \geq \varphi_j(S^j, X^{-j}),$$

where the last inequality follows by individual rationality. Therefore, agent j can deviate to his stand-alone path S^j , which contradicts the fact that X is a NE. Thus, $C(X^j) = s^j$ and $\varphi_j(X) = C(X^j) = s^j$.

Notice that M is a set of consecutive agents. We will show that if $j \in M$, where $j > 1$, then $j - 1 \in M$. Indeed, if $j \in M$, then $s^j < f^j(\lambda^*)$. Thus, $f^j(\lambda^j) < f^j(\lambda^*)$. Therefore, $\lambda^j < \lambda^*$ by the monotonicity of f^j . Since $\lambda^{j-1} \leq \lambda^j$, then $\lambda^{j-1} < \lambda^*$. Therefore, $s^{j-1} = f^{j-1}(\lambda^{j-1}) < f^{j-1}(\lambda^*)$ by the monotonicity of f^{j-1} and the choice of λ^{j-1} .

Now, we see that $\varphi_{j-1} < f^{j-1}(\lambda^*)$. To get a contradiction, assume that $\varphi_{j-1} = f^{j-1}(\lambda^*)$. Then, $\varphi_{j-1} > s^{j-1}$. Since $s^{j-1} \geq \varphi_{j-1}(S^{j-1}, X^{j-1})$, then $\varphi_{j-1}(X) > \varphi_{j-1}(S^j, X^{j-1})$. This contradicts the fact that X is a NE. Hence, $\varphi_{j-1} < f^{j-1}(\lambda^*)$, thus $j - 1 \in M$.

Denote $M = \{1, \dots, m\}$. For this set, conditions i and ii are satisfied. Now, we see that iii also holds.

Assume that $\lambda^* > \lambda^{m+1}$ then $f^{m+1}(\lambda^*) > f^{m+1}(\lambda^{m+1}) = s^{m+1}$ by the monotonicity of f^{m+1} and because λ^{m+1} is the maximal value that satisfies $f^{m+1}(\lambda^{m+1}) = s^{m+1}$. Thus, $\varphi_{m+1} > s^{m+1}$. Since $\varphi_{m+1}(S^{m+1}, X^{-(m+1)}) \leq s^{m+1}$ by individual rationality, then $\varphi_{m+1} > \varphi_{m+1}(S^{m+1}, X^{-(m+1)})$. Therefore, agent $m + 1$ can decrease his cost-share at profile X by deviating to S^{m+1} . This contradicts X to be a NE. Hence, $\lambda^* \leq \lambda^{m+1}$.

Finally, since $C(X^i) = s^i$ for $i = 1, \dots, m$, and $\varphi_i = s^i$, then $f^i(\lambda^i) = s^i < f^i(\lambda^*)$. Thus, by the monotonicity of f^i , we have that $\lambda^i < \lambda^*$. \square

Step 2. φ Pareto ranks the equilibria.

Proof Consider any two equilibria $X = (X^1, \dots, X^k)$ and $\tilde{X} = (\tilde{X}^1, \dots, \tilde{X}^k)$. Let (m, λ) and $(\tilde{m}, \tilde{\lambda})$ be the values given by step 1 for equilibrium X and \tilde{X} , respectively.

If $m < \tilde{m}$, then $\lambda < \tilde{\lambda}$. Indeed, by step 1 (part iii), $\lambda \leq \lambda^{m+1} \leq \lambda^{\tilde{m}} < \tilde{\lambda}$. Therefore, by step 1 (parts i and ii), agents $\{1, \dots, m\}$ are indifferent between both equilibria and agents $\{m + 1, \dots, k\}$ strictly prefer equilibrium X to \tilde{X} .

On the other hand, if $m = \tilde{m}$, then agents $\{1, \dots, m\}$ are indifferent between both equilibria, and agents $\{m + 1, \dots, k\}$ rank the equilibrium depending on whether $\lambda < \tilde{\lambda}$ or vice versa. \square

7.2 WPNI implies AEM

We start the proof for two agents.

7.2.1 Proof for two agents

Step 1. The mechanisms satisfy truncation, that is, $\varphi[c; y_1, y_2] = \varphi[c; \tilde{y}_1, \tilde{y}_2]$ for any $(\tilde{y}_1, \tilde{y}_2) \geq \varphi[c; y_1, y_2]$.

Proof Consider a feasible profile $(c; y_1, y_2)$ such that $y_1 + y_2 > c$. Let (p_1, p_2) be such that $\varphi(c; y_1, y_2) = (p_1, p_2)$ and assume without loss of generality that $p_1 < y_1$. Consider \tilde{y}_1 such that $p_1 < \tilde{y}_1 < y_1$.

Construct the network depicted in Fig. 10 such that agent 1 has two strategies with costs y_1 and \tilde{y}_1 , agent 2 also has two strategies with the same cost y_2 , and the costs of the top and bottom problems are $(c + \epsilon; y_1, y_2)$ and $(c; \tilde{y}_1, y_2)$, respectively.

Clearly, the graphs that generate $(c + \epsilon; y_1, y_2)$ and $(c; \tilde{y}_1, y_2)$ are a NE for small ϵ .

By WPNI, $\varphi(c + \epsilon; y_1, y_2) \geq (c; \tilde{y}_1, y_2)$.

As ϵ tends to zero, and using continuity

$$\varphi(c; y_1, y_2) \geq \varphi(c; \tilde{y}_1, y_2). \tag{12}$$

Similarly, consider the network in Fig. 11 such that agent 1 has two strategies with costs y_1 and \tilde{y}_1 , agent 2 also has two strategies with the same cost y_2 , and the costs of the top and bottom problems are $(c; y_1, y_2)$ and $(c + \epsilon; \tilde{y}_1, y_2)$, respectively.

Clearly, those two problems are Nash equilibria for small ϵ .

By WPNI $\varphi(c + \epsilon; \tilde{y}_1, y_2) \geq \varphi(c; y_1, y_2)$.

Fig. 10 WPNI implies AEM (part 1)

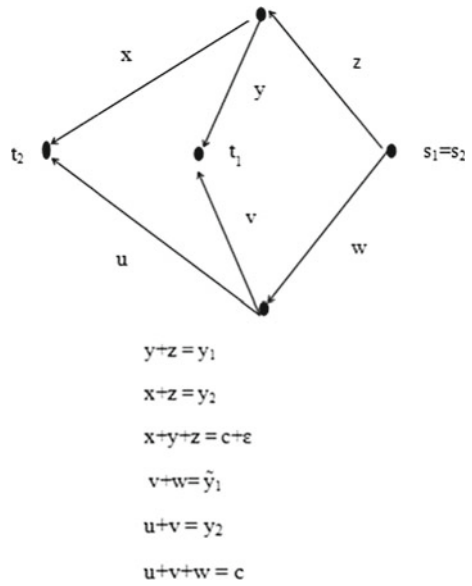
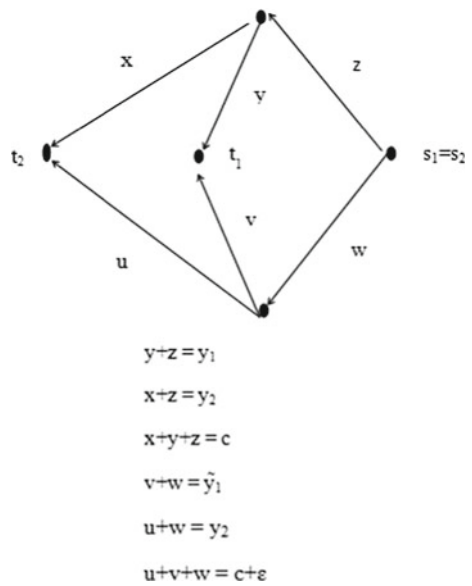


Fig. 11 WPNI implies AEM (part 2)



As ε tends to zero, and using continuity

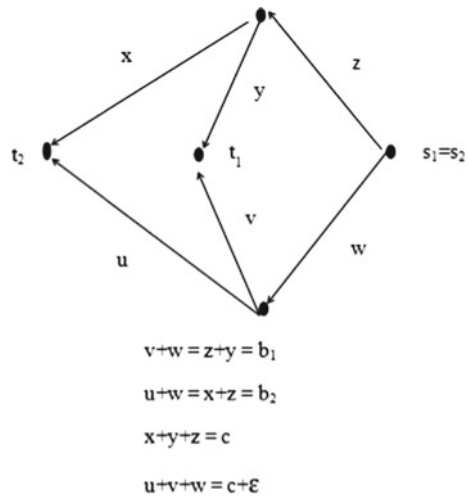
$$\varphi(c; \tilde{y}_1, y_2) \geq \varphi(c; y_1, y_2). \tag{13}$$

by Eqs. 12 and 13

$$\varphi(c; \tilde{y}_1, y_2) = \varphi(c; y_1, y_2).$$

□

Fig. 12 Monotonicity at truncation point



For any $c > 0$, let $g(c) = \varphi[c; c, c]$.

Step 2. $\varphi[c; y_1, y_2] = g(c)$ if $(y_1, y_2) \geq g(c)$; $= (y_1, c - y_1)$ if $y_1 < g_1(c)$; $= (c - y_2, y_2)$ if $y_2 < g_2(c)$.

Proof By step 1 and continuity, $\varphi[c; y_1, y_2] = g(c)$ if $(y_1, y_2) \geq g(c)$.

Consider $y = (y_1, c)$ such that $y_1 < g_1(c)$; and let $(p_1, p_2) = \varphi[c; y_1, c]$. Assume that $p_1 < y_1$.

By continuity, $\varphi_1(y_1 + \epsilon, c) \rightarrow p_1$ as ϵ tends to zero. Let $\tilde{\epsilon}$ be such that $\tilde{p}_1 = \varphi_1(y_1 + \tilde{\epsilon}, c) < y_1$.

Consider the demand $(\frac{\tilde{p}_1+y_1}{2}, c)$, by truncation $\varphi(c; \frac{\tilde{p}_1+y_1}{2}, \tilde{y}_2) = (\tilde{p}_1, c - \tilde{p}_1)$, for any $y_2 > c - p_1$.

Similarly, $\varphi(c; \frac{\tilde{p}_1+y_1}{2}, \tilde{y}_2) = (p_1, c - p_1)$, which is a contradiction.

By truncation, $\varphi_1(y_1 + \tilde{\epsilon}, c) < y_1$. □

Step 3. The mechanism is weakly monotonic at the truncation point. That is, $g(c) < g(\tilde{c})$ for $c < \tilde{c}$.

Proof Suppose that the mechanism is not weakly monotonic at the truncation point. Then, for any small ϵ , we can find c and $c + \epsilon$ such that $g_2(c) > g_2(c + \epsilon)$ and $g_1(c) > g_1(c + \epsilon)$ (or vice versa).

Pick small ϵ and $b \gg \max\{g(c), g(c + \epsilon)\}$ and $c > b_1 + b_2$.

Consider the network depicted in Fig. 12 such that every agent has two strategies of equal cost b_1 and b_2 and generates problems $(c; b_1, b_2)$ and $(c + \epsilon; b_1, b_2)$. Clearly, there are only two equilibria with costs c and $c + \epsilon$, but they are not Pareto ranked since $\varphi(c; b_1, b_2) = g(c)$ and $\varphi(c + \epsilon; b_1, b_2) = g(c + \epsilon)$. □

Step 4. The mechanism can be represented by the above functions.

Proof Consider the AEM represented by the functions $g_i(c)$ as above. It is easy to show that this mechanism generates the mechanism as above.

Indeed, consider $(c; y)$ feasible. Then, if $(y_1, y_2) \geq g(c)$, then $\varphi(c; y) = g(c)$.
 If $y_1 < g_1(c)$, then by truncation $\varphi_1(c; y) = g_1(c)$ and $\varphi_2(c; y) = c - g_1(c)$.
 If $y_2 < g_2(c)$, then by truncation $\varphi_2(c; y) = g_2(c)$ and $\varphi_1(c; y) = c - g_2(c)$.

□

7.2.2 Extension to more than two agents

Proof Consider any parametric solution with k agents, $k > 2$. We can replicate the above arguments for a network of any two agents $\{i, j\}$ by setting $y_l = 0$ for $l \neq i, j$ (demanding independent demands with cost zero). Thus, by the previous case, $F_i(\lambda, y_i) = \min\{y_i, g_i(\lambda)\}$ for some non-decreasing function $g_i(\lambda)$. □

8 Proof of Corollary 4

8.1 $POS(EG) = H(k)$

Consider the efficient profile $P = (P^1, \dots, P^k)$ with cost c^* . Assume without loss of generality that $C(P^1) \geq C(P^2) \geq \dots \geq C(P^k)$. Let S^i be a stand-alone path of agent i with cost s^i . Let $p = EG(c^*; C(P^1), C(P^2), \dots, C(P^k))$. Clearly $p_1 \geq p_2 \geq \dots \geq p_k$, and $p_i \leq \frac{c^*}{i}$ for $i = 1, \dots, k$. Let λ^* and m be such that $p_k \leq p_{k-1} \leq \dots \leq p_{m+1} < \lambda^* = p_m = \dots = p_1$. Let $\tilde{K} = \{i | s^i < \lambda^*\}$. That is, \tilde{K} is the set of agents with a stand-alone cost less than λ^* . Consider the profile $Q = (P_{(\tilde{K})^c}, S_{\tilde{K}})$; that is, the strategy from each agent in \tilde{K} is replaced by his stand-alone path.

Clearly, if $i > m$ then $i \in \tilde{K}$, since $s^i \leq p_i < \lambda^*$. Therefore, Q contains at least all agents who are paying their demand at P , but might include others.

Let $\tilde{k} = |\tilde{K}|$ be the cardinality of \tilde{K} . First, notice that

$$C(S_{\tilde{K}}) \leq (\tilde{k} - m)\lambda^* + s^{m+1} + \dots + s^k \leq (\tilde{k} - m)\frac{c^*}{m} + \frac{c^*}{m+1} + \dots + \frac{c^*}{k}.$$

Therefore $C(S_{(\tilde{K})^c}) \leq \frac{c^*}{k-\tilde{k}+1} + \dots + \frac{c^*}{k}$.

Hence, $C(Q) \leq c^* + C(S_{(\tilde{K})^c}) \leq c^* + \frac{c^*}{k-\tilde{k}+1} + \dots + \frac{c^*}{k}$.

We repeat the above algorithm consecutively to the profile Q . That is, we find λ and move all agents with stand-alone cost less than λ to their stand-alone path. Since there is at most k agents, this algorithm finishes in at most k steps. Let R be the final profile of this algorithm. From the above arguments, $C(R) \leq H(k)c^*$. Let $\tilde{\lambda}$ be the solution to the problem $EG(C(R); C(R_1), \dots, C(R_k))$. If $EG_i(C(R); C(R_1), \dots, C(R_k)) = \tilde{\lambda}$ then $s_i \geq \tilde{\lambda}$. On the other hand, if $EG_i(C(R); C(R_1), \dots, C(R_k)) < \tilde{\lambda}$ then $R^i = S^i$. Similar to the existence of equilibrium for AEM, the br tatonnement would converge to an equilibrium starting from profile R , since λ and the cost would decrease at every step.

Indeed, if an agent is paying his stand-alone cost, the only way to decrease his payment is by increasing his demand, and thus decreasing his cost. Therefore, $\tilde{\lambda}$ should decrease. At his br, his stand-alone cost should be larger than the new λ .

On the contrary, if an agent is paying $\tilde{\lambda}$, then his br should decrease $\tilde{\lambda}$ because his stand-alone cost is larger than $\tilde{\lambda}$.

8.2 For any AEM ξ such that $\xi \neq EG$ we have $POS(\xi) > H(k)$

Proof Consider the AEM ξ generated by the functions f^1, \dots, f^k . Since $\xi \neq EG$, then there is i, j such that $f^i \neq f^j$.

Let λ^* be such that $f^i(\lambda^*) \neq f^j(\lambda^*)$, and c^* such that $c^* = f^1(\lambda^*) + \dots + f^k(\lambda^*)$.

There is an agent l such that $f^l(\lambda^*) > \frac{c^*}{k}$, without loss of generality, assume that this agent is agent k . That is, $f^k(\lambda^*) > \frac{c^*}{k}$. Let $\varphi_k^* = \varphi_k[c^*; c^*, \dots, c^*] = f^k(\lambda^*) > \frac{c^*}{k}$.

Consider the problem $[c^* + f^k(\lambda^*); c^*, c^*, \dots, f^k(\lambda^*)]$. Since

$$\varphi_k[c^* + f^k(\lambda^*); c^*, c^*, \dots, f^k(\lambda^*)] \leq f^k(\lambda^*),$$

then there is an agent l , where $l \neq k$, such that

$$\varphi_l[c^* + f^k(\lambda^*); c^*, c^*, \dots, f^k(\lambda^*)] \geq \frac{c^*}{k-1}.$$

Without loss of generality, assume that this agent is agent $k-1$. Let

$$\varphi_{k-1}^* = \varphi_{k-1}[c^* + f^k(\lambda^*); c^*, c^*, \dots, f^k(\lambda^*)],$$

thus $\varphi_{k-1}^* \geq \frac{c^*}{k-1}$.

Consider the problem

$$[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)].$$

Since

$$\varphi_k[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \leq f^k(\lambda^*)$$

and

$$\varphi_{k-1}[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \leq \varphi_{k-1}^*.$$

Then there is an agent l such that

$$\varphi_l[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)] \geq \frac{c^*}{k-2}.$$

Without loss of generality, assume that this agent is agent $k - 2$. Let

$$\varphi_{k-2}^* = \varphi_{k-2}[c^* + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{k-1}^*, f^k(\lambda^*)],$$

thus $\varphi_{k-2}^* \geq \frac{c^*}{k-2}$.

Continuing the same way, at step i , consider the problem

$$[c^* + \varphi_{i+1}^* + \dots + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{i+1}^*, \dots, \varphi_{k-1}^*, f^k(\lambda^*)].$$

Since

$$\varphi_k[c^* + \varphi_{i+1}^* + \dots + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{i+1}^*, \dots, \varphi_{k-1}^*, f^k(\lambda^*)] \leq f^k(\lambda^*)$$

and

$$\varphi_j[c^* + \varphi_{i+1}^* + \dots + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{i+1}^*, \dots, \varphi_{k-1}^*, f^k(\lambda^*)] \leq \varphi_j^*,$$

for $j = k - 1, \dots, i + 1$. Then, there is an agent l such that

$$\varphi_l[c^* + \varphi_{i+1}^* + \dots + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{i+1}^*, \dots, \varphi_{k-1}^*, f^k(\lambda^*)] \geq \frac{c^*}{i}.$$

Without loss of generality, assume that this agent is agent i . Let

$$\varphi_i^* = \varphi_i[c^* + \varphi_{i+1}^* + \dots + \varphi_{k-1}^* + f^k(\lambda^*); c^*, c^*, \dots, c^*, \varphi_{i+1}^*, \dots, \varphi_{k-1}^*, f^k(\lambda^*)].$$

Thus, $\varphi_i^* \geq \frac{c^*}{i}$.

Consider the network in Fig. 13.

Since $\varphi_i^* \geq \frac{c^*}{i}$ for $i = 1, \dots, k - 1$ and $\varphi_k^* > \frac{c^*}{k}$, then the only equilibrium is where agent i demands the link (s_i, t) with cost φ_i^* . This equilibrium is inefficient and has a cost equal to $\sum_{i=1}^k \varphi_i^* > H(k)(c^* + \epsilon)$, for small ϵ . □

8.3 Any IR mechanism has a PoS at least $H(k)$

Proof We show by an example that any IR cost-sharing mechanism must have a PoS of at least $H(k)$. Consider a situation as shown in Fig. 14. Here, every agent i has two strategies—either connect his demand nodes directly where the cost of the path is $1/i$ or connect through the path where link costs are 0 and $1 + \epsilon$. Consider any arbitrary cost-sharing mechanism ξ that satisfies individual rationality. We will show that if there exists an equilibrium, then this is where every agent is using his direct path to t . We prove this by contradiction.

Case 1. Assume that all agents use a free link to v and then the common link of cost $1 + \epsilon$ to t . But then at least one of the agents must be paying more than $1/k$. We

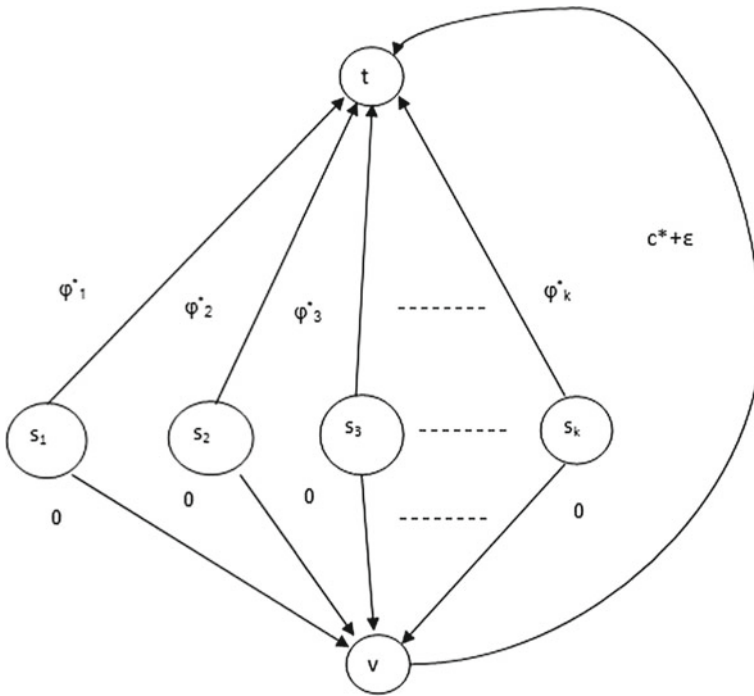


Fig. 13 Optimality of the EG mechanism

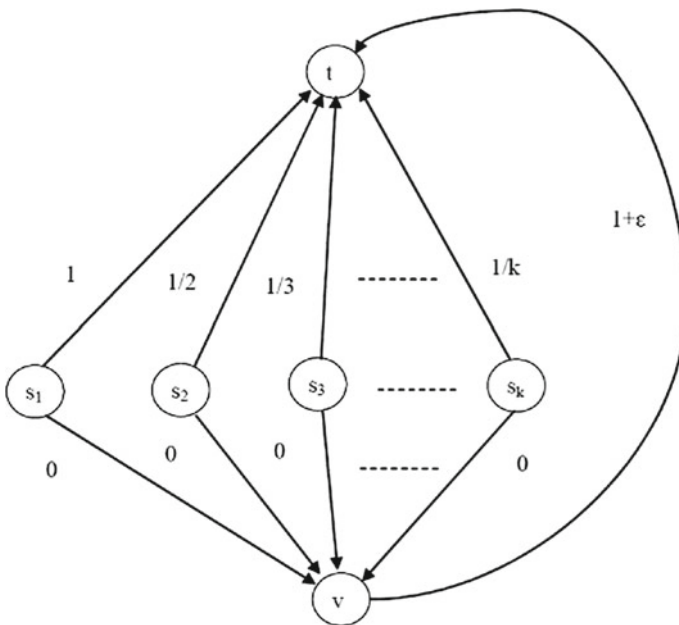


Fig. 14 Incompatibility of EFF and IR

assume that this agent is the k th agent in some configuration²² of the graph. Then, he will have a profitable deviation to go to the direct link of cost $1/k$ under any IR mechanism.

Case 2. Assume that s agents are using their direct link and $k - s$ agents are sharing the common link to v . Then, it follows from the individual rationality of the s agents that at least one of the remaining $k - s$ agents must be paying more than $1/(k - s)$. Notice that in this case, there exists an unused direct link, say, $s_j \rightarrow t$, of cost $1/s_j$ which is at most $1/(k - s)$. Now in some configuration of the graph, agent j will be the agent who is paying the above-mentioned amount of more than $1/(k - s)$ and thus he would like to deviate.

We have just shown that no configuration different from the direct connection is a NE. If the equilibrium exists, then it must be the direct connection and has a cost equal to $H(k)$, whereas the efficient graph has a cost equal to $1 + \epsilon$ (everyone uses a costless link to node v and then the common link to t). As ϵ goes to zero, the PoS approaches to $H(k)$.

Finally, if there is no equilibrium, then the PoS equals infinity. □

8.4 Lower bound for PoS(PR)

Proof Consider the network as shown in Fig. 15. We show that the unique equilibrium of the PR is of order k . Let the costs of links $s_i \rightarrow t$ be x_i . Straightforward computations show that the k th agent will deviate from the efficient graph of cost $1 + \epsilon$ if $x_k \leq \frac{1-k+\sqrt{(k-1)^2+4k(k-1)}}{2k}$. As k grows, x_k converges to the golden number $\frac{\sqrt{5}-1}{2}$ in contrast to $1/k$ for the uniform mechanism, which goes to zero. Also $x_{t-1} > x_t$ for all $t = 2, 3, \dots, k$ and $x_1 = 1$. Thus, the lower bound on the PoS of the PR is $\sum_{i=1}^k x_i$, which is of order k . □

9 Other proofs

9.1 Proof of Proposition 2

The proof is very similar to the existence of a NE for an AEM in Lemma 2.

Consider the asymmetric EG φ generated by the functions f^1, \dots, f^k .

Let $s^i = \min_{P_i \in \Pi_i(N)} C(P_i)$ be the stand-alone cost of agent i .

Let λ^i be the largest number such that $f^i(\lambda^i) = s^i$. Without loss of generality, assume that $\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^k$.

Consider a strategy profile $T = (T^1, T^2, \dots, T^k)$ and let $t^i = C(T^i)$ be the cost of strategy T^i . Let λ^* be a solution to the system $\sum_i \min\{t^i, f^i(\lambda^*)\} = C(T)$.

²² It is important to note that just one such configuration is enough, since PoS is a measure of the performance of the best NE in the worst case example.

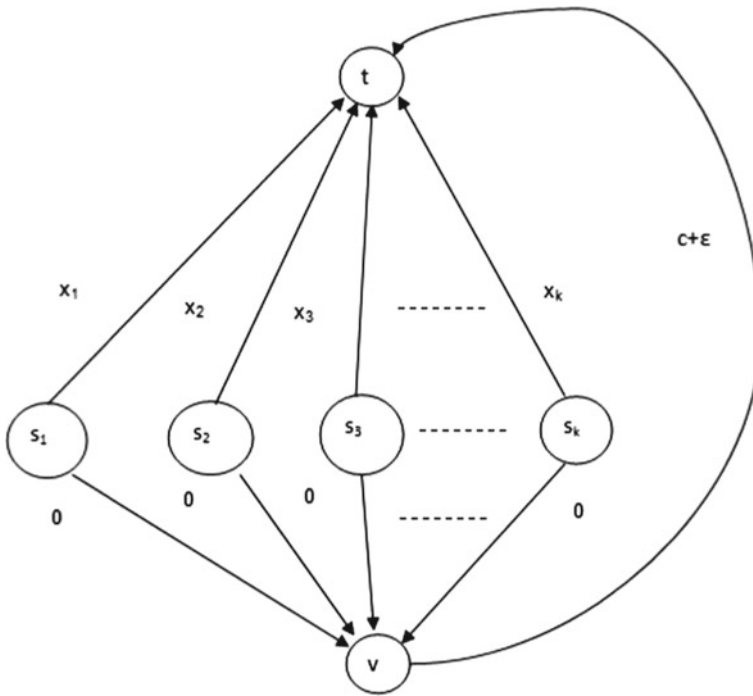


Fig. 15 PoS(PR) is of order k

We say that the strategy profile T is *Nash-convergent* if for every agent i : (a) $\varphi_i(T) = s^i$, or (b) $\varphi_i(T) = f^i(\lambda^*)$ and $\lambda^* \leq \lambda^i$.

Step 1. Consider a Nash-convergent profile T . Suppose that coalition S has a deviation Y^S that weakly improves all agents in S and strictly improves agent $j \in S$. Then, the profile (Y^S, T^{-S}) is also a Nash-convergent profile with smaller λ^* .

Proof We divide this step into steps 1.1 and 1.2.

Let $\tilde{\lambda}$ be the smallest solution to

$$\sum_{k \in S} \min\{y^k, f^k(\tilde{\lambda})\} + \sum_{i \notin S} \min\{t^i, f^i(\tilde{\lambda})\} = C(Y^S, T^{-S}).$$

Step 1.1. The relation $\tilde{\lambda} < \lambda^*$ holds.

Since T is Nash-convergent, then $\varphi_j(T) = s^j$, or $\varphi_j(T) = f^j(\lambda^*)$ and $\lambda^* \leq \lambda^j$.

Case 1. $\varphi_j(T) = s^j$.

Note that

$$\min\{t^j, f^j(\lambda^*)\} = s^j > \min\{y^j, f^j(\tilde{\lambda})\}.$$

Since s^j is the stand-alone cost of agent j , then $y^j \geq s^j$. Therefore, $\min\{y^j, f^j(\tilde{\lambda})\} = f^j(\tilde{\lambda})$. Thus,

$$\min\{t^j, f^j(\lambda^*)\} > f^j(\tilde{\lambda}).$$

This implies, $f^j(\lambda^*) > f^j(\tilde{\lambda})$. By the monotonicity of f^j , we have that $\lambda^* > \tilde{\lambda}$.

Case 2. $\varphi_j(T) = f^j(\lambda^*)$ and $\lambda^* \leq \lambda^i$.

Note that

$$\min\{t^j, f^j(\lambda^*)\} = f^j(\lambda^*) > \min\{y^j, f^j(\tilde{\lambda})\}. \tag{14}$$

We show that $y^j \geq f^j(\tilde{\lambda})$. Indeed, assume the contrary, that is $f^j(\tilde{\lambda}) > y^j$. Then, $\min\{y^j, f^j(\tilde{\lambda})\} = y^j$. Thus, from Eq. 14,

$$f^j(\lambda^*) > y^j \geq s^j. \tag{15}$$

Since $\lambda^j \geq \lambda^*$, then $s^j = f^j(\lambda^j) \geq f^j(\lambda^*)$. This contradicts equation 15. Therefore, $y^j \geq f^j(\tilde{\lambda})$.

Now, we show that $\lambda^* > \tilde{\lambda}$. Since $y^j \geq f^j(\tilde{\lambda})$, then from Eq. 14, $f^j(\lambda^*) > f^j(\tilde{\lambda})$. Thus, $\lambda^* > \tilde{\lambda}$ by the monotonicity of f^j .

Step 1.2. The profile (Y^S, T^{-S}) is Nash-convergent.

- First, we consider an agent i , where $i \notin S$, that satisfies case (a). That is, $\varphi_i(T) = \min\{t^i, f^i(\lambda^*)\} = s^i$.

Since $\lambda^* > \tilde{\lambda}$, then

$$s^i = \min\{t^i, f^i(\lambda^*)\} \geq \min\{t^i, f^i(\tilde{\lambda})\}. \tag{16}$$

If $\varphi_i(Y^S, T^{-S}) = t^i$, then $\min\{t^i, f^i(\tilde{\lambda})\} = t^i$. Therefore, by Eq. 16, $s^i = t^i$. Hence,

$$\varphi_i(Y^S, T^{-S}) = s^i.$$

On the other hand, if $\varphi_i(Y^S, T^{-S}) = f^i(\tilde{\lambda})$, then $\min\{t^i, f^i(\tilde{\lambda})\} = f^i(\tilde{\lambda})$. Therefore, by Eq. 16, $s^i \geq f^i(\tilde{\lambda})$. Since $s^i = f^i(\lambda^i)$, then $f^i(\lambda^i) \geq f^i(\tilde{\lambda})$. Hence, $\lambda^i \geq \tilde{\lambda}$ by the monotonicity of f^i and because λ^i is the maximal value that satisfies $f^i(\lambda^i) = s^i$.

- Second, we consider an agent i , where $i \notin S$, that satisfies case (b) $\varphi_i(T) = f^i(\lambda^*)$ and $\lambda^* \leq \lambda^i$.

Since $\varphi_i(T) = f^i(\lambda^*)$, then $\min\{t^i, f^i(\lambda^*)\} = f^i(\lambda^*)$. Thus, $t^i \geq f^i(\lambda^*)$. Also, note that $f^i(\lambda^*) \geq f^i(\tilde{\lambda})$ because $\lambda^* > \tilde{\lambda}$. Hence, $t^i \geq f^i(\tilde{\lambda})$. Thus, $\varphi_i(Y^S, T^{-S}) = \min\{t^i, f^i(\tilde{\lambda})\} = f^i(\tilde{\lambda})$.

Since $\lambda^* \leq \lambda^i$ and $\tilde{\lambda} \leq \lambda^*$, then $\tilde{\lambda} \leq \lambda^i$ as desired.

- Third, we consider the agent $j \in S$ who changed his strategy. first note that because the profile T is Nash-convergent, then

$$\varphi_j(T) \leq s^j.$$

This is obvious if agent j satisfies case a (in Nash-convergence definition). If agent j satisfies case b, then $\varphi_j(T) \leq f^j(\lambda^*)$ and $\lambda^* \leq \lambda^j$. By the monotonicity of f^j , we have $f^j(\lambda^*) \leq f^j(\lambda^j)$. Therefore, $\varphi_j(T) \leq f^j(\lambda^j) = s^j$.

Since agent j 's cost-share does not increase by deviating, then

$$\varphi_j(Y^S, T^{-S}) \leq \varphi_j(T) \leq s^j.$$

If $\varphi_j(Y^S, T^{-S}) = \min\{y^S, f^j(\tilde{\lambda})\} = y^j$, then $y^j \leq s^j$. Since s^j is the stand-alone cost then $s^j \leq y^j$. Therefore, $y^j = s^j$ and case (a) is satisfied.

On the other hand, if $\varphi_j(Y^S, T^{-S}) = \min\{y^j, f^j(\tilde{\lambda})\} = f^j(\tilde{\lambda})$, then

$$f^j(\tilde{\lambda}) \leq s^j = f^j(\lambda^j).$$

Therefore $\tilde{\lambda} \leq \lambda^j$ by the monotonicity of f^j and because λ^j is the maximal value that satisfies $f^j(\lambda^j) = s^j$. Thus, case (b) is satisfied. □

Step 2. The equilibrium with the minimal cost is Nash-convergent.

Proof Consider the equilibrium with the minimal cost X . Let λ^* is such that

$$\sum_i \min\{C(X^i), f^i(\lambda^*)\} = C(X^1, \dots, X^k).$$

By step 1 of Sect. 7.1, there exists an index m such that:

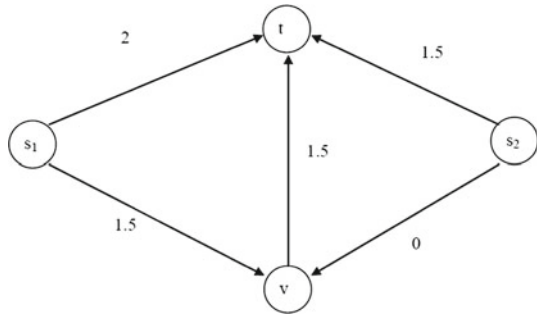
- i. $\varphi_i(X) = s^i$ for $i = 1, \dots, m$,
- ii. $\varphi_h(X) = f^h(\lambda^*)$ for $h > m$, and
- iii. $\lambda^m < \lambda^* \leq \lambda^{m+1}$.

If $\varphi_i(X) = s^i$, then case (a) holds.

If $\varphi_i(X) = f^i(\lambda^*)$, then $\lambda^* \leq \lambda^i$. Therefore, case (b) holds. □

Step 3. Finally, we show the existence of equilibrium. Consider the better response tatonnement where at every step a coalition of agents picks a path that weakly decrease their cost-share of the deviating agents and strictly decrease the cost-share of at least one of them. Then, the better response tatonnement starting from the NE with the minimal cost converges in a finite number of iterations. This is because at every step the value λ^* decreases, and there are a finite number of strategies. If the number of steps is greater than zero, then the limit profile is a NE with a smaller λ . By step 2 of Sect. 7.1, the equilibria are Pareto ranked and a smaller λ implies the equilibrium has smaller cost. This is a contradiction.

Fig. 16 Network illustrating EG does not admit a potential



9.2 The game generated by EG does not admit an ordinal potential

Consider a network shown in Fig. 16 below. Here there are two agents, agent 1 and agent 2, their demand nodes are $\{s_1, t\}$ and $\{s_2, t\}$, respectively. Both agents have two strategies each. One of the strategies of agent i ($i = 1, 2$) is to connect through the direct link, i.e $s_i \rightarrow t$, and her other strategy is to connect indirectly through the node v , that is, $s_i \rightarrow v \rightarrow t$. We denote the two strategies of agent 1 as a and b and the two strategies of agent 2 as c and d where $a := s_1 \rightarrow v \rightarrow t$, $b := s_1 \rightarrow t$, $c := s_2 \rightarrow t$ and $d := s_2 \rightarrow v \rightarrow t$. Given the EG mechanism, the game induced by the network on the set of agents can be represented in normal form by the following matrix, where agent 1 is the row player and agent 2 is the column player. The first number in each cell of the matrix corresponds to the cost-share of agent 1, and the second number corresponds to the cost-share of agent 2.

	c	d
a	3, 1.5	1.5, 1.5
b	2, 1.5	2, 1.5

Suppose that this is an ordinal potential game. Then, there must exist an ordinal potential function $P : \{a, b\} \times \{c, d\} \rightarrow \mathbb{R}$ satisfying $P(a, c) > P(b, c) = P(b, d) > P(a, d) = P(a, c)$, which is impossible.

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