Profit-Sharing and Efficient Time Allocation *

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Abstract

Agents are endowed with time that is invested in projects that generate profit. A mechanism divides the profit generated among agents depending on the allocation of time as well as the amount of profit that every project generates.

We study mechanisms that incentivize agents to contribute their time to the level that generates the maximal aggregate profit at the Nash equilibrium regardless of the production functions (*efficiency*). Our main result is the characterization of all the mechanisms that satisfy efficiency. Furthermore, within this class, we characterize the class of mechanisms that are monotone in the payoffs of the agents with respect to technological improvements in the generation of the profit, the addition of time to the agents, as well as mechanisms that are immune to group manipulations.

The class of efficient mechanisms depend on the type of available projects. It expands earlier profit/cost-sharing mechanisms that are independent on the generation of the profit.

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1 Introduction

Profit-sharing mechanisms are widely used by companies in order to increase profits. Such mechanisms include direct cash bonuses to employees that are awarded based on their performance, either individually or collectively. For instance, under an employee stock ownership plan, companies allocate shares of its stock to employees, hence rewarding employees based on the aggregate profit of the company.[1] Although several empirical studies address the effectiveness of profit sharing mechanisms to increase the productivity of a company, little has been said theoretically about the direct relationship between the profitability of companies and their rewards to employees.[2] Our paper is the first study in this area, characterizing a large class of mechanisms that align the benefits of the company with the payments of their employees.

In order to address this, we consider a model where agents decide how to allocate their fixed endowment of time to different projects. Every project could have different production functions that generate profits depending on the allocation of time by agents. We focus on the case of asymmetric information: a planner (such as the owner of the company or manager) does not know the production functions while the employees have perfect information about them. A mechanism (designed by the planner) assigns the payments depending only on the final aggregated profit generated by each project and the time allocation of the agents to the different projects.[3] An application of this problem is the division of the end of year surplus in companies.

The central issue is that although the planner tries to maximize the total profit of the company, he might not know the production functions. That is, the planner's goal is to select a payment scheme that implements the time allocation which generates the maximum total profit at the Nash equilibrium for any set of production functions. We call this property *efficiency*. Thus, even though the planner has a disadvantage in the information with respect to the agents, efficiency leads to the first-best outcome for the planner as if he had full information.

For instance, consider the proportional sharing mechanism that divides the total profit of each project among the agents in proportion to their time allocation to such projects. This mechanism is not necessarily efficient because agents could have the incentive to put more time in the projects that give the agents a larger proportion of time in order to increase a larger share of the profits of such projects, whereas the company might produce a larger share when agents invest their time collectively into a single project. To see this, consider the following example. There are three agents named 1, 2 and 3 and three collaborative projects between these agents, $\{1, 2\}$, $\{1, 3\}$, $\{1, 2, 3\}$. Suppose that each agent is endowed with 1 unit of time that they split between the projects in which they belong to, and every project generates profit based on the time allocated by the agents. For instance, suppose that the production functions of such projects are $\alpha(t_1^{12} + t_2^{12})$ for the project 12, $\beta min(t_1^{13}, t_3^{13})$ for the project 13 and $\gamma min(t_1^{123}, t_2^{123}, t_3^{123})$ for the project 123. When $\alpha = 2.5$, $\beta = 3$ and $\gamma = 6$, the efficient time allocation $((t_1^{12}, t_1^{13}, t_1^{123}), (t_2^{12}, t_2^{123}), (t_3^{13}, t_3^{123})) = ((0, 0, 1), (0, 1), (0, 1))$ requires all agents to invest their time into the project 123 and generates a profit equal to 6 units. Under proportional sharing, each agent receives 2 units of the profit each. However, this is not an equilibrium under the proportional sharing mechanism since both agents 1 and 2 have the incentive to allocate all resources to the project 12, where they can receive 2.5

[1] The employee stock ownership plan is widely spread in silicon valley start-ups and has created several millionaire employees, for instance at Google, Facebook, and Yahoo. This mechanism resembles the average-profit mechanisms or other asymmetric variations discussed in the paper.

[2] Empirical studies of profit sharing in companies include Kruse (1992), Bhargava (1994) and Kraft and Ugarković (2006), while Weitzman and Kruse (1990) and Prendergast (1999) provide surveys of profit sharing in companies.

[3] This asymmetric information is natural in large companies, where the owner (board) of the company sets general profit-sharing policies for employees before the actual production functions are realized.

units of profit instead of 2.

On the other hand, consider the average profit-sharing mechanism, where each agent gets a fixed share of the total profit generated by the company.[4] This mechanism is efficient because if an agent deviates from the equilibrium that generates the maximum total profit (efficient equilibrium), then the total profit of the company does not increase and neither does his payment.

Alternatively, consider the Shapley mechanism, where the profit of every project is distributed equally among the agents who belong to each project irrespective of their time allocation. When the set of feasible projects have the same size, for instance, if all the projects have size 2, the Shapley mechanism is also efficient. This is because if an agent deviates from the efficient equilibrium, then the total profit of the company does not increase. Thus, since the profit of the company is just the sum of the profit of each project, the aggregated profit of the projects in which this agent participates does not increase, and neither does his payment. However, in general, the Shapley mechanism is not efficient when the projects have different size.[5]

More generally, consider a mechanism where the payoff of an agent only depends positively on the total (aggregate) profit generated by the projects in which the agent participates, as well as the time contributions and profit generated by the projects in which the agent does not participate. We call these mechanisms *separable*. Those mechanisms are efficient because if an agent decides to change his time allocation, then his payoff can only influence the total output of his projects. If the total output of his projects decreases, so does the total profit generated by the company; thus, the agent is worse off. The average profit and Shapley mechanisms are particular cases of separable mechanisms.

Our main result is the characterization of all the mechanisms that satisfy efficiency. The class of efficient mechanisms coincides exactly with the class of separable mechanisms (Theorem 1). The class of efficient, anonymous and time-independent mechanisms is narrow and can be easily described as linear combinations of Shapley and average-profit mechanisms, where the weight assigned to the mechanisms depend on the size of the available projects that can form (Corollary 1).

We also look at the monotonicity of the payments of the agents with respect to the increase of time (*time-monotonicity*) and with respect to increases in the production functions (*technology-monotonicity*).[6] The class of efficient mechanisms that are robust to these monotonicity properties depends on the connectedness of projects (via the non-empty pairwise intersection). The agents who are connected (directly or indirectly via pairwise intersection of projects to which they belong) should get a payment that is monotonic in the aggregate profits generated by connected projects (Corollary 2). Finally, we characterize the class of mechanisms where agents cannot gain by jointly coordinating their time allocation at the efficient equilibrium (Corollary 3).

[4]The mechanism can be interpreted as the stock awarding mechanism, in which agents are given a fixed share of stocks in the company, and thus their final allocation of profit depends on the aggregate profit generated by the company.

[5]This can be seen in the example above, where projects of size 2 and size 3 exist. Herein, agents 1 and 2 have the incentive to deviate from the efficient equilibrium.

[6]In particular, time-monotonicity requires that increases in the allocated time of the agents should not make them worse off. A particular case of this property is agent-monotonicity, that requires that agents are not worse off as new agents arrive at the company. On the other hand, technology-monotonicity requires that improvements in the technology of projects (represented as increases in the production function at any time allocation) should not make agents worse off.

1.1 Related Literature

The concept of implementation of the efficient allocation in a Nash equilibrium has been explored widely in the literature. Maskin and Sjöström (2002) surveys full implementation of efficient outcomes in different production functions. However, the literature of implementation in very general economies has typically lead to impossibilities. We contrast with this literature by finding a specific economy where several mechanisms can implement the efficient allocation.

We focus on the case where agents need to contribute their full-time allocation and the entire profit is allocated to the agents, therefore the traditional issues of moral hazards are ruled out (e.g., Holmstrom (1982)). This restriction is similar to allocation mechanisms for a fixed divisible resource (such as a dollar) depending on the report of the agents (e.g., de Clippel et al. (2008); Tideman and Plassmann (2008); Mackenzie (2015)).

A closely related work studies cost sharing allocation mechanisms that implement the cost-minimizing network. For instance, Juarez and Kumar (2013) focus on implementing the efficient allocation in connection network, where agents should be provided with the incentive to select the cost minimizing network. Hougaard and Tvede (2012, 2015), characterize truthfully implementing cost-minimizing networks by changing the announcement rule. Juarez et al. (2016) studies the axiomatic implementation of efficient outcomes in sequential sharing. Kumar (2013) investigates secure implementation in more general cost-sharing models. Hougaard et al. (2017) study the axiomatic sharing of profits in trees and Moulin and Laigret (2011) study the division of costs/profits of non-redundant items. The narrow classes of mechanisms characterized by Juarez and Kumar (2013); Hougaard and Tvede (2012, 2015); Juarez et al. (2016); Kumar (2013) are a subset of the mechanisms characterized in Corollaries 2 and 3 and resemble the efficient mechanisms when the grand coalition is a feasible project. The classes of mechanisms studied in Hougaard et al. (2017) and Moulin and Laigret (2011) are efficient when the profits (costs) are generated by non-collaborative projects. In contrast with this literature, our profit-sharing rules are dependent on the set of available projects and encompasses most rules studied in this literature.

To our knowledge, this is the first paper to introduce the implementation of the efficient time allocation for any set of production functions.

2 The Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents in an economy. Groups of agents from N can collaborate in projects that generate profit depending on the time allocation of the agents on each project.[7] Formally, let $L \subset 2^N \setminus \{\emptyset\}$ be the set of different projects. For instance, if L contains all subsets of 2 agents, then every group of two agents can collaborate in a project. If $L = 2^N \setminus \{\emptyset\}$ then any potential coalition can collaborate in a project. Let L_i be the set of groups from L that contains agent i and L_{-i} be the set of groups from L that do not contain agent i.

For a given set A and $T \geq 0$, let $\Delta(T, A) = \{x \in \mathbb{R}^A_+ | \sum_{i\in A} x_i = T\}$ be the T-simplex over the set A. Every agent i is endowed with T_i units of time which he can split among the projects in which he participates. The set of time allocations of agent i is the set $D_i = \Delta(T_i, L_i)$. Let $\mathbb{D} = \Pi_{i\in N} D_i$ be the set of all time allocations for all agents. For a given time allocation $t \in \mathbb{D}$, the amount t_i^K is interpreted as the amount of time that agent i spends in project K. For this section, we fix the group of agents N, group of projects L and time endowments $T_1, T_2, \ldots, T_n$. Section 3 will look at the possibility of changes with respect in N, L and $T_1, T_2, \ldots, T_n$.

[7] For simplicity, we assume that there is no repetition in the projects, although a similar argument can be made when projects repeat.

Every project generates profit. Let $\mathbb{F} = \mathbb{R}_+^L$ be the vector of profits for all projects. For a given $F \in \mathbb{F}$, the amount F^K is the profit generated by project $K \in L$.

Definition 1 (Mechanisms)**.** *Fix N, L and $T_1, T_2, \ldots, T_n$. A* ***mechanism*** *is a continuous function $\varphi : \mathbb{D} \times \mathbb{F} \to \mathbb{R}_+^n$ such that*

$$\sum_{i=1}^{n} \varphi_i (t, F) = \sum_{K \in L} F^K.$$

The inputs on a mechanism are the different time allocations and profits of every project. The output is a full distribution of the total profit to the agents.

Example 1. *A.* ***Average profit mechanism****: the final profit of the entire economy is divided equally among all members. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \frac{1}{n} \sum_{K \in L} F^K$$

B. ***Shapley mechanism****: the final profit produced by the project $K \in L$ is shared equally among the agents in K. The total share of every agent is the sum of his shares in the projects. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \sum_{K \in L_i} \frac{F^K}{|K|},$$

where $|K|$ is the number of agents in the project K.

C. ***Proportional sharing mechanism****: the final profit produced by the project K is shared in proportion to the contribution of time of the agents in K. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \sum_{K \in L_i} Pr_i^K(t^K) F^K, \text{ where } Pr_i^K(t^K) = \begin{cases} \frac{t_i^K}{\sum_{j \in K} t_j^K} & \text{if} \quad \sum_{j \in K} t_j^K > 0 \\ 0 & \text{if} \quad \sum_{j \in K} t_j^K = 0 \end{cases}$$

D. ***Generalized Shapley Mechanism****: The share of every agent is a fixed proportion of the profit generated by the projects in which he participates and the unallocated profit of the projects in which the agent does not participate. Formally, consider any collection of projects $L \subset 2^N \setminus \{\emptyset, N\}$ and positive individual shares $\gamma_1, \gamma_2 \ldots, \gamma_n$ such that for any project $K \in L$, the aggregate shares in this project does not exceed 1, $\sum_{j \in K} \gamma_j \leq 1$. For any individual agent i,*

$$\varphi_i(t, F) = \gamma_i \sum_{K \in L^i} F^K + \sum_{M \in L^{-i}} \frac{\gamma_i}{\sum_{l \in N \setminus M} \gamma_l} \left(1 - \sum_{j \in M} \gamma_j\right) F^M.$$

Notice that the average profit, Shapley and generalized Shapley mechanisms are independent of time allocation. On the other hand, the proportional sharing mechanism is dependent on the time allocation. Any convex combination of mechanisms is also a mechanism.

2.1 Efficiency and other desirable properties

The strategy of agent i is an allocation of his time resource between different projects in which he participates. Let $f = (f^K)_{K \in L}$ be the vector of production functions, where $f^K : \mathbb{R}_+^k \to \mathbb{R}_+$ is a continuous and non-decreasing function in each variable. Let $\mathcal{F}$ be the set of vectors of production function.

We study the perfect information non-cooperative game where the strategy of agent i is a function from $\mathcal{F}$ to D_i that assigns to every set of production functions a time allocation for each project. Let $\mathcal{S}_i$ be the set of functions from $\mathcal{F}$ to D_i. The payoff of an agent depends on its own and others' time allocation, production functions and the outcome given by the mechanism.

Definition 2 (Non-cooperative Game G^φ)**.** *Given a mechanism φ, we study the **non-cooperative game** $G^\varphi = [N, (\mathcal{S}_1, \ldots, \mathcal{S}_n), (\pi_1^\varphi, \ldots, \pi_n^\varphi)]$ where*

- *the **strategy space** of agent i is $\mathcal{S}_i$;*
- *the **payoff function** of agent i at the vector of strategies (S_i, S_{-i}) and production function vector $f \in \mathcal{F}$ is*

$$\pi_i^\varphi(S_i, S_{-i}, f) = \varphi_i\left((t_i, t_{-i}), \left[f^K(t^K)\right]_{K \in L}\right), \quad \textit{where } t_j = S_j(f) \; \forall j \in N$$

We are interested in Nash equilibrium strategies, where agents have no incentive to deviate, assuming the strategies of the other agents remain fixed. Under a vector of production functions, a set of strategies generates outputs for the different projects. An efficient strategy dominates any other strategy for any set of production functions. We define an efficient mechanism if an efficient strategy can be supported as a Nash equilibrium.

Definition 3 (Nash Equilibrium and Efficiency)**.**

- *A strategy profile $(S_1^*, S_2^*, \ldots, S_n^*)$ is a **Nash equilibrium** of the game G^φ if for any production function vector $f \in \mathcal{F}$,*

$$\pi_i^\varphi(S_i^*, S_{-i}^*, f) \geq \pi_i^\varphi(S_i, S_{-i}^*, f), \quad \forall S_i \in \mathcal{S}_i$$

- *A **set of strategies** $(S_1, S_2, \ldots, S_n)$ **is efficient** if for any f and for any other strategy $\tilde{S}$*

$$\sum_{K \in L} f^K(t^K) \geq \sum_{K \in L} f^K(\tilde{t}^K), \quad \textit{where } t_i = S_i(f) \textit{ and } \tilde{t}_i = \tilde{S}_i(f)$$

- *A **mechanism is efficient** if a set of efficient strategies is a Nash equilibrium.*

As discussed in the introduction, the average profit mechanism is efficient. The Shapley and the generalized Shapley mechanisms are efficient for some projects L.

2.2 Separable mechanism and the main result: implementing the efficient time allocation

In this section, we characterize the mechanisms that are efficient. There are several restrictions that efficiency imposes on a mechanism. The first restriction is that the payoff of an agent should depend on the aggregate profit generated by the projects in which he participates, instead of the profits of individual projects. The second restriction is that the time allocation of an agent should not influence his payoff (but it might influence the payoff of other agents). The separable mechanisms discussed below include these two restrictions.

Definition 4. *A* ***mechanism*** φ ***is separable*** *if there exist functions* $(g_1, g_2, \ldots, g_n)$ *which are non-decreasing in the first coordinate such that*

$$\varphi_i(t, F) = g_i\left(\sum_{K \in L_i} F^K, \times_{B \in L_{-i}}(t^B, F^B)\right) \quad \forall i \in N.$$

A mechanism is separable if the payoff of agent i only depends on the total aggregated profit generated by his projects, $\sum_{K \in L_i} F^K$, as well as the profits and time allocations that do not contain agent i, $\times_{B \in L_{-i}}(t^B, F^B)$. The class of separable mechanisms is large. We provide below some examples.

Example 2. *A. The average profit mechanism is a separable mechanism generated by the functions*

$$g_i^{AP}\left(\sum_{k \in L_i} F^K, \times_{B \in L_{-i}}(t^B, F^B)\right) = \frac{\sum_{H \in L} F^H}{n} \quad \forall i$$

B. Assume that the grand coalition is not feasible, that is $N \notin L$. *Consider the mechanism* $\varphi_i^*(t, F) = \sum_{B \in L^{-i}} \frac{F^B}{|N \setminus B|}$, *for any* $i \in N$, *where every agent gets paid the average profit of the projects in which he does not belong. The mechanism is separable and generated by the functions,*

$$g_i^*\left(\sum_{K \in L_i} F^K, \times_{B \in L_{-i}}(t^B, F^B)\right) = \sum_{B \in L_{-i}} \frac{F^B}{|N \setminus B|} \quad \forall i$$

C. Shapley is a separable mechanisms only when the set L *contains coalitions of the same size* c. *In this case,*

$$g_i^{Sh}\left(\sum_{K \in L_i} F^K, \times_{B \in L_{-i}}\left(t^B, F^B\right)\right) = \frac{1}{c}\sum_{K \in L_i} F^K \quad \forall i$$

D. The generalized Shapley mechanism is a separable mechanism generated by the functions

$$g_i^{GSh}\left(\sum_{K \in L_i} F^K, \times_{B \in L_{-i}}(t^B, F^B)\right) = \gamma_i \sum_{K \in L^i} F^K + \sum_{M \in L^{-i}} \frac{\gamma_i}{\sum_{l \in N \setminus M} \gamma_l}\left(1 - \sum_{j \in M} \gamma_j\right) F^M \quad \forall i$$

Note that the convex combination of separable mechanisms is also a separable mechanism generated by the convex combination of the g functions. The proportional sharing mechanism is not separable, because the payoff of an agent depends on his allocation of time to different projects. Corollary 1 provides the entire class of separable mechanisms under two additional assumptions.

Theorem 1. *A mechanism is efficient if and only if it is separable.*

The proof is in Appendix A.

We say that a mechanism is **anonymous** if it is independent of the name of the agents. This means that agent i could be replaced with agent j, but the allocations and outputs are the same. We say that a mechanism is **time-independent** if the mechanism only depends on the profit generated by the different projects and not on the time allocated to different projects. The class of efficient and anonymous mechanism is large. We characterize below the class of efficient, anonymous and time-independent mechanisms.

Corollary 1.
- *Consider an integer c such that $0 < c < n$ and let $L^c = \{S \subset N \mid |S| = c\}$ be the set of projects that are the same size c. A mechanism is efficient, anonymous and time-independent in L^c if and only if it is a convex combination of Shapley and φ^*.*
- *Consider $L^T = \cup_{c \in T} L^c$ for some $T \subseteq \{1 \dots, n-1\}$. A mechanism φ is efficient, anonymous and time-independent in L^T if and only if φ is a generalized Shapley mechanism with the same weight for every agent. That is, there exists $0 \leq \alpha \leq min_{l \in L} \frac{1}{|l|}$ such that $\varphi_i(F_i, F_{-i}) = \alpha \sum_{S \in L_i^T} F^S + \sum_{R \in L_{-i}^T} \frac{1-|R|\alpha}{n-|R|} F^R$.*
- *Consider L such that $L = L^T \cup \{N\}$ for some $T \subseteq \{1 \dots, n-1\}$. A mechanism is anonymous, efficient and time independent if and only if it is the average profit mechanism.*

The implications of this corollary are that whenever agents are substitutes and the grand coalition is not a feasible project, none of mechanisms is anonymous and time independent. On the other hand, when the grand coalition is feasible, only the average profit mechanisms meet these properties.

3 Monotonicity and group manipulations

So far, our model has fixed the number of agent $N = \{1, \dots, n\}$, the time allocations $T = (T_1, \dots, T_n)$ and the set of projects L. For each problem $[N, T, L]$, Section 2 has found the class of efficient mechanisms. In this section, we focus on the robustness of such efficient mechanisms with respect to changes in N, T and L.

Given the problem $[N, T, L]$, we denote by $Eff^{\varphi}[N, T, L]$ the set of efficient Nash equilibria under φ.

Increases in time allocation often occur in companies. Such changes typically benefit the agents who receive the increase but may harm those agents whose time does not change. Our next property, requires that such an increase in the time should not harm agents at the efficient equilibrium, regardless of whether they received an increase in their time allocation or not.

Definition 5 (Time-monotonicity)**.** *A efficient mechanism φ is **time-monotonic at** $[N, T, L]$ if for any agent i, time $\tilde{T}_i > T_i$, any efficient equilibrium $S^* \in Eff^{\varphi}[N, T, L]$ and any efficient equilibrium $\tilde{S} \in Eff^{\varphi}[N, (\tilde{T}_i, T_{-i}), L]$, then $\varphi(S^*(f), f(S^*(f)) \leq \varphi(\tilde{S}(f), f(\tilde{S}(f))$.*

We say that the vector of production functions $\tilde{f}$ is a **technological improvement** of f if for any project $K \in L$ and time allocation t^K we have that $\tilde{f}^K(t^K) \geq f^K(t^K)$. Under a technological improvement, the same level of output (profit) requires no more input (time) than before the improvement.

Technological improvements can be observe in companies, especially those making allocations regularly over longer time horizons (e.g., distribution of end of year surplus). Such improvements might harm agents, especially those who do not belong to the improving project. The next property requires that such improvements should not harm any agent at equilibrium, regardless of whether or not they participate in the project that improved.[8]

Definition 6 (Technology-monotonicity)**.** *An efficient mechanism φ is **technology-monotonic at** $[N, T, L]$ if for any efficient equilibrium $S^* \in Eff^{\varphi}[N, T, L]$, any production functions f and any technological improvement $\tilde{f}$ of f, then $\varphi(S^*(f), f(S^*(f)) \leq \varphi(S^*(\tilde{f}), \tilde{f}(S^*(\tilde{f}))$.*

[8]Since we require this property at equilibrium, it is typically stronger than other monotonicity properties studied in the literature such as *strict resource monotonicity* in allocation problems (Thomson (1994, 1995, 2003, 2011); Moulin and Thomson (1988)).

A particular case of technology-monotonicity occurs for a technological improvement from the zero-technology, $f^K(t^K) = 0$ for all t^K, to a non-zero technology $\tilde{f}^K(t^K) > 0$ for some $\tilde{t}^K$. In this case, we can interpret this as **project-monotonicity**, where agents should not get worse off as more projects are available.

We define that two projects H and K are **connected** in L, denote $H \sim K$, if $H = K$ or there is a sequence of projects in L, $(P_0, P_1, ..., P_m)$, such that $H = P_0$, $K = P_m$, and $P_{r-1} \cap P_r \neq \emptyset$ for all $r = 1, ..., m$. That is, two projects are connected if we can find a sequence of feasible projects where every two consecutive projects have a non-empty intersection of agents.

It is easy to check that the *connectedness* is a *equivalence relation*. Then, the **equivalence class** of P under $\sim$ is defined as $[P] := \{H \in L | P \sim H\}$. Moreover, there is a unique partition of L which groups projects together if only if they are connected. We will denote this partition by

$$L/_{\sim} := \{[P] | P \in L\}.$$

Definition 7. *The separable mechanism φ is an **equivalent-class** mechanism if there exists functions $g_i : \mathbb{R}_+^{L/\sim} \to \mathbb{R}_+ \quad$ for $i = 1, 2, \ldots, n$ such that*

$$\sum_{i \in N} g_i(A) = \sum_{[P] \in L/_{\sim}} A_{[P]} \quad \forall A \in \mathbb{R}_+^{L/\sim}$$

and for all t and F,

$$\varphi(t, F) = (g_1(A), g_2(A), \ldots, g_n(A)),$$

where

$$A_{[P]} = \sum_{H \in [P]} F^H.$$

When L contains the grand coalition N, then all the projects are connected. Thus, the equivalent-class mechanisms allocate payments to the agents only depending on the total profit produced. There is, however, a large class of equivalent-class mechanisms that are highly dependent on the set of available projects, as illustrated in the examples below.

Example 3. *Suppose that the set of available projects is*

$$L = \{\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7, 8\}\}.$$

Then, there are two indifferent classes, $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8\}$. Thus, the equivalent-class mechanisms are such that the allocation for agents depends solely on the values $F^{12} + F^{123}$ and $F^{45} + F^{56} + F^{678}$.

For instance, one plausible mechanism is such that the allocation for agents in $\{1, 2, 3\}$ can be in proportion of the number of project they participate:

$$\varphi_{[1,2,3]}(t, F) = \left(\frac{2(F^{12} + F^{123})}{5}, \frac{2(F^{12} + F^{123})}{5}, \frac{F^{12} + F^{123}}{5} \right),$$

and the allocation to agents $\{4, 5, 6, 7, 8\}$ can be an equal-sharing division of the profits that their projects generate:

$$\varphi_i(t, F) = \frac{F^{45} + F^{56} + F^{678}}{5} \quad \text{for } i \in \{4, 5, 6, 7, 8\}.$$

When the set of feasible projects is increased to include the project $\{3, 4\}$, that is

$$\bar{L} = \{\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7, 8\}, \{3, 4\}\},$$

then all the projects are connected, therefore the only plausible equivalent-class mechanisms allocate a share to the agents depending solely in the total value produced $F^{12} + F^{123} + F^{45} + F^{56} + F^{678} + F^{34}$.

Since the equivalent-class mechanisms are separable, then the function $g_i(A_i, A_{-i})$ is non-decreasing in A_i, the profit generated by the component that contains agent i. However, such agent might be negatively affected once the profits of a different component increases. This is rule out under monotonicity.

The equivalent-class mechanism is **monotonic** if no agent gets worse off once either indifference class increases its profit. Formally, for the equivalent-class mechanism generated by g, we require that $g(A) \leq g(\bar{A})$ for any $A, \bar{A} \in \mathbb{R}_+^{L/\sim}$ such that $A \leq \bar{A}$.

Example 4 (Sharing the Proceeds of a Hierarchical Venture)**.** *Hougaard et al. (2017) study the sharing of profits when agents are not symmetric, but instead have a hierarchy (e.g., a linear order of subordinates or a more general hierarchy represented by a tree). The paper does not deal with the question on how the profits are generated, instead, it assumes that they are already given.*

The geometric rule φ^λ for agents with hierarchy $n \succ n-1 \succ \ldots, \succ 1$, profits $(F^1, F^2, \ldots, F^n)$ and parameter $\lambda \in [0,1]$ assigns:

$$\varphi_i^\lambda = \lambda(F^i + (1-\lambda)F^{i-1} + \cdots + (1-\lambda)^{i-1}F^1) \text{ for } i = 1, \ldots n-1, \text{ and}$$

$$\varphi_n^\lambda = F^n + (1-\lambda)F^{n-1} + \cdots + (1-\lambda)^{n-1}F^1$$

We show that depending on how the profits are generated, the class of geometric rules characterized by Hougaard et al. (2017) might be an equivalent-class mechanism that is monotonic.

Indeed, when the profits are generated independently by every agent, that is $L = \{\{1\}, \ldots, \{n\}\}$, then the indifferent classes equal L and for any λ, the respective geometric mechanism is an equivalent-class mechanism that is monotonic.

However, this is not true when the profits are generated collaboratively. For instance, consider the set of projects $L = \{\{1, 2, \ldots, n\}, \{2, 3, \ldots, n\} \ldots, \{n\}\}$, where agent n belongs to n projects, agent $n-1$ belongs to $n-1$ projects, etc. Since all the projects are connected by agent n, the grand coalition is the unique indifference class and the payment of every agent should only depend on $F^1 + F^2 + \cdots + F^n$. Thus, the only geometric mechanism that is an equivalent-class mechanism is when $\lambda = 1$, that is, when agent n obtains all the profit.

Corollary 2. *The following three properties are equivalent for the efficient mechanism φ:*

(i) *φ is time-monotonic*

(ii) *φ is technology-monotonic*

(iii) *φ is an equivalent-class mechanism that is monotonic.*

The proof of this result is in Appendix A.

As projects in companies are often collaborative, agents might know each other and have the possibility of individually gaining by coordinating their time at the expense of overall efficiency. The following property rules out the coordination of time by the agents. We do so by requiring that the equilibrium generated by our mechanism be a strong Nash equilibrium (Aumann (1959)).

Definition 8 (Strong Nash equilibrium)**.** *We say that the Nash equilibrium $(S_1^*, \ldots, S_n^*)$ of the game G^φ is a **strong Nash equilibrium** if for any group of agents $T \subset N$ and strategies $\tilde{S}_T$ of them, if there exists a production function f such that $\varphi_i(S^*(f), f(S^*(f)) < \varphi_i(\tilde{S}(f), f(\tilde{S}(f))$ for some $i \in T$, then there exists $j \in T$ such that $\varphi_j(S^*(f), f(S^*(f)) > \varphi_j(\tilde{S}(f), f(\tilde{S}(f))$, where $\tilde{S} = (S_T, S^*_{-T})$.*

This property is similar to group (coalitional) strategyproofness, studied in other cost sharing models (Moulin (1999, 1993); Juarez (2013)). Unlike these models, we require for the efficient equilibrium to be strong.

The equivalent-class mechanism is **non-bossy (NB)** if whenever the agents in any coalition can maintain their payments constant by decreasing the profit of their equivalent class, the payment of the agents outside the coalition cannot increase. Formally, for the equivalent-class mechanism generated by g, we require that for any $S \subset N$ and $A, \bar{A} \in \mathbb{R}_+^{L/\sim}$ such that $\bar{A}_i \leq A_i$ for any $i \in S$ and $\bar{A}_i = A_i$ for any $i \notin S$, if $g_i(A) = g_i(\bar{A})$ for all $i \in S$, then $g_j(\bar{A}) \leq g_j(A)$ for all $j \notin S$.

Note that NB is weaker than monotonicity. Indeed, consider a equivalent-class mechanism such that for any agent i, his payment $g_i(A_i, A_{-i})$ is strictly increasing in A_i. This mechanism satisfies NB, but unlike monotonicity, imposes no restriction on the change of g_i when projects different than A_i change.

Corollary 3. *Let φ be an efficient mechanism. There exists an efficient equilibrium in the game G^{φ} that is a strong Nash equilibrium if only if φ is an equivalent-class mechanism that satisfies NB.*

The proof of this result is in Appendix A.

4 Conclusion

We have introduced a large class of mechanisms that implement the efficient time allocation. Our main result shows that the class of efficient mechanisms coincide with the class of separable mechanisms, where the payoff of an agent only depends on the total profit generated by his own projects, as well as the time allocations and profits generated by projects in which the agent does not participate. This large class of efficient mechanisms is shrunk substantially when more robust monotonicity properties (on time and technology) are imposed. This is also true in focusing on mechanisms that prevent agent to coordinate their time.

More work needs to be done to understand the sharing of profits in dynamics problems. Juarez et al. (2016) has started such study by focusing in the axiomatic division of finite and sequential benefits in companies.

Appendix A. Proofs

Proof of Theorem 1

Proof. First, we show that if a mechanism is separable, then the mechanism is efficient.

Suppose that an agent, say agent i, deviates from an efficient strategy under a separable mechanism. Then, the deviation by agent i does not lead to an increase in total profit of his projects in any production functions due to the definition of efficient strategy. Thus, agent i cannot increase his payoff because g_i is a non-decreasing function in the first coordinate. This is a contradiction.

Next, we show that if a mechanism is efficient, then the mechanism is separable for any set of production functions.

Step 1: We show that if a mechanism is efficient, then the payoff of agent i does not depend on his time allocation. That is,

$$\varphi_i(t_i, t_{-i}, F) = h_i(t_{-i}, F),$$

where t_i is the strategy of agent i and $t_{-i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ is the collection of all agents' strategies but agent i.

First, consider $\bar{k} \in L^i$ and define the production functions as follows:

$$\begin{aligned} f^k(t) &= c^k, \quad \text{for } k \neq \bar{k}; \\ f^{\bar{k}}(t) &= c^{\bar{k}} + \epsilon\left(t_i^{\bar{k}}\right); \end{aligned}$$

where $c^k \in \mathbb{R}_+$ is a constant for each $k \in L$.

By definition of efficient strategy, agent i allocates his full resource to the project $\bar{k}$. That is, $t_i = E_i^{\bar{k}} \in D_i$, where $E_{i,k}^{\bar{k}} = T_i$ if $k = \bar{k}$, and $E_{i,k}^{\bar{k}} = 0$ if $k \in L_i \setminus \{\bar{k}\}$.

Then, for all $\tilde{t}_i \in D_i$ we have that

$$\varphi_i\left(E_i^{\bar{k}}, t_{-i}, f(E_i^{\bar{k}}, t_{-i})\right) \geq \varphi_i\left(\tilde{t}_i, t_{-i}, f(\tilde{t}_i, t_{-i})\right).$$

Let $c = [c^k]_{k\in L}$. As ϵ goes to 0, $f(t) \to c$. Therefore, by continuity of φ we have that

$$\varphi_i\left(E_i^{\bar{k}}, t_{-i}, c\right) \geq \varphi_i(\tilde{t}_i, t_{-i}, c), \quad \text{for any } \tilde{t}_i \in D_i. \tag{1}$$

Alternatively, fix $\tilde{t}_i \in D_i$ and consider the production functions as follows:

$$\begin{aligned} \tilde{f}^k(t^k) &= c^k + \epsilon\left(\min\{\tilde{t}_i^k, t_i^k\}\right), \quad \text{for } k \in L_i; \\ \tilde{f}^k(t^k) &= c^k, \quad \text{for } k \in L \setminus L_i. \end{aligned}$$

Notice that the optimal profile for agent i equals $\tilde{t}_i$. Since φ is efficient, we have that

$$\varphi_i\left(\tilde{t}_i, t_{-i}, \tilde{f}\left(\tilde{t}_i, t_{-i}\right)\right) \geq \varphi_i\left(E_i^{\bar{k}}, t_{-i}, \tilde{f}\left(E_i^{\bar{k}}, t_{-i}\right)\right).$$

As ϵ goes to 0, $f(t) \to c$. Therefore, by continuity of φ we have that

$$\varphi_i\left(\tilde{t}_i, t_{-i}, c\right) \geq \varphi_i\left(E_i^{\bar{k}}, t_{-i}, c\right). \tag{2}$$

Hence, by the inequalities (1) and (2),

$$\varphi_i\left(\tilde{t}_i, t_{-i}, c\right) = \varphi_i\left(E_i^{\bar{k}}, t_{-i}, c\right).$$

Thus, the payoff of agent i is independent of his time allocation. Similarly, the payoffs of the other agents are also independent of their own time allocation.

Step 2: We show that the share of agent i depends on the sum of the profits of the projects in which he belongs.

First, consider $\bar{k} \in L_i$ and define the production functions as follows:

$$\begin{aligned} f^{\bar{k}}(t) &= c^{\bar{k}} + t_i^{\bar{k}} + \gamma \sum_{j\in\bar{k}\setminus\{i\}} t_j^{\bar{k}}; \\ f^k(t) &= c^k + \gamma\left(\sum_{j\in k} t_j^k\right), \quad \text{for any } k \in L \setminus \{\bar{k}\} \end{aligned}$$

where $\gamma < 1$. and $c^k \in \mathbb{R}_+$ are arbitrary constants.

By efficiency, for all $\tilde{t}_i \in D_i$ we have that

$$\varphi_i\left(E_i^{\bar{k}}, t_{-i}, f\left(E_i^{\bar{k}}, t_{-i}\right)\right) \geq \varphi_i\left(\tilde{t}_i, t_{-i}, f\left(\tilde{t}_i, t_{-i}\right)\right).$$

Therefore, as γ goes to 1,

$$f\left(E_i^{\bar{k}}, t_{-i}\right) \to \left(c^{\bar{k}} + T_i + \sum_{j \in \bar{k} \setminus \{i\}} t_j^{\bar{k}}, \left[c^k + \sum_{j \in k \setminus \{i\}} t_j^k\right]_{k \in L_i \setminus \bar{k}}, \left[c^k + \sum_{j \in k} t_j^k\right]_{k \in L \setminus L_i}\right) = F^*,$$

and

$$f\left(\tilde{t}_i, t_{-i}\right) \to \left[c^k + \sum_{j \in k} t_j^k\right]_{k \in L} = G^*.$$

Thus, by the continuity of φ_i,

$$\varphi_i\left(E_i^{\bar{k}}, t_{-i}, F^*\right) \geq \varphi_i\left(\tilde{t}_i, t_{-i}, G^*\right).$$

Therefore, transferring all the time of i to the project $\bar{k}$ does not decrease the share of agent i.
Alternatively, for a given $\tilde{t}_i \in D_i$, consider the following production functions,

$$\begin{aligned}
\tilde{f}^k(t) &= c^k + \gamma \min\left\{\tilde{t}_i^k, t_i^k\right\} + \sum_{j \in k} t_j^k, \quad \text{for any } k \in L_i; \\
\tilde{f}_{jk}(t) &= c^k + \left(\sum_{j \in k} t_i^k\right) \text{ where } k \in L \setminus L_i.
\end{aligned}$$

For $\gamma < 1$, the optimal profile requires $t_i = \tilde{t}_i$. Comparing this with the suboptimal profile $t_i = (T_i, 0, \ldots, 0)$, and making γ converge to zero, we have that:

$$\varphi_i\left(E_i^{\bar{k}}, t_{-i}, F^*\right) \leq \varphi_i\left(t_i, t_{-i}, G^*\right).$$

Therefore, we have that the payoff of agent i is invariant to the reallocation of profits in the projects that contain agent i.

Hence, by Steps 1 and 2, φ_i depends on the aggregate profit of the projects in which i belongs, as well as others' time allocations and profits only.

Step 3: We show that a mechanism is non-decreasing function on the total output of its own projects.

Consider the following production functions:

$$\begin{aligned}
\tilde{f}^{\bar{k}}(t) &= c^T + (\gamma + \delta) t_i^{\bar{k}} + \gamma \sum_{j \in \bar{k} \setminus \{i\}} t_j^{\bar{k}}; \\
\tilde{f}^k(t) &= c^k + \gamma \sum_{j \in \bar{k}} t_j^k, \quad \text{for any } k \in L \setminus \{\bar{k}\}.
\end{aligned}$$

where $\gamma < 1$ and $\delta > 0$.

Then, at the optimal profile, agent i contributes his full time allocation to the project $\bar{k}$. Therefore, for any arbitrary profile t_i:

$$\varphi_i\left(\sum_{k\in L_i} c^k+(\gamma+\delta)T_i+\gamma\sum_{k\in L_i}\sum_{j\in k\setminus\{i\}} t_j^{\bar{k}}, F_{-i}, t_{-i}\right)\geq \varphi_i\left(\sum_{k\in L_i} c^k+(\delta)t_i^{\bar{k}}+\gamma\sum_{k\in L_i}\sum_{j\in k} t_j^{\bar{k}}, F_{-i}, t_{-i}\right).$$

As γ goes to 0, we have that

$$\varphi_i\left(\sum_{k\in L_i} c^k+\delta T_i, F_{-i}, t_{-i}\right)\geq\left(\sum_{k\in L_i} c^k+\delta t_i^{\bar{k}}, F_{-i}, t_{-i}\right).$$

Therefore, the step follows immediately since $\delta\in[0,1]$, $\{c^k\}_{k\in L_i}$ and $t_i^{\bar{k}}\leq T_i$ are arbitrary numbers. $\square$

Proof of Corollary 1

Proof. Consider the mechanism φ that is efficient, anonymous and time-independent.

Step 1: We show that φ is linear.

Note that by Theorem 1, anonymity and time independence, there exists a function g such that φ_i can be written as

$$\varphi_i(F)=g\left(\sum_{K\in L_i^T} F^K, (F^B)_{B\in L_{-i}^T}\right)\tag{3}$$

for all $i\in N$ and $F\in\mathbb{F}$.

It is important to remark that by anonymity the output of $g\left(\sum_{K\in L_i^T} F^K, (F^B)_{B\in L_{-i}^T}\right)$ is the same for any permutation on profits of projects with equal cardinality in L_{-i}^T.

In fact, if $\bar{M}=(M,M,...,M)$ for any $M\in\mathbb{R}$, by anonymity and budget balance

$$\varphi_i(\bar{M})=g(|L_i^T|M,M,...,M)=\frac{|L^T|}{|N|}M\quad\forall i\in N.$$

We proceed to show that g is linear. Let F be an arbitrary vector of profits in L^T.

Without loss of generality, there are projects $R\cup\{1\}$ and $R\cup\{2\}$ in L^T, where $|R|+1\in T$ and $1\notin R$, $2\notin R$. We denote $\tilde{F}$ the vector such that

$$\tilde{F}^K=\begin{cases} F^{R\cup\{1\}}, & \text{if } K=R\cup\{2\}\\ F^{R\cup\{2\}}, & \text{if } K=R\cup\{1\}\\ F^K, & \text{otherwise.}\end{cases}$$

Note that $\sum_{K\in L^T} F^K=\sum_{K\in L^T}\tilde{F}^K$. Moreover, by anonymity we have $\varphi_i(F)=\varphi_i(\tilde{F})$ for all $i\in N\setminus\{1,2\}$. Therefore,

$$\begin{aligned}\varphi_1(F)+\varphi_2(F) &= \sum_{K\in L^T} F^K-\sum_{i\in N\setminus\{1,2\}}\varphi_i(F)\\ &= \sum_{K\in L^T} F^K-\sum_{i\in N\setminus\{1,2\}}\varphi_i(\tilde{F})\\ &= \varphi_1(\tilde{F})+\varphi_2(\tilde{F})\end{aligned}$$

by budget balance property. Then, $\varphi_1(F) - \varphi_1(\tilde{F}) = \varphi_2(\tilde{F}) - \varphi_2(F)$.

Hence, if

$$A = \sum_{K \in L_1^T} F^K, \quad x = F^{R\cup\{2\}}, \quad z_{-1} = (F^B)_{B \in L_{-1}^T \setminus R\cup\{2\}},$$
$$B = \sum_{K \in L_2^T} F^K, \quad y = F^{R\cup\{1\}}, \quad z_{-2} = (F^B)_{B \in L_{-2}^T \setminus R\cup\{1\}}.$$

by (3) we have

$$g(A, x, z_{-1}) - g(A + x - y, y, z_{-1}) = g(B + y - x, x, z_{-2}) - g(B, y, z_{-2}).$$

Thus, for $\alpha = x - y$ and $C = B - \alpha$ we have

$$g(A, x, z_{-1}) - g(A + \alpha, x - \alpha, z_{-1}) = g(C, x, z_{-2}) - g(C + \alpha, x - \alpha, z_{-2}).$$

This mean that the change in the output of g when we transfer an amount of a project to the first entry, just depend of the amount transferred and the cardinality of the project.

Therefore, the function $h_b(\alpha) = g(A, x) - g(A + \alpha, x_b - \alpha, x_{-b})$ is additive, due to

$$\begin{aligned} h_b(\alpha) &= g(A, x) - g(A + \alpha, x_b - \alpha, x_{-b}) \\ h_b(\beta) &= g(A + \alpha, x_b - \alpha, x_{-b}) - g(A + \alpha + \beta, x_b - \alpha - \beta, x_{-b}), \end{aligned}$$

and then,

$$\begin{aligned} h_b(\alpha) + h_b(\beta) &= g(A, x) - g(A + \alpha + \beta, x_b - \alpha - \beta, x_{-b}) \\ &= h_b(\alpha + \beta). \end{aligned}$$

Since g is continuous, $h_b : \mathbb{R} \to \mathbb{R}$ is continuous and additive, then there is $\lambda_b \in \mathbb{R}$ such that, $h_b(z) = \lambda_b z$.

We list all projects in L_{-i}^T by $(b_1, b_2, \ldots, b_r)$. If $\bar{F} = \frac{1}{|L^T|} \sum_{k \in L^T} F^k$ then,

$$\begin{aligned} h_{b_j}(F^{b_j} - \bar{F}) &= g\left(\sum_{k \in L_i^T} F^k + \sum_{p=1}^{j-1} (F^{b_p} - \bar{F}), \bar{F}, \cdots, \bar{F}, F^{b_j}, \cdots, F^{b_r}\right) \\ &- g\left(\sum_{k \in L_i^T} F^k + \sum_{p=1}^{j} (F^{b_p} - \bar{F}), \bar{F}, \bar{F}, \cdots, \bar{F}, F^{b_j+1}, \cdots, F^{b_r}\right) \end{aligned}$$

Therefore, we have

$$\sum_{j=1}^{r} h_{b_j}(F^{b_j} - \bar{F}) = g\left(\sum_{k \in L_i^T} F^k, F^{b_1}, \cdots, F^{b_r}\right) - g\left(|L_i^T|\bar{F}, \bar{F}, \cdots, \bar{F}\right).$$

Then,

$$g\left(\sum_{k \in L_i^T} F^k, F^{b_1}, \cdots, F^{b_r}\right) = \sum_{j=1}^{r} \lambda_{b_j}(F^{b_j} - \bar{F}) + \frac{|L^T|}{|N|}\bar{F},$$

and we conclude that g is linear. Then, φ is linear.

Step 2: We show that

$$\varphi_i(F) = \alpha \sum_{S \in L_i^T} F^S + \sum_{R \in L_{-i}^T} \frac{1 - |R|\alpha}{n - |R|} F^R.$$

As we showed g and φ are linear, then

$$\varphi_i(F) = \alpha \sum_{S \in L_i^T} F^S + \sum_{R \in L_{-i}^T} \beta_R F^R.$$

Let E_P be the vector of profits such that $E_P^K = 1$ if $K = P$ and 0 otherwise. By anonymity $\varphi_i(E_P) = \alpha$ for all $i \in P$ and $\varphi_j(E_P) = \beta_P$ for all $j \notin P$.

Now, by budget balance property we have

$$|P|\alpha + (n - |P|)\beta_P = 1.$$

Then,

$$\beta_P = \frac{1 - |P|\alpha}{n - |P|} \quad \text{for any } P \in L^T.$$

Therefore, the step follows immediately.

Step 3: We show that if $N \in L^T$, then φ is the average profit mechanism.

If $L = \{N\}$ then,

$$g\left(F^N\right) = \frac{1}{|N|} F^N,$$

which is clear by budget balance.

Now, if $T \neq \emptyset$ then there are projects $R \cup \{1\}$ and $R \cup \{2\}$ in L^T, where $1 \notin R$, $2 \notin R$ and we can proceed as in the previos step to obtain

$$\varphi_i(F) = \alpha \sum_{S \in L_i^T} F^S + \sum_{R \in L_{-i}^T} \frac{1 - |R|\alpha}{n - |R|} F^R,$$

for some α.

By anonymity $\varphi_i(E_N) = \alpha$ for all $i \in N$. Thus, $\alpha|N| = 1$ by budget balance and then $\alpha = 1/n$ and

$$\varphi_i(F) = \frac{1}{n} \sum_{K \in L^T} F^K.$$

□

Proof of Corollary 2

Proof. **Property (iii)** $\Rightarrow$ **Property (i)** is obvious.

We now show that **Property (i)** $\Rightarrow$ **Property (iii).**

Step A1. φ_i is independent of time.

By Theorem 1, the payoff of agent i, depends only on the time allocations of others. Then,

$$\varphi_i(t, F) = \varphi_i\left(F; t_{-i}\right), \quad \forall i.$$

Now, we show φ_i is independent of the other's time allocations t_{-i}.
Consider the following constant production functions:

$$f^k(t^k) = \alpha^k, \quad \forall k \in L.$$

Note that under these production functions, any allocation of time is an efficient Nash equilibrium. Suppose that the time for any agent $j \neq i$ increases, $\tilde{T}_j = T_j + \epsilon$.

Then for any project k by time-monotonicity, we have

$$\varphi_i\left(\alpha; \tilde{t}_{-\{i,j\}}, \tilde{t}_j^k + \epsilon, \left(\tilde{t}_j^p\right)_{p\in L_j\setminus\{k\}}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By taking the limit as ϵ tends to zero,

$$\varphi_i\left(\alpha; \tilde{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By exchanging the roles of t_{-i} and $\tilde{t}_{-i}$ we have that

$$\varphi_i\left(\alpha; t_{-i}\right) \geq \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Thus,

$$\varphi_i\left(\alpha; t_{-i}\right) = \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Step A2. Now, we show that the payoff of an agent is invariant to transfers of profit between connected projects.

Consider two connected projects S and T. Let j be a player in $S \cap T$. Furthermore, consider the following set of production functions, for any $\beta \in \mathbb{R}_+$:

$$\begin{aligned} f^T(t^T) &= \alpha^T + \beta t_j^T; \\ f^S(t^S) &= \alpha^S + \beta t_j^S; \\ f^K(t^K) &= \alpha^K, \quad \forall K \neq S, T \end{aligned}$$

Note that under efficiency, agent j assigns any distribution of his time to the projects S and T while the other agents assign their time arbitrarily.

Now, consider that the time for the agent j increases, $\tilde{T}_j = T_j + \epsilon$.

By time-monotonicity, the payoff for any player i satisfies that

$$\varphi_i\left(\alpha^{-S,T}, \alpha^S + \beta T_j, \alpha^T\right) \leq \varphi_i\left(\alpha^{-S,T}, \alpha^S, \alpha^T + \beta(T_j + \epsilon)\right).$$

The limit as ϵ goes to 0 is

$$\varphi_i\left(\alpha^{-S,T}, \alpha^S + \beta T_j, \alpha^T\right) \leq \varphi_i\left(\alpha^{-S,T}, \alpha^S, \alpha^T + \beta T_j\right).$$

By exchanging the assignment of time, we have

$$\varphi_i\left(\alpha^{-S,T}, \alpha^S, \alpha^T + \beta T_j\right) \leq \varphi_i\left(\alpha^{-S,T}, \alpha^S + \beta(T_j + \epsilon), \alpha^T\right).$$

If ϵ approaches 0,

$$\varphi_i\left(\alpha^{-S,T}, \alpha^S, \alpha^T + \beta T_j\right) \leq \varphi_i\left(\alpha^{-S,T}, \alpha^S + \beta T_j, \alpha^T\right).$$

Then,

$$\varphi_i\left(\alpha^{-S,T}, \alpha^S, \alpha^T + \beta T_j\right) = \varphi_i\left(\alpha^{-S,T}, \alpha^S + \beta T_j, \alpha^T\right),$$

Hence, the payoff of any agent i is invariant to transfers of profit between connected projects.

By repeating the argument for all pair of connected projects completes the proof.

Step A3. Now, we will prove that g_i is monotonic for all $i \in N$.

Let $S \in L$ be an arbitrary project. Consider the following set of production functions,

$$\begin{aligned} f^S(t^S) &= a^S + t_j^S, \quad \text{for some agent } j \in S, \\ f^K(t^K) &= a^K, \quad \forall K \neq S, \end{aligned}$$

where a^K is a constant for each $K \in L$, such that

$$A_{[P]} = \sum_{K \in [P]} a^K$$

Note that under efficiency, agent j assigns his time to the project S while the other agents can assign their time arbitrarily.

Consider that the time T_j of agent j is such that $T_j = 0$ and there is an increases to $T_j = \epsilon > 0$. By time-monotonicity and Step A2, the payoff for any agent i satisfies that

$$g_i(A_{[S]}, A_{-[S]}) \leq g_i(A_{[S]} + \epsilon, A_{-[S]}), \quad \forall i \in N.$$

Hence, the function g_i is non-decreasing in every entry. Then, the equivalent-class mechanism g is monotonic.

Property (iii) $\Rightarrow$ **Property (ii)** is obvious.

We now show that **Property (ii)** $\Rightarrow$ **Property (iii).**

Step B1. φ_i is independent of time.

By Theorem 1, the payoff of agent i, depends only on the time allocations of others. That is,

$$\varphi_i(t, F) = \varphi_i\left(F; t_{-i}\right), \quad \forall i.$$

Now, we show φ_i is independent of the other's time allocations t_{-i}.
Consider the following constant production functions:

$$f^k(t^k) = \alpha^k, \quad \forall k \in L.$$

Note that under these production functions, any allocation of time is an efficient Nash equilibrium. Suppose that the technology in the project S has been improved.

$$\tilde{f}^S(t^S) = \alpha^S + \epsilon.$$

Then, by technology monotonicity,

$$\varphi_i\left(\left(\alpha^k\right)_{k \in L \setminus S}, \alpha^S + \epsilon; \tilde{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By taking the limit as ϵ tends to zero,

$$\varphi_i\left(\alpha; \tilde{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By exchanging the roles of t_{-i} and $\tilde{t}_{-i}$ we have that

$$\varphi_i\left(\alpha; t_{-i}\right) \geq \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Thus,

$$\varphi_i\left(\alpha; t_{-i}\right) = \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Step B2. In this step, we show that the payoff of an agent is invariant to transfers of profit between connected projects. Consider two connected projects S and T. Let j be a player in $S \cap T$. Furthermore, consider the following set of production functions:

$$\begin{aligned} f^T(t^T) &= c^T + \beta t_j^T; \\ f^S(t^S) &= c^S + \beta t_j^S; \\ f^K(t^K) &= c^K, \quad \forall K \neq S, T \end{aligned}$$

Note that under efficiency, agent j assigns any distribution of his time to the projects S and T while the other agents can assign their time arbitrarily.

Consider the following technology improvement of f^T:

$$\begin{aligned} f^T(t^T) &= c^T + \beta t_j^T + \epsilon; \\ f^S(t^S) &= c^S + \beta t_j^S; \\ f^K(t^K) &= c^K, \quad \forall K \neq S, T. \end{aligned}$$

By technology monotonicity, the payoff for any player i satisfies that

$$\varphi_i\left(c^{-S,T}, c^S + \beta T_j, c^T\right) \leq \varphi_i\left(c^{-S,T}, c^S, c^T + \beta T_j + \epsilon\right).$$

At the limit, when ϵ tends to 0, we have that

$$\varphi_i\left(c^{-S,T}, c^S + \beta T_j, c^T\right) \leq \varphi_i\left(c^{-S,T}, c^S, c^T + \beta T_j\right).$$

Alternatively, consider the production functions.

$$\begin{aligned} f^S(t^S) &= c^S + \beta t_j^S + \epsilon; \\ f^T(t^T) &= c^T + \beta t_j^T; \\ f^K(t^K) &= c^K, \quad \forall K \neq S, T \end{aligned}$$

By repeating the above argument we have that

$$\varphi_i\left(c^{-S,T}, c^T + \beta T_j, c^S\right) \leq \varphi_i\left(c^{-S,T}, c^T, c^S + \beta T_j + \epsilon\right).$$

As ϵ tends to zero, this leads to

$$\varphi_i\left(c^{-S,T}, c^T + \beta T_j, c^S\right) \leq \varphi_i\left(c^{-S,T}, c^T, c^S + \beta T_j\right).$$

Hence,

$$\varphi_i\left(c^{-S,T}, c^S, c^T + \beta T_j\right) = \varphi_i\left(c^{-S,T}, c^S + \beta T_j, c^T\right),$$

Hence, the payoff of any agent i is invariant to transfers of profit between connected projects.

By repeating the argument for all pair of connected projects completes the proof.

Step B3. We prove that g_i is monotonic for all $i \in N$.

Consider the following constant production functions:

$$f^k(t^k) = a^k, \quad \forall k \in L,$$

such that

$$A_{[P]} = \sum_{K \in [P]} a^K$$

Note that any allocation of time is an efficient Nash equilibrium.

Suppose that the technology in the project S has been improved.

$$\tilde{f}^S(t^S) = \alpha^S + \epsilon.$$

Then, by technology monotonicity,

$$g_i(A_{[S]}, A_{-[S]}) \leq g_i(A_{[S]} + \epsilon, A_{-[S]}), \quad \forall i \in N.$$

Hence, the function g_i is non-decreasing in every entry. Thus, the mechanism g is monotonic.

□

Proof of Corollary 3

Proof. The only if part is clear because under such a function g, every agent allocates their time resources to achieve the maximum (efficient) level of the total profit in his equivalence class. In other words, even if some agents form a coalition, they cannot increase their individual payoff because the aggregate profit does not increase. But, if the payoffs of a set of agents do not change if they decrease the profits of some projects, then the payoff of all other agents does not increase.

Let us prove the if part:

Step 1: We show that φ_i is independent of time allocations, t_i and t_{-i}.

By Theorem 1, φ is such that

$$\varphi_i(t, F) = \varphi_i\left(\sum_{k \in L_i} F^k, \left(F^b\right)_{b \in L_{-i}}, t_{-i}\right), \quad \forall i.$$

Consider the time-independent production functions:

$$f^k(t^k) = \alpha^k, \quad \forall k \in L \text{ and arbitrary constants } \alpha^k,$$

Suppose that φ_i depends on t_{-i}. Then, one agent, j, can help agent i to receive a higher payoff by changing his time allocation, because the other allocations are constant. This is a violation to the definition of strong efficiency. Therefore, we have

$$\varphi_i(t, F) = \varphi_i\left(\sum_{K \in L_i} F^K, \left(F^B\right)_{B \in L_{-i}}\right), \quad \forall i.$$

Step 2: We show that the payoff of any agent is invariant to transfers of profit between connected projects.

Consider two connected projects S and T. Let j be a player in $S \cap T$. Furthermore, consider the following set of production functions:

$$\begin{aligned} f^T(t^T) &= c^T + \beta t_j^T; \\ f^S(t^S) &= c^S + \beta t_j^S; \\ f^K(t^K) &= c^K, \quad \forall K \neq S, T \end{aligned}$$

for any $\beta > 0$.

Note that under efficiency, agent j assigns any distribution of his time to the projects S and T while the other agents can assign their time arbitrarily.

By Theorem 1 and Step 1, we have that the payoff of agent j is invariant to transfers of profit between S and T, due to the first entry of g_j do not change.

Then, by strong Nash equilibrium, we have that for any agent $i \in N$,

$$\varphi_i\left(c^{-S,T}, c^S + \beta T_j, c^T\right) = \varphi_i\left(c^{-S,T}, c^S, c^T + \beta T_j\right).$$

Hence, the payoff of any agent i is invariant to transfers of profit between connected projects.

By repeating the argument for all pair of connected projects we have that φ is a equivalent-class mechanism g.

Step 3: We show that g satisfies NB.

Conversely, suppose that there are $S \subset N$ and $A \in \mathbb{R}_+^{L/\sim}$ such that $\bar{A}_S \leq A_S$, $g_i(A) = g_i(\bar{A}_S, A_{-S})$ and for some $j \in N \setminus S$, $g_i(\bar{A}_S, A_{-S}) > g_i(A)$. Then, the coalition $S \cup \{j\}$ violates the strong Nash property.

□

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