Implementing efficient allocations in bilateral networks*

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Abstract

Agents are endowed with time that is invested in different bilateral projects. Projects generate profit depending on their time allocation. A mechanism divides the profit generated by the projects among agents. We study mechanisms that incentivize agents to contribute their time to the level that generates the maximal profit for society at the Nash equilibrium (we call this efficiency). We focus on the case of bilateral projects, that is when every project requires time allocation from only two agents. Our main result is the characterization of all the mechanisms that satisfy efficiency for any set of production functions. Furthermore, we characterize efficient mechanisms that satisfy other desirable properties such as monotonicity in the payoffs, strong Nash equilibrium of the efficient outcome, or additivity in the number of participating agents.

Keywords: *Profit-sharing, efficiency, implementation.*

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1 Introduction

We study the problem of profit sharing in companies. Agents collectively own a company and are endowed with a fixed amount of time that they invest in different projects that generate profit. We study the case where each project only requires time contributions by a pair of agents, which we call a bilateral project. Every project could have different production functions in the way that they generate profit. We focus on the case where a planner does not know the production functions and must assign the payments depending only on the final profit generated by each project and the time allocation of the agents. Examples of this problem include law firms, cooperative ventures or division of the end of year surplus in companies.

The central issue is that although the planner tries to maximize the total profit of the company, the planner might or might not know the production functions. Therefore, the planner's goal is to find a mechanism which implements the efficient time allocation at the Nash equilibrium for any set of production functions. We call this property efficiency.

For instance, consider the mechanisms that divide the total profit of each project in proportion to the time allocated to different projects. This mechanism is not necessarily efficient because agents have the incentive to put more time in the projects that generate the largest profit in order to get a larger share of the profit, whereas that might not be the most efficient way to allocate time in the society. To see this, consider the following example. There are three agents named 1, 2, and 3. Each agent is endowed with 1 unit of time. Profit is generated by collaborative projects. The production functions are $\alpha(t^1_{12} + t^2_{12})$ for the project with agents 1 and 2, $\beta min(t^1_{13}, t^3_{13})$ for the project with agents 1 and 3, and $\gamma min(t^2_{23}, t^3_{23})$ for the project with agents 2 and 3 (see Figure 1). When $\alpha = 1, \beta = 1.2$ and $\gamma = 1.5$, the efficient time allocation is $((t^1_{12}, t^1_{13}), (t^2_{12}, t^2_{23}), (t^3_{13}, t^3_{23})) = ((1, 0), (0, 1), (0, 1))$ by which they generate 2.5 in total, where agent 1 receives 1, while agent 2 and 3 receive 0.75 each. However, this is not an equilibrium since agent 2 would allocate all resources to the project with agent 1. Therefore, they produce 2 in total, where agent 1 and 2 receive 1 each, while agent 3 receives nothing.

On the other hand, consider the average profit mechanism, where each agent gets a fixed share

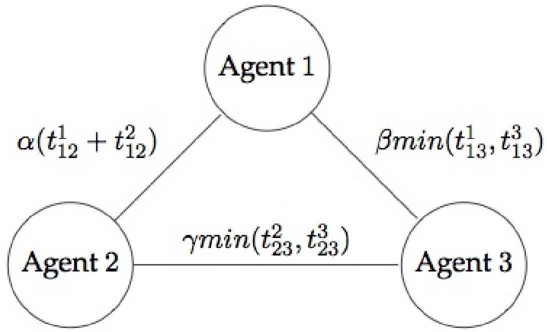


Figure 1: A network of 3 agents

of the total profit generated by the society. This mechanism is efficient because if an agent deviates from the equilibrium that generates the maximum total profit, then the total profit of the society will not increase and neither does his payment.

Alternatively, consider the Shapley mechanism, where every agent is assigned half of the profit of the projects in which he belongs. This mechanism is also efficient because if an agent deviates from the equilibrium that generates the maximal profit, then the total profit of the society will not increase. Since he is only affecting the projects in which he belongs, then the total profit of his projects will not increase and neither does his payment.[1]

More generally, consider a mechanism where the payoff of an agent only depends on the total profit generated by the projects in which the agent belongs, as well as the time contributions and profit generated by the projects in which the agent does not belong. We call these separable mechanisms. Separable mechanisms are efficient because if an agent decides to change his time allocation, then his payoff can only influence the total output of his projects. If the total output of his projects decreases, so does the total profit generated by the society; thus, the agent is worse off.

The main result of the paper is the characterization of all the mechanisms that satisfy efficiency. The class of efficient mechanisms coincide exactly with the class of separable mechanisms (Theorem 1). We also look at mechanisms that satisfy other desirable properties such as

[1]The importance of the bilateral network is reflected in this mechanism. For instance, if we allow at least one trilateral project, then the mechanism that divides the profit of a project equally between its members is not efficient. Indeed, we can easily find an example with linear production functions where the efficient allocation requires an agent to contribute his full time allocation to a trilateral project, where he gets a third of the profit that this project generates, but he is better off switching his time to a bilateral project, where he gets half of the profit. We discuss more about multilateral projects in Section 4.

strong monotonicity at the efficient equilibrium (SMEQ) and strong efficiency (SE). To understand SMEQ, suppose that the technology of a project becomes more productive. Then no agent should be harmed by that. In other words, the payoff at the equilibrium by all the agents should not decrease. Under SE, agents cannot gain by jointly coordinating their time allocations. That is, the efficient equilibrium should be a Strong Nash equilibrium (see, Aumann, 1959).

Proposition 2 characterizes all the mechanisms that are strongly efficient, or strongly monotonic at equilibrium and efficiency. These mechanisms take the shape similar to the average profit function, where the payoff of the agents only depends on the total profit generated by society. That is, these mechanisms disregard any information on the time contributions of the agents or the profits generated by specific projects.

Finally, we investigate the efficient mechanisms that are additive, which explores issues related to redistribution of profits after only a subset of projects have been completed.[2] In particular, suppose that the projects containing agent i are completed first, and the projects that do not contain agent i are completed next. Additivity is an ordering-free axiom that requires that if partial payments were made after the rounds of completed work, then the overall payment should not change. The class of efficient mechanisms are described in Proposition 3. These mechanisms take the form of a seniority mechanism, where agents get a constant share of the profit generated by their own projects and the remainder of the profit is split between the rest of the agents with some arbitrarily function.

The rest of the paper is as follows: Section 1.1 surveys the relevant literature. Section 2 describes the model and the main result of the paper. In Section 3, we explore the efficient mechanisms that are additive. In Section 4, we extend our main result to the multilateral case. Finally, we conclude in Section 5.

[2]This property is also reminiscent to the traditional consistency axiom, by re-interpreting the axiom as agents leaving the society (see, Thomson, 2007).

1.1 Related Literature

The concept of implementation of the efficient allocation in a Nash equilibrium has been explored widely in the literature. Maskin and Sjöström (2002) survey full implementation of efficient outcomes in different production functions. However, the literature of implementation in very general economies has typically lead to impossibilities. We contrast with this literature by finding a specific economy where several mechanisms can implement the efficient allocation.

We focus on the case where agents need to contribute their full time allocation and the entire profit is allocated to the agents, therefore the traditional issues of moral hazards are ruled out (e.g., Holmstrom, 1982). This restriction is similar to allocation mechanisms for a fixed divisible resource (such as a dollar) depending on the report of the agents (e.g., de Clippel, Moulin and Tideman, 2008 and Tideman and Plassmann, 2008).

A closely related work studies cost sharing allocation mechanisms that implement the cost minimizing network. For instance, Juarez and Kumar (2013) focus on implementing the efficient allocation in connection network, where agents should be provided with incentive to select the cost minimizing network. This equilibrium should Pareto dominate all the other equilibriums. Furthermore, the class of mechanisms characterized in Juarez and Kumar (2013) are closely related to the mechanisms characterized in Proposition 2. Other closely related work, Hougaard and Tvede (2012, 2013) (Memo: This should be 2015! see Reference) characterize truthfully implementing cost minimizing networks by changing the announcement rule. Their main results also lead to characterizing rules related to Proposition 2. In contrast with this literature, the bilateral setting allows us to implement a larger class of rules that have not been explored in the implementation literature.

To our best knowledge, our work is the first to introduce the implementation of the efficient time allocation for any set of production functions.

2 Bilateral network model

We investigate mechanisms in a bilateral network. Each pair of agents has a project which generates a profit depending on their time allocation. Let $N = \{1, 2, \dots, n\}$ be the set of agents and $L = \{(1,2), (1,3), (1,4), \dots, (n-1,n)\}$ be the set of links (or projects) for each pair of agents. The number of links is $l = \frac{1}{2}(n-1)n$. Furthermore, let L^i be the set of links that agent i belongs to, and L^{-i} be the set of links that agent i does not belong to. Let T^i be the time resource of agent i. Every project generates profit depending on the time allocated by the agents associated to that project. The total profit of the society is the sum of the profit generated at every project. Let (t^i_j, t^j_i, F^{ij}) be a vector, where t^i_j is the time resource allocated by agent i to the project with agent j, t^j_i is the time resource allocated by agent j to the project with agent i, and F^{ij} is the profit generated by the project. Let $\times_{(i,j)\in L}(t^i_j, t^j_i, F^{ij})$ be the concatenation of all these vectors.[3]

Definition 1. *A mechanism in a bilateral network is a continuous function $\varphi : (\mathbb{R}^2_+ \times \mathbb{R}_+)^L \to \mathbb{R}^n_+$ such that*

$$\sum_{i=1}^{n} \varphi_i \left(\times_{(k,l)\in L}(t^k_l, t^l_k, F^{kl}) \right) = \sum_{(k,l)\in L} F^{kl}$$

The inputs on a mechanism are the different time allocations and profits of every project. The output is a full distribution of the total profit to the agents.

Example 1. A. *Average profit mechanism: the final profit of the entire society is divided equally among all members. That is, for any $i \in N$,*

$$\varphi_i(\times_{(k,l)\in L}(t^k_l, t^l_k, F^{kl})) = \frac{1}{n} \sum_{(k,l)\in L} F^{kl}$$

B. *Shapley mechanism: the final profit produced by the project of agents i and j is shared equally between i and j. The total share of every agent is the sum of his shares in the*

[3] For instance,

$$\times_{(i,j)\in\{(1,2),(1,3),(2,3)\}}(t^i_j, t^j_i, F^{ij}) = (t^1_2, t^2_1, F^{12}) \times (t^1_3, t^3_1, F^{13}) \times (t^2_3, t^3_2, F^{23}).$$

projects. That is, for any $i \in N$,

$$\varphi_i(\times_{(k,l)\in L}(t^k_l, t^l_k, F^{kl})) = \frac{1}{2}\sum_{(i,j)\in L^i} F^{ij}$$

C. *Proportional sharing mechanism: the final profit produced by the project of agents i and j is shared between them, depending on the proportion of their contribution of time. That is, for any $i \in N$,*

$$\varphi_i(\times_{(k,l)\in L}(t^k_l, t^l_k, F^{kl})) = \begin{cases} \sum_{(i,j)\in L^i} Pr^i_j(t^i_j, t^j_i)F^{ij} & \textit{if} \quad t^i_j > 0; \\ 0 & \textit{otherwise}, \end{cases}$$

where the proportion of agent i with agent j is $Pr^i_j(t^i_j, t^j_i) = \frac{t^i_j}{t^i_j + t^j_i}$.

Notice that average profit and Shapley mechanisms are independent of time allocation, but proportional sharing mechanism is dependent on the time allocation. Any convex combination of mechanisms is also a mechanism.

2.1 Efficiency and other desirable properties

The strategy of agent i is an allocation of his time resource between different projects to which he belongs, where the sum of all his time allocations is not larger than his total time resource T^i. Let $f = (f_{ij})_{(i,j)\in L}$ be the vector of production functions, where $f_{ij} : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a continuous non-decreasing function on both coordinates. Let $\mathcal{F}$ be the set of vectors of production function, and $v^i = (t^i_j)_{j\neq i}$ where $\sum_{j\neq i} t^i_j = T^i$.

Definition 2. *Agent i's strategy profile, S^i, is a function*

$$S^i : \mathcal{F} \to v^i.$$

We study the case of perfect information. The strategy of agent i is such that for every set of production functions in the bilateral network, it assigns a time allocation for each project.

Definition 3. *For the mechanism φ and the vector of production functions f, the payoff function of agent i at the vector of strategies (S^i, S^{-i}) is:*

$$\pi_i(S^i, S^{-i}, f, \varphi) = \varphi_i\left(\times_{(k,l)\in L}\left(t^k_l, t^l_k, f_{kl}(t^k_l, t^l_k)\right)\right), \quad \textit{where } S^i = (t^i_j)_{j\neq i}.$$

The payoff of an agent depends on its own and others' time allocation, production functions and the outcome given by the mechanism.

We are interested in Nash equilibrium strategies, where agents have no incentive to deviate, assuming the strategies of the other agents remain fixed.

Definition 4. *Given a mechanism φ and the production function vector f, a strategy profile $(S^{*1}, S^{*2}, \ldots, S^{*n})$ is a Nash equilibrium if:*

$$\pi_i(S^{*i}, S^{*-i}, f, \varphi) \geq \pi_i(S^i, S^{*-i}, f, \varphi) \quad \forall S^i.$$

Under a production function vector, a set of strategies generates output. An efficient strategy dominates other strategies for any production functions.

Definition 5. *A set of strategies $(S^1, \ldots, S^n)$ is efficient if for any f and for any $\tilde{S}$*

$$\sum_{(i,j)\in L} f_{ij}(t^i_j, t^j_i) \geq \sum_{(i,j)\in L} f_{ij}(\tilde{t}^i_j, \tilde{t}^j_i), \quad \textit{where } (t^i_j)_{j\neq i} = S^i(f) \textit{ and } (\tilde{t}^i_j)_{j\neq i} = \tilde{S}^i(f).$$

Also, we define the efficiency for a mechanism, where the mechanism can implement the set of efficient strategy as a Nash equilibrium.

Definition 6. *A mechanism is efficient if the set of efficient strategies is a Nash equilibrium.*

As discussed in the introduction, the Shapley and the average profit mechanisms are efficient, but proportional sharing mechanism is not.

2.2 Separable mechanism and the main result: implementing the efficient time allocation

In this section, we characterize the mechanisms that are efficient. There are several restrictions that efficiency imposes on a mechanism. The first restriction is that the payoff of an agent should depend on the aggregate profit generated by the projects in which he belongs, instead of individual projects. The second restriction is that the time allocation of an agent should not influence his payoff (but it might influence the payoff of other agents). The separable mechanisms discussed below include these two restrictions.

Definition 7. *A mechanism φ is separable if there exist functions $g_1, g_2, \ldots, g_n$ non-decreasing in the first coordinate such that for every agent i,*

$$\varphi_i\left(\times_{(k,l)\in L}(t_l^k, t_k^l, F^{kl})\right) = g_i\left(\sum_{(i,j)\in L^i} F^{ij}, \times_{(k,l)\in L^{-i}}(t_l^k, t_k^l, F^{lk})\right).$$

A mechanism is separable if the payoff of agent i only depends on the total aggregated profit generated by his projects (denoted $\sum_{(i,j)\in L^i} F^{ij}$), as well as the profits and times that do not contain agent i (denoted $\times_{(k,l)\in L^{-i}}(t_l^k, t_k^l, F^{lk})$). The class of separable mechanisms is large. We provide below some examples. Corollary 1 provides the entire class of separable mechanisms under two additional assumptions.

Example 2. *A. Shapley is a separable mechanisms generated by the functions,*

$$g_i^{Sh}\left(\sum_{(i,j)\in L^i} F^{ij}, \times_{(k,l)\in L^{-i}}(t_l^k, t_k^l, F^{lk})\right) = \frac{1}{2}\sum_{(i,j)\in L^i} F^{ij} \text{ for any } i.$$

B. The average profit mechanism is a separable mechanism generated by the functions,

$$g_i^{AP}\left(\sum_{(i,j)\in L^i} F^{ij}, \times_{(k,l)\in L^{-i}}(t_l^k, t_k^l, F^{lk})\right) = \frac{\sum_{(k,l)\in L} F^{kl}}{n} \text{ for any } i.$$

C. The convex combination of the Shapley mechanism and the average profit mechanism is also a separable mechanism generated by the convex combination of g^{Sh} and g^{AP}.

D. Consider the mechanism $\varphi_i^*\left(\times_{(k,l)\in L}(t_l^k, t_k^l, F^{kl})\right) = \frac{2}{(n-1)(n-2)}\sum_{(k,l)\in L^{-i}} F^{lk}$, for any $i \in N$, where every agent gets paid the average profit of the projects in which he does not belong. The mechanism is separable and generated by the functions,

$$g_i^*\left(\sum_{(i,j)\in L^i} F^{ij}, \times_{(k,l)\in L^{-i}}(t_l^k, t_k^l, F^{lk})\right) = \frac{2}{(n-1)(n-2)}\sum_{(k,l)\in L^{-i}} F^{lk} \text{ for any } i.$$

The proportional sharing mechanism is not separable, because the payoff of an agent depends on his allocation of time to different projects.

Theorem 1. *A mechanism is efficient if and only if it is separable.*

The proof is in Appendix A.

We say that a mechanism is *symmetric* if it is independent of the name of the agents. We say that a mechanism is *time-independent* if the mechanism only depends on the profit generated by the different projects. The class of efficient and symmetric mechanism is large. We characterize below the class of efficient, symmetric and time-independent mechanisms for 3 agents. This class contains Shapley, φ^* and AC mechanisms in the middle.

Corollary 1. *Suppose $n = 3$. A mechanism is efficient, symmetric and time-independent if and only if it is a convex combination of Shapley and φ^*.*

Proof. Consider the mechanism φ that is efficient, symmetric and time-independent.

First, note that by Theorem 1, symmetry and time independence, there exists a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that φ_i can be written as $\varphi_i(F^{il}, F^{ik}, F^{lk}) = g(F^{il}+F^{ik}, F^{lk})$ for any $(F^{il}, F^{ik}, F^{lk})\in\mathbb{R}_+^3$.

Second, we show that g is linear. That is, there exists constants $\alpha \geq 0$ and $\beta \geq 0$ such that $g(A, B) = \alpha A + \beta B$ for any $A, B \geq 0$.

To see this, note that by efficiency,

$$\varphi_1+\varphi_2+\varphi_3 = g(F^{12}+F^{13},F^{23})+g(F^{21}+F^{23},F^{13})+g(F^{31}+F^{32},F^{12}) = F^{12}+F^{13}+F^{23}$$

for any $F^{12}\geq 0$, $F^{13}\geq 0$ and $F^{23}\geq 0$.

Consider the profile $(F^{12},F^{13},F^{23})\in\mathbb{R}^3_{++}$ and some small $\epsilon>0$. Then,

$$g(F^{12}+F^{13},F^{23})+g(F^{21}+F^{23},F^{13}) = g(F^{12}+F^{13}-\epsilon,F^{23}+\epsilon)+g(F^{21}+F^{23}+\epsilon,F^{13}-\epsilon).$$

Let $A=F^{12}+F^{13}$, $B=F^{21}+F^{23}$ and $X=F^{12}+F^{13}+F^{23}$. Then,

$$g(A,X-A)+g(B,X-B) = g(A-\epsilon,X-A+\epsilon)+g(B+\epsilon,X-B-\epsilon).$$

When $A=B$, we have that $G(A,X-A)=\frac{g(A-\epsilon,X-A+\epsilon)+g(A+\epsilon,X-A-\epsilon)}{2}$ for any $X\geq A\geq 0$ and any small $\epsilon>0$. This implies that $g(A,X-A)=\alpha A+\beta(X-A)+\gamma$ for some constants α, β, and γ. Since $g(0,0)+g(0,0)+g(0,0)=0$, then $\gamma=0$. Furthermore, notice that $g(A,X-A)\geq 0$, thus $\alpha\geq 0$ and $\beta\geq 0$.

On the other hand, consider the profile where $F^{12}=F^{13}=F^{23}=C\geq 0$. Then, $3g(2C,C)=3(\alpha 2C+\beta C)=3C$. Therefore, $\beta=1-2\alpha$. Since $\alpha\geq 0$ and $\beta\geq 0$, we have that $\frac{1}{2}\geq\alpha\geq 0$.

When $\alpha=0$ the mechanism φ generates φ^*. When $\alpha=\frac{1}{2}$, the mechanism φ generates Sh. Moreover, when $\alpha=\beta=\frac{1}{3}$, the mechanism φ generates AC. □

2.3 Strong Nash Implementation

In this section, we consider the case where agents can coordinate with other agents. In particular, we look at mechanisms where agents have no incentive to form coalitions. That is, if a group of agents jointly coordinates their time and at least one of the agents of the group strictly increases his payoff, then it must be that another agent of the group strictly decreases his payoff. This is captured in our definition of Strong Nash equilibrium.

Definition 8 (Strong Nash Equilibrium)**.** *Given a mechanism φ and the production function vector f, a strategy profile S^* is a strong Nash Equilibrium if whenever there exists a subset of agents $T \subset N$, a strategy profile S^T, and an agent $i \in T$ such that*

$$\pi_i\left((S^T, S^{*-T}), f, \varphi\right) > \pi_i\left((S^{*T}, S^{*-T}), f, \varphi\right),$$

then there exists another agent $j \in T$ such that

$$\pi_j\left((S^T, S^{*-T}), f, \varphi\right) < \pi_j\left((S^{*T}, S^{*-T}), f, \varphi\right).$$

Definition 9. *A mechanism is strongly efficient if the set of efficient strategies is a strong Nash equilibrium.*

In order to introduce the next definition, consider the case where there is a change in the technology used by one of the projects. In particular, suppose that the technology used by the project with agent i and j increases the profit at any level of time allocations. In this case, the definition of strong monotonicity at equilibrium discussed below requires that the payoff of all the agents in the society at equilibrium should not decrease.[4] In other words, improvements in the quality of the projects should not harm any agents in the society.

Definition 10 (Strong Monotonicity at Equilibrium)**.** *Consider the vector of production functions f and let $\tilde{f}_{ij}$ a production function such that $\tilde{f}_{ij}(s^i_j, s^j_i) \geq f_{ij}(s^i_j, s^j_i), \forall (s^i_j, s^j_i)$. Let $\tilde{f} = (\tilde{f}_{ij}, f_{-ij})$. Consider any equilibrium strategies $\left\{(t^i_j, t^j_i)\right\}_{(i,j)\in L}$ under f and $\left\{(\tilde{t}^i_j, \tilde{t}^j_i)\right\}_{(i,j)\in L}$ under $\tilde{f}$. A mechanism is strongly monotonic at equilibrium if for any agent k:*

$$\varphi_k\left(\times_{(i,j)\in L}\left(\tilde{t}^i_j, \tilde{t}^j_i, \tilde{f}_{ij}(\tilde{t}^i_j, \tilde{t}^j_i)\right)\right) \geq \varphi_k\left(\times_{(i,j)\in L}\left(t^i_j, t^j_i, f_{ij}(t^i_j, t^j_i)\right)\right).$$

The next proposition characterizes all mechanisms that meet Strong Nash Equilibrium, or

[4]Thomson (2007) defines a very similar concept, which is called "'strict resource monotonicity" in allocation problems.

Strong Monotonicity at Equilibrium. Surprisingly, these mechanisms only depend on the sum of profits generated by all projects in the society.

Proposition 2. *The following three properties are equivalent for the mechanism φ:*

(i) *φ satisfies Strong Efficiency*

(ii) *φ satisfies Strong Monotonicity at equilibrium and Efficiency*

(iii) *there exists non-decreasing functions: $g_i : \mathbb{R}_+ \to \mathbb{R}_+ \quad$ for $i = 1, \ldots, n$ such that*

$$\sum_i g_i(A) = A \quad \forall A \in \mathbb{R}_+$$

and

$$\varphi(t, F) = \left(g_1 \left(\sum_{(i,j) \in L} F^{ij} \right), \ldots, g_n \left(\sum_{(i,j) \in L} F^{ij} \right) \right) \quad \forall F \text{ and } t.$$

The proof is in Appendix A.

3 Seniority Mechanism

In many firms the compensation of agents vary depending on seniority, where agents with longer history and track record receive a larger share of the profit generated by their projects. Consider for instance the compensation schemes used in law firms, where employees are typically ranked as associate, junior partner or senior partner.[5] Their shares of the profit increase as they move up in the ladder. In this section, we introduce a class of mechanisms that capture features related to seniority, where the sharing mechanism does not treat agents symmetrically.

We introduce a seniority mechanism in a bilateral network. A seniority mechanism distributes the payoffs to the agents based on the profits generated by their own projects and the profits generated by the projects in which the agent does not belong. In particular, each agent has a common

[5] Anecdotally, this project emerged by discussing with attorneys the rules used for sharing their profits. The mechanisms below resembles this discussion.

share rate to the profit generated by his own projects. We denote by $s^i \in [0,1]$ the share of the profits in which agent i belongs. Since the share s^i is common for every project in which agent i belongs, we have that:

$$s^i + s^j \leq 1, \quad \forall i, j. \tag{1}$$

A vector of weights $(s^1, s^2, \ldots, s^n)$ is feasible if it satisfies Equation (1).

For every project $\{i, j\}$, the rest of the profit $(1 - s^i - s^j)F^{ij}$ is distributed to the agents in $N \setminus \{i, j\}$ with an arbitrary function g_{ij} that depend on the time allocation to project (i, j) and the remaining profit $(1 - s^i - s^j)F^{ij}$. We formalize this mechanism below.

Definition 11 (Seniority Mechanism)**.** *Consider a feasible vector of weights $(s^1, s^2, \ldots, s^n)$ and for every $(l, m) \in L$ the function $g_{lm} : \mathbb{R}^3_+ \to \mathbb{R}^{N \setminus \{l,m\}}$ such that for any $(t^l_m, t^m_l, x) \in \mathbb{R}^3_+$:*

$$\sum_{k \neq l,m} g^{lm}_k \left(t^l_m, t^m_l, x\right) = x. \tag{2}$$

The allocation of agent i under the Seniority mechanism φ^{SEN} is:

$$\varphi^{SEN}_i (A) = s^i \sum_{(i,j) \in L^i} F^{ij} + \sum_{(l,m) \in L^{-i}} g^{lm}_i \left(t^l_m, t^m_l, (1 - s^l - s^m)F^{lm}\right).$$

The class of Seniority mechanisms is large. When $s^i = \frac{1}{2}$ for all i, the class of seniority mechanisms leads to the Shapley mechanism, where every agent gets half of the profit he generates. Alternatively, when $s^i = \frac{1}{n}$ for all i and $g^{lm}_k \left(t^l_m, t^m_l, x\right) = \frac{x}{n-2}$, then the class of Seniority mechanisms leads to the average profit mechanism, where every agent is compensated equally. Finally, when $s^i = 0$ for all i, then agents receive a compensation only from the projects in which they do not belong.

In order to characterize Seniority mechanisms, we introduce a notion of additivity. The idea is that additivity separates a problem into two parts depending on whether an agent belongs to a project or not.

Definition 12 (Additivity)**.** *Consider the following problem $A = \left\{\left(t^i_j, t^j_i, \tilde{F}^{ij}\right)\right\}_{i \neq j}$. Let $k \in N$*

and

$$B^k(A) \;=\; \left\{\left(t_j^k, t_k^j, \tilde{F}^{kj}\right)\right\}_{(k,j)\in L^k} \cup \{(0,0,0)\}_{(j,l)\in L^{-k}}$$

be the set of projects from A that contain agent i, and

$$C^k(A) \;=\; \left\{\left(t_j^l, t_l^j, \tilde{F}^{lj}\right)\right\}_{(j,l)\in L^{-k}} \cup \{(0,0,0)\}_{(k,j)\in L^k}.$$

be the set of projects from A that do not contain agent k.

The mechanism φ is additive if for any problem A and any agent k,

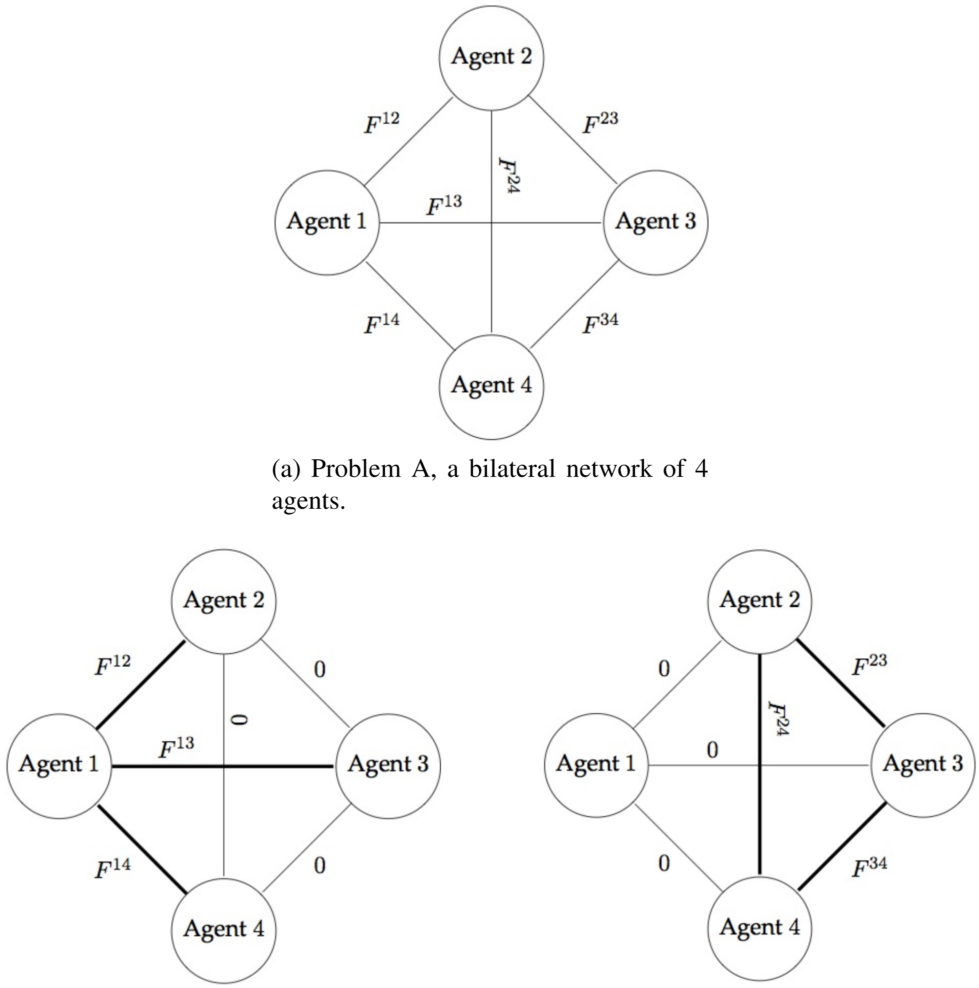
$$\varphi(A) = \varphi\left(B^k(A)\right) + \varphi\left(C^k(A)\right).$$

Consider the problem A and agent 1 (see Figure 2). Additivity on agent 1 divides a problem A into two parts. First, the projects that contain agent 1 are assigned the same profit and same time allocation as in problem A, while the projects that do not contain agent 1 are assigned a zero profit and zero time allocation (we call this problem $B^1(A)$). Second, the projects that contain agent 1 are assigned zero profit and zero time allocation, while the projects that do not contain agent 1 are assigned the same profit and same time allocation as in problem A (we call this problem $C^1(A)$).

Additivity requires that the order in which the projects are completed should not influence the final payoff of the agents. That is, if the projects that involve agent 1 are completed first ($B^1(A)$) and the rest of the projects are completed next $(C^1(A))$, or alternatively, all the projects are completed simultaneously, then the final payment of the agents should not change. This axiom is a very compelling for companies where agents have the freedom to complete tasks in any order that they choose while payments are made sequentially after individual projects are completed.[6]

Notice that seniority mechanisms are additive because the full profit generated by every project

[6] As we will see in the proof of the next result, the current additivity axiom implies a much stronger version of additivity, where the function needs to be separable in every individual project.



(a) Problem A, a bilateral network of 4 agents.

(b) Problem $B^1(A)$.

(c) Problem $C^1(A)$,

Figure 2: Additivity: 4 agents case

is redistributed back to the agents only depending on the time and profit related to that project. In particular, for project (i, j), agents i and j get a share s^i and s^j of the profit F^{ij}. The rest of the profit $(1 - s^i - s^j)F^{ij}$ is split between the rest of the agents using the function g_{ij}.

The following proposition characterizes the class of additive and efficient mechanisms.

Proposition 3. *An efficient mechanism is additive if and only if it is a Seniority mechanism.*

The proof is in Appendix A.

Finally, we characterize asymmetric variations of the Average Cost mechanisms, where every agent is assigned a fixed share of the total profit produced by the society.

Corollary 2. *The following three properties are equivalent for the mechanism φ.*

(i) φ *satisfies Efficiency, Additivity and Strong Monotonicity at Equilibrium*

(ii) φ *satisfies Additivity and Strong Efficiency*

(iii) There exists $s^1, s^2, \ldots, s^n$ *such that* $\sum_i s^i = 1$*, and* $\varphi_i\left(t, F\right) = s^i \left(\sum_{(k,l)\in L} F^{kl}\right)$.

The proof is in Appendix A.

4 Multilateral projects and Efficiency

The main result of the paper can be easily extended to the case of multilateral projects, where coalitions of size larger than two agents can form to collaborate in a project. While the class of efficient mechanisms resembles the class of mechanisms characterized in Theorem 1, some mechanisms discussed above, such as the Shapley mechanism, will no longer be efficient.

Let $M \subsetneq 2^N$ be the set of potential coalitions that can form. Let M^i be the set of coalitions from M that contains agent i. If M only contains all subsets of 2 agents, then it is the bilateral case discussed above. If $M = 2^N \setminus \{\emptyset\}$ then any potential coalition can collaborate in a project.

Definition 13. *A mechanism for multilateral projects is separable if there exist functions* $g_1, g_2, \ldots, g_n$ *non-decreasing in the first coordinate such that:*

$$\varphi_i\left(\times_{S\in M}(t^S, F^S)\right) = g_i\left(\sum_{S\in M^i} F^S, \times_{T\in M^{-i}}(t^T, F^T)\right), \quad \forall i.$$

Similarly as above, a mechanism is separable if the payoff of agent i only depends on the total aggregated profit generated by his projects, and the profits and times of the projects that do not contain agent i.

The average profit mechanisms and asymmetric variations discussed in Proposition 2 are separable for multilateral projects. The Shapley mechanism, which divides the profit of the projects equally between its agents, is separable if and only if M only contains coalitions of the same cardinality.

The seniority mechanisms discussed above, where every agent receives a fixed share from his projects, can be easily extended to belong to the class of separable mechanisms for multilateral projects as long as the sum of the shares of the agents from every project add up to less than or equal to 1.

The proposition below characterizes the class of efficient mechanisms for multilateral projects.

Proposition 4. *A mechanism for multilateral projects is efficient if and only if it is separable.*

The proof of this proposition is almost identical to the proof of Theorem 1 and available upon request to the authors.

5 Conclusion

In this paper, we have studied robust mechanisms that implement efficient allocations in bilateral network for any production functions. Theorem 1 shows that separable mechanisms, where the payoff of an agent only depends on the total profit generated by his own projects, the time allocations and the profits generated by which he does not belong, are the only efficient mechanisms. We have also introduced strong monotonicity at equilibrium and strong efficiency, and characterized all the mechanisms that satisfy them. We find that the shape of such mechanisms look similar to the average profit mechanism, where the payoff of the agents only depends on the total profit produced by the society (Proposition 2). Furthermore, we introduced the notion of additivity and characterize the efficient mechanisms that are additive (Proposition 3). This study has mainly focused on bilateral networks, where the most interesting mechanisms arise. The extension of the main result to multilateral networks is provided in Proposition 4.

Appendix A. Proofs

A.1 Proof of Theorem 1

Proof. First, we show that if a mechanism is separable, then the mechanism is efficient.

Suppose that an agent, say agent i, deviates from an efficient strategy under a separable mechanism. Then, the deviation by agent i does not lead to an increase in total profit of his projects in any production functions due to the definition of efficient strategy. Thus, agent i cannot increase his payoff because g_i is a non-decreasing function in the first coordinate. This is a contradiction.

Next, we show that if a mechanism is efficient, then the mechanism is separable for any set of production functions.

Step 1: We show that if a mechanism is efficient, then the payoff of agent i does not depend on his time allocation. That is,

$$\varphi_i\left(t^i, t^{-i}, F\right) = h_i\left(t^{-i}, F\right),$$

where $t^i = (t^i_1, t^i_2, \ldots, t^i_{i-1}, t^i_{i+1}, \ldots, t^i_n)$ is the strategy of agent i, and $t^{-i} = (t^1, t^2, \ldots, t^{i-1}, t^{i+1}, \ldots, t^n)$ is the collection of all agents' strategies but agent i.

First, consider the following production functions:

$$\begin{aligned}
f_{i1}(t^i_1, t^1_i) &= c^{i1} + \epsilon\left(t^i_1 + t^1_i\right); \\
f_{i2}(t^i_2, t^2_i) &= c^{i2} + \epsilon\left(t^i_2 + t^2_i\right); \\
&\vdots \\
f_{i(n-1)}(t^i_{n-1}, t^{n-1}_i) &= c^{i(n-1)} + \epsilon\left(t^i_{n-1} + t^{n-1}_i\right); \\
f_{in}(t^i_n, t^n_i) &= c^{in} + 2\epsilon t^i_n + \epsilon t^n_i; \\
f_{jk}(t^j_k, t^k_j) &= c^{jk} + \epsilon\left(t^j_k + t^k_j\right), \quad \forall j, k \neq i,
\end{aligned}$$

where $c^{ij} \in \mathbb{R}_+$ is constant for all $(i, j) \in L$.

By definition of efficient strategy, agent i allocates his full resource to the project with agent n.

That is, $t^i = (0, 0, \ldots, 0, T^i)$. Then, for all $\tilde{t^i} \in \mathbb{R}_+^{n-1}$, such that $\sum_j \tilde{t^i_j} = T^j$ we have that

$$\begin{aligned}
&\varphi_i \left((0, 0, \ldots, 0, T^i), t^{-i}, c^{i1} + \epsilon(0 + t_i^1), c^{i2} + \epsilon\left(0 + t_i^2\right), \ldots, c^{in} + 2\epsilon T^i + \epsilon t_i^n, F^{-i}\right) \\
&\quad \geq \varphi_i \left((\tilde{t^i_1}, \tilde{t^i_2}, \ldots, \tilde{t^i_n}), t^{-i}, c^{i1} + \epsilon(\tilde{t^i_1} + t_i^1), c^{i2} + \epsilon\left(\tilde{t^i_2} + t_i^2\right), \ldots, c^{in} + 2\epsilon \tilde{t^i_n} + \epsilon t_i^n, F^{-i}\right),
\end{aligned}$$

for any $\epsilon > 0$ and letting F^{-i} be the output vector in which agent i does not belong. As ϵ goes to 0, by the continuity of φ we have that

$$\varphi_i \left((0, 0, \ldots, 0, T^i), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right) \geq \varphi_i \left((\tilde{t^i_1}, \tilde{t^i_2}, \ldots, \tilde{t^i_n}), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right). \tag{3}$$

Alternatively, let $\tilde{t^i} \in \mathbb{R}_+^{n-1}$ by a fixed vector such that $\sum_j \tilde{t^i_j} = T^i$, and consider the production functions as follows:

$$\begin{aligned}
f_{i1}(t_1^i, t_i^1) &= c^{i1} + \epsilon\left(\min\{\tilde{t^i_1}, t_1^i\} + t_i^1\right); \\
f_{i2}(t_2^i, t_i^2) &= c^{i2} + \epsilon\left(\min\{\tilde{t^i_2}, t_2^i\} + t_i^2\right); \\
&\vdots \\
f_{i(n-1)}(t_{n-1}^i, t_i^{n-1}) &= c^{i(n-1)} + \epsilon\left(\min\{\tilde{t_{n-1}^i}, t_{n-1}^i\} + t_i^{n-1}\right); \\
f_{in}(t_n^i, t_i^n) &= c^{in} + \epsilon\left(\min\{\tilde{t^i_n}, t_n^i\} + t_i^n\right); \\
f_{jk}(t_k^j, t_j^k) &= c^{jk} + \epsilon\left(t_k^j + t_j^k\right), \quad \text{where } j, k \neq i.
\end{aligned}$$

Notice that at the optimal scenario (Memo: "Profile" would be better?), agent i allocates his resource as $\tilde{t^i} = (\tilde{t^i_1}, \tilde{t^i_2}, \ldots, \tilde{t^i_n})$. The mechanism φ_i is efficient, then we have

$$\begin{aligned}
&\varphi_i \left((\tilde{t^i_1}, \tilde{t^i_2}, \ldots, \tilde{t^i_n}), t^{-i}, c^{i1} + \epsilon(\tilde{t^i_1} + t_i^1), c^{i2} + \epsilon(\tilde{t^i_2} + t_i^2), \ldots, c^{in} + \epsilon(\tilde{t^i_n} + t_i^n), F^{-i}\right) \\
&\quad \geq \varphi_i \left((0, 0, \ldots, 0, T^i), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in} + \epsilon(\tilde{t^i_n} + t_i^n), F^{-i}\right),
\end{aligned}$$

where $\epsilon > 0$. As ϵ goes to 0, by the continuity of φ we have that

$$\varphi_i\left((\tilde{t_1^i}, \tilde{t_2^i}, \ldots, \tilde{t_n^i}), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right) \geq \varphi_i\left((0, 0, \ldots, 0, T^i), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right). \tag{4}$$

By Equations (3) and (4), we have

$$\varphi_i\left((0, 0, \ldots, 0, T^i), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right) = \varphi_i\left((\tilde{t_1^i}, \tilde{t_2^i}, \ldots, \tilde{t_n^i}), t^{-i}, c^{i1}, c^{i2}, \ldots, c^{in}, F^{-i}\right).$$

Thus, the payoff of agent i is independent of his allocation. Similarly, the payoffs of the other agents are also independent of their allocations.

Step 2: We show that the share of agent i depends on the sum of the profits of the projects in which he belongs.

Consider the following production functions:

$$\begin{aligned}
f_{i1}(t_1^i, t_i^1) &= c^{i1} + t_1^i + \gamma t_i^1; \\
f_{i2}(t_2^i, t_i^2) &= c^{i2} + \gamma\left(t_2^i + t_i^2\right); \\
&\vdots \\
f_{i(n-1)}(t_{n-1}^i, t_i^{n-1}) &= c^{i(n-1)} + \gamma\left(t_{n-1}^i + t_i^{n-1}\right); \\
f_{in}(t_n^i, t_i^n) &= c^{in} + \gamma\left(t_n^i + t_i^n\right); \\
f_{jk}(t_k^j, t_j^k) &= c^{jk} + \gamma\left(t_k^j + t_j^k\right), \quad \text{where } j, k \neq i.
\end{aligned}$$

where $\gamma < 1$. By efficiency, for all $t^i \in \mathbb{R}_+^{n-1}$ such that $\sum_j t_j^i = T^i$ we have that

$$\begin{aligned}
&\varphi_i\left((T^i, 0, \ldots, 0), c^{i1} + (T^i + t_i^1), c^{i2} + \gamma t_2^2, \ldots, c^{in} + \gamma t_i^n, F^{-i}, t^{-i}\right) \\
&\quad \geq \varphi_i\left((t_1^i, t_2^i, \ldots, t_n^i), c^{i1} + (t_1^i + t_i^1), c^{i2} + \gamma(t_2^i + t_i^2), \ldots, c^{in} + \gamma\left(t_n^i + t_i^n\right), F^{-i}, t^{-i}\right).
\end{aligned}$$

As γ goes to 1, then

$$
\begin{aligned}
&\varphi_i\left((T^i,0,\ldots,0),c^{i1}+(T^i+t_i^1),c^{i2}+t_i^2,\ldots,c^{in}+t_i^n,F^{-i},t^{-i}\right)\\
&\quad\geq\varphi_i\left((t_1^i,t_2^i,\ldots,t_n^i),c^{i1}+(t_1^i+t_i^1),c^{i2}+(t_2^i+t_i^2),\ldots,c^{in}+(t_n^i+t_i^n),F^{-i},t^{-i}\right). \quad (5)
\end{aligned}
$$

Therefore, transferring all the contribution of agent i to the project with agent 1 does not decrease the share of agent i.

Alternatively, consider the following production functions,

$$
\begin{aligned}
f_{i1}(t_1^i,t_i^1) &= c^{i1}+\min\{\tilde{t_1^i},t_1^i\}+\gamma t_1^i+t_i^1\\
f_{i2}(t_2^i,t_i^2) &= c^{i2}+\min\{\tilde{t_2^i},t_2^i\}+t_i^2\\
&\vdots\\
f_{i(n-1)}(t_{n-1}^i,t_i^{n-1}) &= c^{i(n-1)}+\min\{\tilde{t_{n-1}^i},t_{n-1}^i\}+t_i^{n-1}\\
f_{in}(t_n^i,t_i^n) &= c^{in}+\min\{\tilde{t_n^i},t_n^i\}+t_i^n;\\
f_{jk}(t_k^j,t_j^k) &= c^{jk}+\left(t_k^j+t_j^k\right), \qquad \text{where } j,k\neq i.
\end{aligned}
$$

For $\gamma<1$, the optimal profile requires $t^i=(\tilde{t_1^i},\tilde{t_2^i},\ldots,\tilde{t_n^i})$. Comparing this with the suboptimal profile $t^i=(T^i,0,\ldots,0)$,

$$
\begin{aligned}
&\varphi_i\left((\tilde{t_1^i},\tilde{t_2^i},\ldots,\tilde{t_n^i}),c^{i1}+\tilde{t_1^i}+\gamma\tilde{t_1^i}+t_i^1,c^{i2}+\tilde{t_1^i}+t_i^2,\ldots,c^{in}+\tilde{t_1^i}+t_i^n,F^{-i},t^{-i}\right)\\
&\quad\geq\varphi_i\left((T^i,0,\ldots,0),c^{i1}+\tilde{t_1^i}+\gamma T^i+t_i^1,c^{i2}+t_i^2,\ldots,c^{in}+t_i^n,F^{-i},t^{-i}\right).
\end{aligned}
$$

As γ goes to 1, then

$$
\begin{aligned}
&\varphi_i\left((\tilde{t_1^i},\tilde{t_2^i},\ldots,\tilde{t_n^i}),c^{i1}+2\tilde{t_1^i}+t_i^1,c^{i2}+\tilde{t_1^i}+t_i^2,\ldots,c^{in}+\tilde{t_1^i}+t_i^n,F^{-i},t^{-i}\right)\\
&\quad\geq\varphi_i\left((T^i,0,\ldots,0),c^{i1}+\tilde{t_1^i}+T^i+t_i^1,c^{i2}+t_i^2,\ldots,c^{in}+t_i^n,F^{-i},t^{-i}\right). \quad (6)
\end{aligned}
$$

Therefore, transferring all the contribution of the profit of agent i to the project with agent 1 does not increase the share of agent i.

By Equations (5) and (6), we have that the payoff of agent i is invariant to the reallocation of profits in the projects that contain agent i.

Therefore, by steps 1 and 2, φ_i depends on the aggregate profit of the projects in which i belongs, as well as others' time allocations and profits.

Step 3: We show that a mechanism is non-decreasing function on the total output of its own projects.

Consider the following production functions:

$$\begin{aligned}
f_{i1}(t_1^i, t_i^1) &= c^{i1} + (\gamma+\delta)t_1^i + \gamma t_i^1; \\
f_{i2}(t_2^i, t_i^2) &= c^{i2} + \gamma(t_2^i + t_i^2); \\
&\vdots \\
f_{i(n-1)}(t_{n-1}^i, t_i^{n-1}) &= c^{i(n-1)} + \gamma(t_{n-1}^i + t_i^{n-1}); \\
f_{in}(t_n^i, t_i^n) &= c^{in} + \gamma\left(t_n^i + t_i^n\right); \\
f_{jk}(t_k^j, t_j^k) &= c^{jk} + \gamma\left(t_k^j + t_j^k\right), \quad \text{where } j, k \neq i.
\end{aligned}$$

where $\gamma < 1$, $\delta > 0$. At optimal profile,

$$\begin{aligned}
&\varphi_i\left(c^{i1} + (\gamma+\delta)T^i + \gamma t_i^1 + c^{i2} + \gamma t_i^2 + \cdots + c^{in} + \gamma t_i^n, F^{-i}, t^{-i}\right) \\
&\qquad \geq \varphi_i\left(c^{i1} + (\gamma+\delta)t^i + \gamma t_i^1 + c^{i2} + \gamma(t_2^i + t_i^2) + \cdots + c^{in} + \gamma(t_n^i + t_i^n), F^{-i}, t^{-i}\right).
\end{aligned}$$

for any arbitrary profile t^i.

As γ goes to 0, we have that

$$\varphi_i\left(\sum_{j\neq i} c^{ij} + \delta T^i, F^{-i}, t^{-i}\right) \geq \varphi_i\left(\sum_{j\neq i} c^{ij} + \delta t_1^i, F^{-i}, t^{-i}\right).$$

Therefore, the step follows immediately since $\{c^{ij}\}_{ij}$ and $t_1^i \geq T^i$ are arbitrary numbers. $\square$

A.2 Proof of Proposition 2

Proof. Property (iii) ⇒ Property (i) is clear because under such a function g, every agent allocates their time resources to achieve the maximum (efficient) level of the total profit in the society. In other words, even if some agents form a coalition, they cannot increase their individual payoff because the aggregate profit does not increase.

We prove that Property (i) ⇒ Property (iii).

Step A1: Show φ_i is independent of time allocations, t^i and t^{-i}.

By Theorem 1, φ is such that

$$\varphi_i(F,t) = \varphi_i\left(\sum_{j\neq i} F^{ij}, \left(F^{lk}\right)_{k,l\neq i}, t^{-i}\right), \quad \forall i.$$

Consider the time-independent production functions:

$$f_{ij}(t_j^i, t_i^j) = c^{ij}, \quad \forall i,j \text{ and arbitrary constants } c^{ij},$$

Suppose that φ_i depends on t^{-i}. Then, one agent, j, can help agent i to receive a higher payoff by changing his time allocation, because the other allocations are constant. This is a violation to the definition of strong efficiency. Therefore, we have

$$\varphi_i(t,F) = \varphi_i\left(\sum_{j\neq i} F^{ij}, \left(F^{lk}\right)_{k,l\neq i}\right), \quad \forall i.$$

Step A2: Show φ_i depends on $\sum_{k,j\neq i} F^{kj}$.

Without loss of generality, we focus in the case where $i=1$. Consider the following production functions: $f_{1j}(t_j^1, t_1^j) = t_j^1$ for any $j \neq 1$ and $f_{kj}(t_j^k, t_k^j) = c^{kj}$ for $k, j \neq 1$ and some constants c^{kj}.

Then, the payoff of agent 1 equals:

$$\begin{aligned}\varphi_1(t,F) &= \varphi_1\left(\sum_{j\neq 1} F^{1j}, F^{23}, F^{24}, F^{25}, \cdots, F^{n-1,n}\right)\\ &= \varphi_1\left(\sum_{j\neq 1} F^{1j}, F^{23}+F^{24}, 0, F^{25}\cdots, F^{n-1,n}\right)\\ &= \cdots\\ &= \varphi_1\left(\sum_{j\neq 1} F^{1j}, \sum_{k,l\neq i} F^{kl}, 0, 0\cdots, 0\right).\end{aligned}$$

where at every step, the equality holds by strong efficiency.

Therefore, the payoff of agent 1 only depends on the sum of its own outputs and the other pairs' outputs. We can apply a similar argument to any other agent. Therefore, we have

$$\varphi_i\left(\sum_{j\neq i} F^{ij}, \left(F^{lk}\right)_{k,l\neq i}\right) = \varphi_i\left(\sum_{j\neq i} F^{ij}, \sum_{k,l\neq i} F^{kl}\right), \quad \forall i.$$

Step A3: Show φ_i depends on $\sum_{ij} F^{ij}, \quad \forall i,j$

Consider the following production function: $f_{ij}(t^i_j, t^j_i) = t^i_j + t^j_i$. By Theorem 1 and strong efficiency, we have

$$\varphi_i\left(\sum_{j\neq i} F^{ij}, \sum_{k,l\neq i} F^{kl}\right) = \varphi_i\left(\sum_{j\neq i} F^{ij} + x, \sum_{k,l\neq i} F^{kl} - x\right)$$

for any $x < \sum_{k,l\neq i} F^{kl}$. Therefore, we have

$$\varphi_i\left(\sum_{j\neq i} F^{ij}, \sum_{k,l\neq i} F^{kl}\right) = \varphi_i\left(\sum_{ij} F^{ij}, 0\right)$$

Property (iii) $\Rightarrow$ Property (ii) is obvious.

We new show that **Property (ii)** $\Rightarrow$ **Property (iii).**

Step B1. φ_i is independent of time.

By Theorem 1, the payoff of agent i, depends only on the time allocations of others, the total profit of the projects in which agent i belongs, and the profits of the projects in which agent i does not belong. That is,

$$\varphi_i(F,t) = \varphi_i\left(\sum_{j\neq i} F^{ij}, \{F^{lj}\}_{(l,j)\in L^{-i}}, t^{-i}\right), \quad \forall i.$$

Now, we show φ_i is independent of the other's time allocations t^{-i}.
Consider the following constant production functions:

$$f_{ij}(t_j^i, t_i^j) = c^{ij}, \quad \forall i, j \text{ and arbitrary constants } \{c^{kl}\}_{(k,l)\in L}$$

Note that under these production functions, any allocation of time is an efficient Nash equilibrium. Suppose that the technology in the project with agent i and j has been improved.

$$\tilde{f}_{ij}(t_j^i, t_i^j) = c^{ij} + \epsilon.$$

Then, by strong monotonicity,

$$\varphi_i\left(c^{ij}+\epsilon, c^{-ij}; \tilde{t}^{-i}\right) \geq \varphi_i\left(c^{ij}, c^{-ij}; t^{-i}\right), \quad \forall i, \quad t^{-i} \quad \text{and} \quad \tilde{t}^{-i}$$

By taking the limit as ϵ tends to zero,

$$\varphi_i\left(c^{ij}, c^{-ij}; \tilde{t}^{-i}\right) \geq \varphi_i\left(c^{ij}, c^{-ij}; t^{-i}\right), \quad \forall i, \quad t^{-i} \quad \text{and} \quad \tilde{t}^{-i}$$

By exchanging the roles of t^{-i} and $\tilde{t}^{-i}$ we have that

$$\varphi_i\left(c^{ij}, c^{-ij}; t^{-i}\right) \geq \varphi_i\left(c^{ij}, c^{-ij}; \tilde{t}^{-i}\right), \quad \forall i, \quad t^{-i} \quad \text{and} \quad \tilde{t}^{-i}$$

Thus,

$$\varphi_i\left(c^{ij}, c^{-ij}; t^{-i}\right) = \varphi_i\left(c^{ij}, c^{-ij}; \tilde{t}^{-i}\right), \quad \forall i, \quad t^{-i} \quad \text{and} \quad \tilde{t}^{-i}$$

Step B2. In this step we show that the payoff of an agent is invariant to transfers of profit from a project in which he does not below to a project in which the agent belongs.

Consider the following set of production functions:

$$\begin{aligned} f_{12}(t_2^1, t_1^2) &= t_2^1 + t_1^2; \\ f_{23}(t_3^2, t_2^3) &= t_3^2 + t_2^3; \\ f_{lm}(t_m^l, t_l^m) &= c^{lm}, \quad \forall (l,m) \neq (1,2),(2,3) \end{aligned}$$

Note that under efficiency, agent 2 sends any distribution of his time to the projects with agents 1 and 3, while agents 1 and 3 invest their time in the projects with agent 2.

Further, consider the following production functions:

$$\begin{aligned} \tilde{f}_{12}(t_2^1, t_1^2) &= t_2^1 + (1+\epsilon) t_1^2; \\ \tilde{f}_{23}(t_3^2, t_2^3) &= t_3^2 + t_2^3; \\ \tilde{f}_{lm}(t_m^l, t_l^m) &= c^{lm}, \quad \forall (l,m) \neq (1,2),(2,3) \end{aligned}$$

We look at the payoff of agent 3. Consider the case when agent 2 transfers the profit from the projects in which agent 3 belongs, to the projects that agent 3 does not belong to. That is, agent 2 transfers some amount t^2 from project 23 to project 12. Formally, by strong monotonicity and efficiency an any $t^2 < T^2$,

$$\varphi_3(F^{13}+F^{23}+t^2+\sum_{i>3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) \leq \varphi_3(F^{13}+F^{23}+\sum_{i>3} F^{3i}, F^{12}+(1+\epsilon)t^2, \{F^{ij}\}_{(i,j)\in L^3\setminus(1,2)}).$$

At the limit, when ϵ tends to 0, we have that

$$\varphi_3(F^{13}+F^{23}+t^2+\sum_{i>3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) \leq \varphi_3(F^{13}+F^{23}+\sum_{i>3} F^{3i}, F^{12}+t^2, \{F^{ij}\}_{(i,j)\in L^3\setminus(1,2)}).$$

Alternatively, consider the production functions.

$$\begin{aligned}\tilde{f}_{12}(t_2^1, t_1^2) &= t_2^1 + t_1^2;\\ \tilde{f}_{23}(t_3^2, t_2^3) &= t_3^2 + (1+\epsilon)t_2^3;\\ \tilde{f}_{lm}(t_m^l, t_l^m) &= c^{lm}, \quad \forall (l,m) \neq (1,2),(2,3)\end{aligned}$$

By repeating the above argument we have that

$$\varphi_3(F^{13}+F^{23}+t^2+\sum_{i>3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) \geq \varphi_3(F^{13}+F^{23}+\sum_{i>3} F^{3i}, F^{12}+t^2, \{F^{ij}\}_{(i,j)\in L^3\setminus(1,2)}).$$

The two inequalities above lead to

$$\varphi_3(F^{13}+F^{23}+t^2+\sum_{i>3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) = \varphi_3(F^{13}+F^{23}+\sum_{i>3} F^{3i}, F^{12}+t^2, \{F^{ij}\}_{(i,j)\in L^{-3}\setminus(1,2)}).$$

In particular,

$$\varphi_3(\sum_{i\neq 3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) = \varphi_3(F^{12} + \sum_{i\neq 3} F^{3i}, 0, \{F^{ij}\}_{(i,j)\in L^3\setminus(1,2)}).$$

By repeating the argument for every project that does not contain 3, we have

$$\varphi_3(\sum_{i\neq 3} F^{3i}, \{F^{ij}\}_{(i,j)\in L^3}) = \varphi_3(\sum_{(i,j)\in L^{-3}} F^{ij} + \sum_{i\neq 3} F^{3i}, 0, 0, \ldots, 0) = \varphi_3(\sum_{(i,j)\in L} F^{ij}, 0, 0, \ldots, 0)$$

$\square$

A.3 Proof of Proposition 3

Proof. **Step 1.** The first step decomposes the solution of a give problem A into the sum of subproblems where only one bilateral project has positive contribution in profit and time.

Let $Z = (0,0,0)$. We simplify notation by denoting the concatenation vector $\times_{k\in A} Z =$

$(Z, Z, \ldots, Z)$ simply as Z_A.

By applying additivity repeatedly, where at every step we take an agent different than i and apply additivity to each problem containing that agent, we can decompose the problem A as follows:

$$\begin{aligned}
\varphi_i(A) &= \varphi_i\left((t_1^i, t_i^1, F^{i1}), (t_2^i, t_i^2, F^{i2}), \ldots, (t_{i-1}^i, t_i^{i-1}, F^{i,i-1}), (t_{i+1}^i, t_i^{i+1}, F^{i,i+1}), \ldots, (t_n^i, t_i^n, F^{in}), Z_{L^{-i}}\right) \\
&\quad + \varphi_i\left(\left\{(t_k^l, t_l^k, F^{kl})\right\}_{(k,l)\in L^{-i}}, Z_{L^i}\right) \\
&= \varphi_i\left((t_1^i, t_i^1, F^{i1}), Z_{L^i\setminus(i,1)}, Z_{L^{-i}}\right) \\
&\quad + \varphi_i\left((0,0,0), (t_2^i, t_i^2, F^{i2}), \ldots, (t_{i-1}^i, t_i^{i-1}, F^{i,i-1}), (t_{i+1}^i, t_i^{i+1}, F^{i,i+1}), \ldots, (t_n^i, t_i^n, F^{in}), Z_{(k,l)\in L^{-i}}\right) \\
&\quad \ldots \\
&\quad + \varphi_i\left((t_n^i, t_i^n, F^{in}), Z_{L\setminus(i,n)}\right) \\
&\quad + \varphi_i\left(Z_{(i,n)}, \left\{(t_m^l, t_l^m, F^{lm})\right\}_{L^i\setminus(i,n)}, Z_{L^{-i}}\right) \\
&\quad + \varphi_i\left(\left\{(t_k^l, t_l^k, F^{kl})\right\}_{(k,l)\in L^{-i}}, Z_{L^i}\right) \\
&= \ldots = \\
&= \sum_{k\neq i} \varphi_i\left(\left\{\left(t_k^i, t_i^k, F^{ik}\right), Z_{L\setminus(i,k)}\right\}\right) + \sum_{k,l\neq i} \varphi_i\left(\left\{\left(t_k^l, t_l^k, F^{kl}\right), Z_{L\setminus(k,l)}\right\}\right)
\end{aligned}$$

Step 2: Since φ is efficient, by Theorem 1, φ_i does not depend on $(t_j^i, t_i^j), \quad \forall j \neq i$. Hence,

$$\varphi_i(A) = \sum_{k\neq i} \varphi_i\left(\left\{\left(0,0,F^{ik}\right), Z_{L\setminus(i,k)}\right\}\right) + \sum_{k,l\neq i} \varphi_i\left(\left\{\left(t_k^l, t_l^k, F^{kl}\right), Z_{L\setminus(k,l)}\right\}\right).$$

First, we focus on the first term. First note that by budget balance of the mechanism,

$$\varphi_i\left(\left\{(0,0,0), Z_{(l,m)\neq(i,k)}\right\}\right) = 0. \tag{7}$$

By efficiency and Theorem 1,

$$\varphi_i\left(\left\{\left(0,0,\sum_{k\neq i} F^{ik}\right), Z_{L\setminus(i,k)}\right\}\right) = \sum_{k\neq i} \varphi_i\left(\left\{\left(0,0,F^{ik}\right), Z_{(l,m)\in L\setminus(i,k)}\right\}\right).$$

Hence, since this happens at any level of production and for any agent, then

$$\varphi_i\left(\left\{(0,0,X), Z_{L\setminus(i,\tilde{k})}\right\}\right) = \varphi_i\left(\left\{(0,0,X), Z_{L\setminus(i,k)}\right\}\right).$$

Moreover, by Theorem 1,

$$\begin{aligned}\varphi_i\left(\left\{(0,0,X), Z_{L\setminus(i,k)}\right\}\right) &= \varphi_i\left(\left\{(0,0,X-t), Z_{L\setminus(i,\tilde{k})}\right\}\right) \\ &\quad +\varphi_i\left(\left\{(0,0,t), Z_{L\setminus(i,k)}\right\}\right) \\ &= \varphi_i\left(\left\{(0,0,X-t), Z_{L\setminus(i,k)}\right\}\right) \\ &\quad +\varphi_i\left(\left\{(0,0,t), Z_{L\setminus(i,k)}\right\}\right).\end{aligned}$$

Hence, φ_i is additive in F^{ik} for any $k \neq i$. Since φ_i is continuous and equation (7) holds, then φ_i is linear in F^{ik}. Thus,$\exists s^i \in [0,1]$ such that

$$\varphi_i\left(\left\{(0,0,X), Z_{L\setminus(i,j)}\right\}\right) = s^i X, \quad \forall X \text{ and all } (i,j) \in L^i.$$

Since φ is non-negative, then $s^l + s^m \leq 1, \quad \forall (l,m) \geq 0$.

Finally, consider the function $g^{i,j} : \mathbb{R}^3 \to \mathbb{R}^{N\setminus 2}$ given by

$$g_k^{ij}(t_j^i, t_j^i, X) = \frac{1}{(1-s^i-s^j)}\varphi_k((t_j^i, t_j^i, X), Z_{L\setminus(i,j)})$$

Then, we have that

$$\sum_{k\neq i,j} g_k^{ij}\left(t_j^i, t_j^i, X\right) = \frac{1}{(1-s^i-s^j)}\sum_{k\neq i,j}\varphi_k((t_j^i, t_j^i, X), Z_{L\setminus(i,j)}) =$$

$$= \frac{X - \varphi_i((t_j^i, t_j^i, X), Z_{L\setminus(i,j)}) - \varphi_j((t_j^i, t_j^i, X), Z_{L\setminus(i,j)})}{(1-s^i-s^j)} = \frac{X - s^i X - s^j X}{(1-s^i-s^j)} = X$$

$$\sum_{k\neq i,j} g_k^{ij}\left(t_j^i, t_j^i, F^{ij}\right) = F^{ij}, \quad \forall F^{ij} \geq 0,$$

Therefore,

$$\varphi_i\left(A\right) = s^i \sum_{j\neq i} F^{ij} + \sum_{(l,m)\in L^{-i}} g_i^{lm}\left(t_m^l, t_l^m, (1-s^l-s^m)F^{lm}\right) \quad \forall i.$$

□

A.4 Proof of Corollary 2

Proof. The relations (i) $\iff$ (ii) and (iii) $\Rightarrow$ (i) are obvious by Theorem 1.

We show that (i) $\Rightarrow$ (iii).

By Proposition 3, we have

$$\varphi_i\left(A\right) = s^i \sum_{j\neq i} F^{ij} + \sum_{l,m\neq i} g_i^{lm}\left(t_m^l, t_l^m, (1-s^l-s^m)F^{lm}\right), \quad \forall i.$$

By Proposition 3, the function is invariant to the transfer of profit within two projects (l, m) and (j, k) in which agent i does not belong. Therefore, g_{lm} and g_{jk} are linear. Thus,

$$\varphi_i\left(A\right) = s^i \sum_{j\neq i} F^{ij} + \gamma^i \sum_{l,m\neq i} F^{lm}, \quad \text{and } \gamma^i \in \mathbb{R}.$$

Similarly, by Proposition 3, the function is invariant to the transfer of profit within a project in which i does not belong to a project in which i belongs. Therefore,

$$s^i = \gamma^i, \quad \forall i,$$

where $\sum s^i = 1$ due to the fact that $\gamma^i = (1 - \sum_{j\neq i} s^j)$. □

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