Profit-Sharing and Efficient Time Allocation *

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Abstract

Agents are endowed with time, which in turn is invested in projects that generate profit. A mechanism divides the profit generated by these agents depending on the allocation of time as well as the amount of profit made by every project.

We study mechanisms that incentivize agents to contribute their time to a level that results in the maximal aggregate profit at the Nash equilibrium, regardless of the production functions involved (*efficiency*). Our main finding involves the characterization of all mechanisms that satisfy efficiency. Furthermore, within this class, we characterize the mechanisms that are monotone on the addition of time to agents as well as those monotone on the payoffs of the agents with respect to technological improvements in the generation of profit.

The class of efficient mechanisms depends on the type of available projects and their connectedness. It expands earlier profit-sharing mechanisms that are independent of profit generation.

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1 Introduction

Profit-sharing mechanisms are used widely by companies in order to increase profits. Such mechanisms include direct cash bonuses to employees, who are awarded based on their performance, either individually or collectively. For instance, under an employee stock ownership plan, a company will allocate shares of its stock, hence rewarding its employees based on aggregate profit.[1]

We study profit-sharing mechanisms where a planner (such as the owner, board or manager of the company) is interested in maximizing profits while the agents (employees) are interested in maximizing their own payoff. The design of an effective profit-sharing mechanism by a planner requires the alignment of his interests with the payoffs of the agents. Such an alignment has been widely explored in the well-established mechanism design literature under different constraints of information. On the one hand, issues of information concerning the preferences of the agents have been widely studied. For instance, the traditional VCG mechanisms (Vickrey (1961); Clarke (1971); Groves (1973)) are efficient and incentive-compatible mechanisms when the planner does not have information about the preferences of the agents. On the other hand, issues of information concerning the actions taken by agents have been widely studied in the contract theory literature (see Holmstrom (1982) and related literature below). Our work studies a third informational issue, in which agents are more aware of their collective abilities to generate profits than the planner. This information asymmetry happens, for instance, in large companies, where the owner or board of the company sets general profit-sharing policies for employees before the actual production functions are realized (e.g., when hiring the agents). This also happens in dynamic settings where the ability to generate profits might change on a regular basis, and the agents are aware of this change although the planner is not.

While traditional models of profit-sharing limit their scope to a single project/task that generates profit (e.g., Holmstrom (1982)), we study a setting in which multiple projects that generate profits are available. Such profits can be generated by collaborative projects in which multiple agents participate, and the determination of the profits depends on the level of engagement (e.g., the time invested) of the agents participating in the projects. These collaborative settings are common in companies where agents are given the flexibility to choose the projects in which they can work, from law and consulting firms to academia.

[1] Employee stock ownership plans are spread widely throughout Silicon Valley start-ups and have created several millionaire employees, for instance at Google, Facebook, and Yahoo.

To formalize this, we consider a model in which agents decide how to allocate their fixed endowment of time to different projects. Every project can have different production functions that generate profits, depending on the allocation of time by agents. We focus on the case of asymmetric information, whereby a planner does not know any production functions while employees have perfect information in this regard. A mechanism (designed by the planner) assigns payments depending only on the final profit generated by each project and the time allocation of agents to different projects.[2]

The central issue is that, although the planner tries to maximize the total profit of the company, he might not know anything about the production functions. In other words, the planner's goal is to select a payment scheme that incentivizes agents to allocate their time efficiently at a Nash equilibrium for any set of production functions. We call this property *efficiency*. Thus, even though the planner is disadvantaged through a lack of information with respect to the agents, efficiency leads to the first-best outcome for the planner—as if he had full information from the outset.

Consider the proportional sharing mechanism that divides the total profit of each project among agents in proportion to their time allocation for such projects. This mechanism is not necessarily efficient because agents could be incentivized to put more time into projects that provide them with a greater proportion of time, in order to increase their share of the profits of such projects, whereas the company might produce a greater profit when agents invest their time collectively into a single project. To illustrate this notion, consider the following example. There are three agents, named 1, 2, and 3, and three collaborative projects between these agents, namely 12, 13, and 123. Suppose that each agent is endowed with one unit of time that they split between the projects to which they belong, and every project generates profit based on the time allocated by the agents. For instance, suppose that the production functions of such projects are perfect substitutes $2.5(t_1^{12} + t_2^{12})$ for the project 12, perfect complements $3(min(t_1^{13}, t_3^{13}))$ for the project 13, and also perfect complements $6(min(t_1^{123}, t_2^{123}, t_3^{123}))$ for the project 123. The efficient time allocation $((t_1^{12}, t_1^{13}, t_1^{123}), (t_2^{12}, t_2^{123}), (t_3^{13}, t_3^{123})) = ((0, 0, 1), (0, 1), (0, 1))$ requires all agents to invest their time into the project 123 and generates a profit equal to 6 units. Under proportional sharing, each agent receives 2 units of the profit. However, this is not in equilibrium under the proportional sharing mechanism since agents 1 and 2 both have the incentive

[2]In contrast with the traditional literature in contract theory, e.g., Holmstrom (1982), the planner observes the actions of the agents (their time invested at each project) as well as the profit generated by each project, and this information may be used to determine their share of the profit. We assume that agents' time is costless, or alternatively, they are given a fix amount of hours that they need to work (e.g., they work forty hours per week), therefore, the traditional moral hazard issues that often occur in contract theory do not appear in our setting.

to allocate all resources to the project 12, in which case they will each receive 2.5 units of profit instead of 2.

In contrast, consider the average profit sharing mechanism, whereby each agent receives a fixed share of the total profit generated by the company.[3] This mechanism is efficient because, if an agent deviates from the equilibrium that generates the maximum total profit (efficient equilibrium), the total profit of the company will not increase and neither will his payment.

Alternatively, consider the equal sharing mechanism, whereby the profit of every project is distributed equally among agents belonging to each project, irrespective of their time allocation. When the set of feasible projects is of the same size, for instance, if all the projects have size 2, the equal sharing mechanism is also efficient because, if an agent deviates from the efficient equilibrium, the total profit of the company will not increase. Thus, since the profit of the company is the sum of the profit of each project, the aggregated profit of the projects in which this agent participates does not increase, and neither does his payment; however, in general, the equal sharing mechanism is not efficient when projects are different in size.[4]

More generally, consider a mechanism where the payoff of an agent only depends positively on the total (aggregate) profit generated by the projects in which the agent participates, the profits generated by the projects in which the agent does not participate, and the time allocations of other agents. We call these mechanisms *separable*, and they are efficient because if an agent decides to change his time contributions from the efficient allocation, the total output of his projects does not increase. Furthermore, such a change will not affect the profits of the projects in which the agent does not belong nor the time contributions of other agents. Thus, the efficient allocation is a Nash equilibrium at a separable mechanism. The average profit and equal sharing mechanisms are particular cases of separable mechanisms.

Our main finding is the characterization of all the mechanisms that satisfy efficiency and continuity.[5] The class of efficient and continuous mechanisms coincides exactly with the class of separable mechanisms (Theorem 1). The class of efficient, continuous, anonymous, and time-independent mechanisms is narrow and can be described as the linear combinations of equal sharing and average-profit mechanisms, where the weight as-

[3]The mechanism can be interpreted as the stock awarding mechanism, through which agents are given a fixed share of stocks in the company, and thus their final allocation of profit depends on the aggregate profit generated.

[4]This can be seen in the example above, where project sizes 2 and 3 exist. Herein, agents 1 and 2 have the incentive to deviate from the efficient equilibrium.

[5]*Continuity* with respect to time allocations and profits generated.

signed to the mechanisms depends on the size of available projects that can form (Theorem 2).

We also look at the monotonicity of the payments of agents with respect to increases in their time (*time-monotonicity*) and with respect to increases in the production functions (*technology-monotonicity*).[6] The class of efficient and continuous mechanisms that are robust to these monotonicity properties depends on the connectedness of projects. Agents who are connected (directly or indirectly via the non-empty pairwise intersection of projects to which they belong) should receive a payment that is monotonic in the aggregate profits generated by all connected projects regardless of whether or not the agents belong to such projects (Theorem 3).

1.1 Related Literature

The early work in the contract theory literature identified that when "payoff alone is observable, optimal contracts will be second-best" (Holmstrom et al. (1979)) for individual and team production profit-sharing schemes (Holmstrom (1982)). Therefore, mechanisms that use more information are needed in order to obtain first-best mechanisms. While first-best mechanisms could be achieved in specific settings with limited observability about the individual (Rayo (2007)) or collective performance (Deb et al. (2016)), our model assumes that the actions taken by agents are fully observable and traditional issues of moral hazard do not occur. Instead, our contribution to the contract theory literature comes from studying the case where the planner has no information about the production functions that generate profits. In this setting, our work studies the case where there are multiple projects that generate profit rather than a single one.

The axiomatic and non-axiomatic study of efficient profit-sharing rules in settings without moral hazard has been recently studied. For instance, Juarez et al. (2018) study efficient revenue-sharing rules when the profits are realized in a sequence rather than simultaneously. It characterizes the average profit mechanism and other asymmetric variations that are independent of individual generation and only dependent on the total profit generated. Hougaard et al. (2017) study revenue-sharing when agents are not symmetric, but instead, are ordered in a a hierarchical structure. In Example 4 we discuss the rules characterized by Hougaard et al. (2017) and show that the efficiency of such rules depends on the type of available projects and their interconnectedness.

[6]In particular, time-monotonicity requires that increases in the allocated time of agents should not make them worse off. On the other hand, technology-monotonicity requires that improvements in the technology of projects (represented as increases in the production function at any time allocation) should not make agents worse off, either.

Our profit-sharing model has a dual interpretation as a cost-sharing model. This interpretation is achieved by defining a cost-function as the negative of a profit function and a cost-sharing mechanism as the negative of a profit-sharing mechanism. In this dual interpretation, agents contribute time to different tasks that collectively generate a cost, a cost-sharing mechanism divides this cost among agents, agents are interested in minimizing their cost-share, and the planner is interested in minimizing the total cost. Any profit-sharing mechanism discussed hereafter has a dual cost-sharing mechanism and vice versa.

Within the cost-sharing literature, the implementation of the cost-minimizing (efficient) allocation as a Nash equilibrium has been recently explored in related cost-sharing settings and some of the mechanisms characterized in this cost-sharing literature can be extended to our model. For instance, Juarez and Kumar (2013) study a model where agents select paths in a network, which in turn generate some cost that must be divided among the agents. Juarez and Kumar (2013) characterize the average cost mechanism (the dual of the average profit mechanism) and asymmetric variations as the mechanisms that (Pareto-)Nash implement the cost-minimizing network. Hougaard and Tvede (2015) study a cost-sharing model where the cost-sharing rule reported by the planner is only an estimate and may be changed by the planner depending on realized information. Hougaard and Tvede (2015) characterize asymmetric variations of the average cost mechanism where the division of the cost is shared in fixed proportions. The narrow classes of mechanisms characterized by Juarez and Kumar (2013) and Hougaard and Tvede (2015) can be captured by the mechanisms characterized in Theorem 3 and resemble efficient mechanisms when the grand coalition is a feasible project.

To our knowledge, this is the first paper to introduce the implementation of the efficient time allocation for any set of production functions.

2 The Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents in an economy. Groups of agents from N can collaborate in projects that generate profit depending on the time allocation of the agents for each project.[7] Formally, let $\mathcal{P} \subset 2^N \setminus \{\emptyset\}$ be the set of different projects. For instance, if $\mathcal{P}$ contains all subsets of two agents, then every group of two agents can collaborate in a project. If $\mathcal{P} = 2^N \setminus \{\emptyset\}$, then any potential coalition can collaborate in a project. Let $\mathcal{P}_i$ be the set of groups from $\mathcal{P}$ that contains agent i, and let $\mathcal{P}_{-i}$ be the set of groups from $\mathcal{P}$

[7]For simplicity, we assume that there is no repetition in the projects, although a similar argument can be made when projects repeat.

that do not contain agent i. For the rest of the paper, we fix the agents N and the group of projects $\mathcal{P}$.

Every agent i is endowed with T_i units of time, which he can split between the projects in which he participates. We will denote by T the vector $(T_1, T_2, \ldots, T_n)$. Let t_i^P be the time that agent i spends on project P. The set of time allocations of agent i is the set $D_i^{T_i} = \{t_i \in \mathbb{R}_+^{\mathcal{P}_i} \mid \sum_{P \in \mathcal{P}_i} t_i^P = T_i\}$. Let $\mathbb{D}^T = \Pi_{i \in N} D_i^{T_i}$ be the set of time allocation profiles. In this section, we fix the time endowments $T = (T_1, T_2, \ldots, T_n)$. Section 3 will look at the possibility of changes with respect to time endowments T.

Every project generates profit. We assume that regardless of how agents allocate (or mis-allocate) their time, no project can have a negative profit. Let $\mathbb{F} = \mathbb{R}_+^{\mathcal{P}}$ be the set of vectors of profits for all projects. For a given $F \in \mathbb{F}$, the amount F^K is the profit generated by project $K \in \mathcal{P}$.

Definition 1 (Mechanisms)**.** *Consider a vector of times $T = (T_1, T_2, \ldots, T_n)$. A **mechanism at** T is a function $\varphi^T : \mathbb{D}^T \times \mathbb{F} \to \mathbb{R}_+^n$ such that*

$$\sum_{i=1}^{n} \varphi_i^T(t, F) = \sum_{K \in \mathcal{P}} F^K.$$

When there is no confusion, we write φ^T simply as φ.

A mechanism's inputs are the different time allocations and profits for every project. The output is a full distribution of the total profit given to agents. Despite the agents having full information about the production functions, the planner does not elicit this information. We have several motivations for this approach. From the practical perspective, not obtaining this information from agents imposes fewer communication requirements between the planner and agents. From the applied side, most profit-sharing mechanisms currently used, such as the stock-ownership plan, elicit minimal information from agents and instead rely on realized information (i.e., the profits generated by agents and their time spent in different projects) as well as agents' characteristics (e.g., seniority, type of work, etc.). Our results below will validate our approach since, surprisingly, gathering these reports would be unnecessary paperwork as the planner can achieve the first-best without them.

In contrast with traditional mechanisms in quasi-linear domains, such as the VCG mechanisms (Vickrey (1961); Clarke (1971); Groves (1973)), we restrict our mechanisms to be budget-balanced. In other words, we assume that the full (or a fixed fraction of the) profit generated is divided among the agents.[8] The case when the planner divides

[8] Allocating all profit to agents is assumed without loss of generality. This can be extended easily to the

the full profit among the agents can be interpreted as a joint ownership (e.g., law-firms or a joint business venture). We note that this assumption is widely used in the cost-sharing literature and other body of literature for the division of a fixed divisible resource (such as a dollar) depending on the agents' reports (de Clippel et al. (2008); Tideman and Plassmann (2008); Mackenzie (2015)). In contrast with this literature, our work does not assume that the resource is fixed, but instead the amount to share is determined by the time contributions of the agents.

Example 1. *A.* ***Average profit mechanism****: The final profit of the entire company is divided equally among all members. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \frac{1}{n} \sum_{K \in \mathcal{P}} F^K.$$

B. ***Equal sharing mechanism****: The final profit produced by project $K \in \mathcal{P}$ is shared equally among agents in K. The total share belonging to every agent is the sum of his shares in the projects. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \sum_{K \in \mathcal{P}_i} \frac{F^K}{|K|},$$

where $|K|$ is the number of agents in project K.

C. ***Proportional sharing mechanism****: The final profit produced by project K is shared in proportion to the contribution of time made by the agents in K. That is, for any $i \in N$,*

$$\varphi_i(t, F) = \sum_{K \in \mathcal{P}_i} Pr_i^K(t^K) F^K, \text{ where } Pr_i^K(t^K) = \begin{cases} \frac{t_i^K}{\sum_{j \in K} t_j^K} & \text{if} \quad \sum_{j \in K} t_j^K > 0 \\ 0 & \text{if} \quad \sum_{j \in K} t_j^K = 0 \end{cases}$$

D. ***Generalized equal sharing Mechanism****: Every agent's share is a fixed proportion of the profit generated by the projects in which he participates and the unallocated profit of the projects in which he does not participate. Formally, consider any collection of projects $\mathcal{P} \subset 2^N \setminus \{\emptyset, N\}$ and positive individual shares $\gamma_1, \gamma_2 \dots, \gamma_n$ such that for any project $K \in \mathcal{P}$, the aggregate shares in this project do not exceed 1, $\sum_{j \in K} \gamma_j \leq 1$. For any individual agent i,*

case where agents receive a fixed percentage of the profit, as is the case for yoghurt company Chobani, which ensures a fixed 10% share of the profit is transmitted to its employees.

$$\varphi_i(t,F) = \gamma_i \sum_{K\in\mathcal{P}_i} F^K + \sum_{M\in\mathcal{P}_{-i}} \frac{\gamma_i}{\sum_{l\in N\setminus M}\gamma_l}\left(1-\sum_{j\in M}\gamma_j\right) F^M.$$

E. ***Altruistic Mechanism***: *The profit generated by each project is equally divided among the agents outside the project. For any individual agent i,*

$$\varphi_i(t,F) = \sum_{B\in\mathcal{P}_{-i}} \frac{F^B}{|N\setminus B|}$$

Note that average profit, as well as the equal sharing, generalized equal sharing and altruistic mechanisms, are independent of time allocation. On the other hand, the proportional sharing mechanism is dependent on time allocation. Any convex combination of mechanisms is also a mechanism.

2.1 Efficiency and other desirable properties

In order to formalize our game, let $f = (f^K)_{K\in\mathcal{P}}$ be the vector of production functions, where $f^K : \mathbb{R}_+^K \to \mathbb{R}_+$ is a continuous and non-decreasing function in each variable. Let $\mathcal{F}$ be the set of production function vectors.

We study the perfect information non-cooperative game, where each agent is fully informed of the production functions, and they should allocate their time resource between different projects in which they participate based on these production functions. Formally, the strategy of agent i is a function from $\mathcal{F}$ to $D_i^{T_i}$ that assigns a time allocation to every production function vector. Let $\mathcal{S}_i$ be the set of functions from $\mathcal{F}$ to $D_i^{T_i}$. The payoff of an agent depends on his own and others' time allocation, production functions and the outcome of the mechanism.

Definition 2 (Non-cooperative Game $G^{\varphi,f}$)**.** *Given a mechanism φ and production function vector $f\in\mathcal{F}$, we study the* ***non-cooperative game*** $G^{\varphi,f} = [N, (\mathcal{S}_1,\dots,\mathcal{S}_n), (\pi_1^{\varphi,f},\dots,\pi_n^{\varphi,f})]$ *where*

- *the* ***strategy space*** *of agent i is $\mathcal{S}_i$;*
- *the* ***payoff function*** *of agent i at the vector of strategies (S_i, S_{-i}) is*

$$\pi_i^{\varphi,f}(S_i,S_{-i}) = \varphi_i\left((t_i,t_{-i}), \left[f^K(t^K)\right]_{K\in\mathcal{P}}\right), \quad \text{where } t_j = S_j(f)\ \forall j\in N.$$

Under a production function vector, a profile of strategies generates outputs for different projects. We say a strategy is efficient if it produces a larger output for any set of production functions. We define an efficient mechanism as one that supports the efficient strategy as a Nash equilibrium.

Definition 3 (Nash Equilibrium and Efficiency).

- *A strategy profile* $(S_1^*, S_2^*, \ldots, S_n^*)$ *is a* ***Nash equilibrium*** *of the game* $G^{\varphi,f}$ *if*

$$\pi_i^{\varphi,f}(S_i^*, S_{-i}^*) \geq \pi_i^{\varphi,f}(S_i, S_{-i}^*), \quad \forall i \in N, S_i \in \mathcal{S}_i.$$

- *A* ***profile of strategies*** $(S_1, S_2, \ldots, S_n)$ ***is efficient*** *in the game* $G^{\varphi,f}$ *if for any other strategy* $\tilde{S}$,

$$\sum_{K \in \mathcal{P}} f^K(t^K) \geq \sum_{K \in \mathcal{P}} f^K(\tilde{t}^K), \quad \textit{where } t_i = S_i(f) \textit{ and } \tilde{t}_i = \tilde{S}_i(f).$$

- *A* ***mechanism is efficient*** *if the game* $G^{\varphi,f}$ *has at least one Nash equilibrium formed by a profile of efficient strategies.*

In contrast with stronger notions of efficiency where all Nash equilibria are required to be efficient, we consider a weaker notion where *at least one* Nash equilibrium is efficient. To see that under our definition there could exist some inefficient Nash equilibrium profiles, consider three agents $N = \{1, 2, 3\}$, four projects $\mathcal{P} = \{\{1\}, \{2\}, \{3\}, N\}$, $T = (8, 8, 8)$ and $f^{\{k\}}(t_k^{\{k\}}) = t_k^{\{k\}}$, $k = 1, 2, 3$, $f^N(t_1^N, t_2^N, t_3^N) = (3 + \epsilon)\min\{t_1^N, t_2^N, t_3^N\}$. Under the equal-sharing mechanism, the efficient Nash equilibrium corresponds to the case where each agent devotes the 8 units to project N. Nevertheless if each agent i devotes the 8 units to project $\{i\}$ is another Nash equilibrium, but it is not efficient. In our model, there is no mechanism such that all Nash equilibria are efficient. Therefore, our weaker notion of efficiency is necessary.

As discussed in the introduction, the average profit mechanism is efficient, while the equal sharing, generalized equal sharing and altruistic mechanisms are efficient only for narrow sets of available projects. This will be formalized in Section 2.2.

Another basic requirement for a mechanism is continuity, which states that in every problem, small changes on time allocations or profit generation should have a small impact on the redistribution of profits. Such small changes might happen, for instance, due to measurement errors, and we require for our mechanism to be robust with respect to these errors.

Definition 4. *A mechanism* φ *is* ***continuous*** *if the function* $\varphi : \mathbb{D}^T \times \mathbb{F} \to \mathbb{R}_+^n$ *is continuous.*

All the mechanisms studied in this paper are continuous.

2.2 Separable mechanism and the main result: Implementing efficient time allocation

In this section, we characterize a number of efficient and continuous mechanisms. There are several restrictions that efficiency imposes on a mechanism, the first of which is that the payoff of an agent should depend on the aggregate profit generated by the projects in which he participates, instead of the profits of individual projects. The second restriction is that the time allocation of an agent should not influence his payoff (though it might influence the payoff of other agents). The separable mechanisms discussed below include these two restrictions.

Definition 5 (Separability)**.** *A **mechanism** φ **is separable** if there exist functions $(g_1, g_2, \ldots, g_n)$, which are non-decreasing in the first coordinate such that*

$$\varphi_i(t, F) = g_i\left(\sum_{K\in\mathcal{P}_i} F^K, \left(F^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) \quad \forall i \in N.$$

A mechanism is separable if the payoff of agent i depends only on the total aggregated profit generated by his projects, $\sum_{K\in\mathcal{P}_i} F^K$, as well as the profits of projects that do not contain agent i, $(F^B)_{B\in\mathcal{P}_{-i}}$, and the others agents' time allocations t_{-i}. The class of separable mechanisms is large, as seen in the following examples.

Example 2. *A. The average profit mechanism is a separable mechanism generated by the functions*

$$g_i^{AP}\left(\sum_{k\in\mathcal{P}_i} F^K, \left(F^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) = \frac{\sum_{H\in\mathcal{P}} F^H}{n}, \quad \forall i \in N.$$

B. Equal sharing is a separable mechanism only when the set of projects $\mathcal{P}$ contains coalitions of the same size c. In this case,

$$g_i^{ES}\left(\sum_{K\in\mathcal{P}_i} F^K, \left(F^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) = \frac{1}{c}\sum_{K\in\mathcal{P}_i} F^K, \quad \forall i \in N.$$

C. The generalized equal sharing mechanism is a separable mechanism generated by the functions

$$g_i^{GES}\left(\sum_{K\in\mathcal{P}_i} F^K, \left(F^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) = \gamma_i \sum_{K\in\mathcal{P}_i} F^K + \sum_{M\in\mathcal{P}_{-i}} \frac{\gamma_i}{\sum_{l\in N\setminus M}\gamma_l}\left(1 - \sum_{j\in M}\gamma_j\right) F^M, \forall i \in N.$$

D. The altruistic mechanism is a separable mechanism generated by the functions

$$g_i^{AL}\left(\sum_{K\in\mathcal{P}_i}F^K,\left(F^B\right)_{B\in\mathcal{P}_{-i}},t_{-i}\right)=\sum_{B\in\mathcal{P}_{-i}}\frac{F^B}{|N\setminus B|},\quad \forall i\in N.$$

Note that the convex combination of separable mechanisms is also a separable mechanism generated by the convex combination of the g functions. The proportional sharing mechanism is not separable, because the payoff of an agent depends on his allocation of time to different projects.

Theorem 1. *A mechanism is efficient and continuous if—and only if—it is separable.*

The proof is in Appendix A.

This result shows that a planner interested in implementing the efficient outcome has a fairly large flexibility when choosing an efficient and continuous mechanism. Indeed, the share of an agent may depend on the total profit of projects in which the agent belongs, as well as those individual profits in which the agent does not belong. While his share cannot depend on his own time allocation, it can depend on others' time allocations. This fairly large flexibility is remarkable as our efficiency requirement is very robust, since it works for each conceivable production function. Further studies that restrict the knowledge of the planner to particular types of production functions could provide an even larger class of efficient and continuous mechanisms.

The structure of available projects also plays an important role in the determination of the class of efficient and continuous mechanisms. The payoff of a central agent i who belongs to all available projects (i.e., when $\mathcal{P}_i=\mathcal{P}$) should only depend on the combined profit of all projects, and the time allocations t_{-i} of the agents different than i. On the other hand, the payoff of an isolated agent j who belongs to a single project may depend on all variables of the mechanism except his own time allocation —this includes the total profits generated by all projects F and the time allocations t_{-j} of the agents different than j.

In light of the large class of efficient and continuous mechanisms discovered in Theorem 1, we now provide a restriction of the efficient and continuous mechanisms under two additional conditions.

Definition 6. *A mechanism φ is* ***time-independent*** *if for all $t,\widetilde{t}\in\mathbb{D}^T$,*

$$\varphi(t,F)=\varphi(\widetilde{t},F),\ \ \forall F\in\mathbb{F}.$$

A time-independent mechanism only depends on the profit generated by the different projects and not on the time allocations. Time-independence is desirable for companies that do not have the ability to monitor the time contributions of the agents to each project. However, they can only observe the aggregate time of each agent to all projects. Even if companies have the ability to observe the times allocated to each project, it may not be desirable to use them in the absence of information about the production functions.[9]

Another property that we consider is anonymity. In order to define it, we now introduce notation needed for this definition. Let c be an integer such that $0 < c < n$, the set $\mathcal{P}^c = \{S \subset N \mid |S| = c\}$ denotes the set of all projects of size c. Moreover, let $C \subseteq \{1, \ldots, n-1\}$ be a set of cardinalities. The set $\mathcal{P}^C = \bigcup_{c \in C} \mathcal{P}^c$ denotes the set of all projects with sizes in C.

Let $S \in \mathcal{P}^C$ and let $i, j \in N$. We define the permuted project as:

$$S_{(ij)} = S_{(ji)} = \begin{cases} S \setminus \{i\} \cup \{j\}, & \text{if } i \in S \\ S \setminus \{j\} \cup \{i\}, & \text{if } j \in S \\ S, & \text{if } i, j \notin S \end{cases}$$

Note that for every project $S \in \mathcal{P}^C$ the permuted project $S_{(ij)}$ satisfies $S_{(ij)} \in \mathcal{P}^C$. Now, consider the profit vector $F \in \mathbb{F}$ and time contribution vector $t \in \mathbb{D}$ and construct the permuted vectors, $F_{(ij)}$ and $t_{(ij)}$ as follows:

$$(F_{(ij)})^S = \begin{cases} F^{S_{(ij)}}, & \text{if } i \in S \text{ or } j \in S \\ F^S, & \text{if } i, j \notin S \end{cases}$$

and,

$$(t_{(ij)})^S_k = \begin{cases} t_j^{S_{(ij)}}, & \text{if } k = i \\ t_i^{S_{(ij)}}, & \text{if } k = j \\ t_k^{S_{(ij)}}, & \text{if } k \neq i \text{ and } k \neq j \end{cases}$$

Definition 7. *Let $\mathcal{P}^C$ be the set of projects. A mechanism φ is **anonymous** if for all $i, j \in N$, then*

$$\varphi_i(t, F) = \varphi_j(t_{(ij)}, F_{(ij)}).$$

A mechanism is **anonymous** if it is independent of the name of the agents.

[9] Indeed, consider a planner whose ideal solution is to distribute the profit of each project in proportion to agents' individual marginal contributions to such project. As any combinations of production functions are possible and the planner is unaware of these, the time allocation is a bad proxy of the marginal contributions of agents. Furthermore, using the times as such proxy can only lead to unfair outcomes when agents are highly diverse in their abilities.

We characterize below the class of efficient, continuous, anonymous, and time-independent mechanisms.

Theorem 2.
- *A mechanism is efficient, continuous, anonymous, and time-independent in $\mathcal{P}^c$ if—and only if—it is a convex combination of the equal sharing and altruistic mechanisms.*
- *A mechanism φ is efficient, continuous, anonymous, and time-independent in $\mathcal{P}^C$ if—and only if—φ is a generalized equal sharing mechanism with the same weight for every agent. In other words, there exists $0 \leq \alpha \leq min_{K\in\mathcal{P}}\frac{1}{|K|}$, such that*

$$\varphi_i(F_i, F_{-i}) = \alpha \sum_{S\in\mathcal{P}_i^C} F^S + \sum_{R\in\mathcal{P}_{-i}^C} \frac{1-|R|\alpha}{n-|R|} F^R.$$

- *Consider $\mathcal{P}$ such that $\mathcal{P} = \mathcal{P}^C \cup \{N\}$ for some $C \subseteq \{1,\ldots,n-1\}$. A mechanism is efficient, continuous, anonymous, and time-independent if—and only if—it is the average profit mechanism.*

The proof is in Appendix A. Note that the available projects severely narrow down the class of efficient, continuous, anonymous, and time-independent mechanisms, with the average profit mechanism belonging to each of these characterizations.[10]

3 Monotonic Mechanisms

So far, our model has fixed the time endowments $T = (T_1, \ldots, T_n)$. Section 2 shows the class of efficient and continuous mechanisms. In this section, we focus on the robustness of these efficient and continuous mechanisms with respect to changes in the time endowments T and the vector of production functions f. Given the vector of times T, we denote through $Eff^{\varphi,f}[T]$ the set of efficient Nash equilibria under φ.

Our first robustness check is with respect to time. Increases in time allocation often occur in companies and typically benefit agents who receive this increase, though they may harm those agents whose time does not change. Our next property requires that such an increase in time should not harm agents in an efficient equilibrium, regardless of whether or not they receive an increase in their time allocation.

[10]Average profit mechanism is clearly a convex combination of the equal sharing mechanism and the altruistic mechanism, and it is also generated by the generalized equal sharing mechanism when $\alpha = \frac{1}{n}$.

Definition 8 (Efficient and continuous extension). *Let φ be an efficient and continuous mechanism at T. For any agent i and for any time $\tilde{T}_i > T_i$, an **efficient and continuous extension of φ over** $[T_i, \tilde{T}_i]$ is a function*

$$\Phi : \bigcup_{\tau \in [T_i, \tilde{T}_i]} \mathbb{D}^{(\tau, T_{-i})} \times \mathbb{F} \to \mathbb{R}^n_+,$$

such that:

- *The restriction of Φ to $\mathbb{D}^\tau \times \mathbb{F}$, denoted $\Phi|_{\mathbb{D}^\tau \times \mathbb{F}}$, is an efficient and continuous mechanism at (τ, T_{-i}) for all $\tau \in [T_i, \tilde{T}_i]$.*
- *$\Phi|_{\mathbb{D}^T \times \mathbb{F}} = \varphi$.*
- *Φ is a continuous function.*

When there is no confusion, we write $\Phi|_{\mathbb{D}^\tau \times \mathbb{F}}$ simply as Φ^τ.

Definition 9 (Time-monotonicity). *An efficient mechanism φ is **time-monotonic at** T if for any agent i and for any time $\tilde{T}_i > T_i$, there is an efficient and continuous extension Φ of φ over $[T_i, \tilde{T}_i]$ such that for any $\tau \in [T_i, \tilde{T}_i]$ and any two efficient equilibria $S^* \in Eff^{\varphi,f}[T]$ and $\tilde{S} \in Eff^{\Phi^\tau,f}[(\tau, T_{-i})]$ then,*

$$\varphi(S^*(f), f(S^*(f))) \leq \Phi^\tau(\tilde{S}(f), f(\tilde{S}(f))), \quad \forall f.$$

The second robustness check is with respect to changes in production functions. We say that the vector of production functions $\tilde{f}$ is a **technological improvement** of f if for any project $K \in \mathcal{P}$ and time allocation t^K we have that $\tilde{f}^K(t^K) \geq f^K(t^K)$. Following a technological improvement, the same level of output (profit) requires no more input (time) than prior to the improvement.

Technological improvements can be observed in companies, especially those redistributing on a regular basis (e.g., the distribution of any end-of-year surplus to employees). Such improvements might harm agents, though, particularly those who do not belong to an improving project. The next property requires that such improvements not harm any agent at equilibrium, regardless of whether or not they participate in the project that has shown improvements.[11]

Definition 10 (Technology-monotonicity). *An efficient mechanism φ is **technology-monotonic at** T if for any production functions f and any technological improvement $\tilde{f}$ of f and any efficient equilibrium $S^* \in Eff^{\varphi,f}[T] \cap Eff^{\varphi,\tilde{f}}[T]$, then $\varphi(S^*(f), f(S^*(f)) \leq \varphi(S^*(\tilde{f}), \tilde{f}(S^*(\tilde{f}))$.*

[11] Since we require this property at equilibrium, it is typically stronger than other monotonicity properties studied in the literature, such as *strict resource monotonicity* in allocation problems (Thomson (1994, 1995, 2003, 2011); Moulin and Thomson (1988)).

The class of efficient and continuous mechanisms that are robust to either of the above changes is large and dependent on a set of available projects that are directly or indirectly connected. In particular, we say that two projects H and K are **connected** in $\mathcal{P}$, denoted as $H \sim K$, if $H = K$ or there is a sequence of projects in $\mathcal{P}$, $(P_0, P_1, ..., P_m)$, such that $H = P_0$, $K = P_m$, and $P_{r-1} \cap P_r \neq \emptyset$ for all $r = 1, ..., m$. That is, two projects are connected if we can find a sequence of feasible projects where every two consecutive projects have a non-empty intersection of agents.

It is easy to check that *connectedness* is an *equivalence relation*, in which case the **equivalence class** of P under $\sim$ is defined as $[P] := \{H \in \mathcal{P} | P \sim H\}$. Moreover, there is a unique partition of $\mathcal{P}$, which groups projects together if—and only if—they are connected. We will denote this partition by

$$\mathcal{P}/_{\sim} := \{[P] | P \in \mathcal{P}\}.$$

Definition 11 (Equivalence-Class Mechanism)**.** *The separable mechanism φ is an* ***equivalence-class mechanism*** *if there exist functions $g_i : \mathbb{R}_+^{\mathcal{P}/\sim} \to \mathbb{R}_+ \quad$ for $i = 1, 2, \ldots, n$, such that*

$$\sum_{i \in N} g_i(A) = \sum_{[P] \in \mathcal{P}/_{\sim}} A_{[P]}, \quad \text{for all } A \in \mathbb{R}_+^{\mathcal{P}/\sim}$$

and for all t and F,

$$\varphi(t, F) = (g_1(A), g_2(A), \ldots, g_n(A)),$$

where

$$A_{[P]} = \sum_{H \in [P]} F^H.$$

Note that all equivalence-class mechanisms are time-independent. When $\mathcal{P}$ contains the grand coalition N, all the projects are connected. Thus, these equivalence-class mechanisms allocate payments to agents only depending on the aggregate profit generated by all projects. There is, however, a large class of equivalence-class mechanisms that are highly dependent on the set of available projects, as illustrated in the examples below.

Example 3. *Suppose that the set of available projects is*

$$\mathcal{P} = \{\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{5, 6\}, \{6, 7, 8\}\},$$

in which case there are two equivalence classes, namely $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8\}$. Thus, the equivalence-class mechanisms are such that the allocation to agents depends solely on the values $F^{12} + F^{123}$ and $F^{45} + F^{56} + F^{678}$.

For instance, one plausible mechanism is such that the allocation to agents in $\{1,2,3\}$ can be in proportion to the number of projects in which they participate:

$$\varphi_{[1,2,3]}(t,F) = \left(\frac{2(F^{12}+F^{123})}{5}, \frac{2(F^{12}+F^{123})}{5}, \frac{F^{12}+F^{123}}{5}\right),$$

and the allocation to agents $\{4,5,6,7,8\}$ can be an equal-sharing division of the profits generated by their projects:

$$\varphi_i(t,F) = \frac{F^{45}+F^{56}+F^{678}}{5} \quad \textit{for } i \in \{4,5,6,7,8\}.$$

When the set of feasible projects is increased to include the project $\{3,4\}$, i.e.,

$$\bar{\mathcal{P}} = \{\{1,2\},\{1,2,3\},\{4,5\},\{5,6\},\{6,7,8\},\{3,4\}\},$$

all of the projects are connected. Thus, the only plausible equivalence-class mechanisms allocate a share to the agents, depending solely on total profit $F^{12}+F^{123}+F^{45}+F^{56}+F^{678}+F^{34}$.

Let $A_{[i]}$ be the profit generated by the component that contains agent i. Since the equivalence-class mechanisms are separable, the function $g_i(A_{[i]}, A_{-[i]})$ is non-decreasing in $A_{[i]}$. In general, agent i might be negatively affected once an element in $A_{-[i]}$ increases; monotonicity prevents this from happening.

The equivalence-class mechanism is **monotonic** if no agent becomes worse off once either equivalence class increases its profit. Formally, for the equivalence-class mechanism generated by g, monotonicity requires that $g(A) \leq g(\bar{A})$ for any $A, \bar{A} \in \mathbb{R}_+^{\mathcal{P}/\sim}$, such that $A \leq \bar{A}$.

Example 4. *[Sharing the Proceeds of a Hierarchical Venture]*

Hougaard et al. (2017) study the sharing of profits when agents are not symmetric but instead have a hierarchy (e.g., a linear order of subordinates or a more general hierarchy represented by a tree). This paper does not deal with the question on how profits are generated; instead, it assumes that they are already given.

The geometric rule φ^λ for agents with hierarchy $n \succ n-1 \succ \cdots \succ 1$, profits $(F^1, F^2, \ldots, F^n)$, and parameter $\lambda \in [0,1]$ assigns:

$$\varphi_i^\lambda = \lambda(F^i + (1-\lambda)F^{i-1} + \cdots + (1-\lambda)^{i-1}F^1) \textit{ for } i = 1, \ldots n-1, \textit{ and}$$

$$\varphi_n^\lambda = F^n + (1-\lambda)F^{n-1} + \cdots + (1-\lambda)^{n-1}F^1$$

We show that depending on how the profits are generated, the class of geometric rules characterized by Hougaard et al. (2017) may be an equivalence-class monotonic mechanism. Indeed, when profits are generated independently by every agent, i.e., $\mathcal{P} = \{\{1\}, \ldots, \{n\}\}$, the equivalence classes equal $\mathcal{P}$, and for any λ, the geometric rule φ^λ is an equivalence-class monotonic mechanism.

However, the geometric rule φ^λ may not be an equivalence-class monotonic mechanism when the profits are generated collaboratively. For instance, consider the set of projects

$$\mathcal{P} = \{\{1, 2, \ldots, n\}, \{2, 3, \ldots, n\}, \ldots, \{n\}\},$$

where agent n belongs to n projects, agent $n-1$ belongs to $n-1$ projects, etc. Since all of the projects are connected by agent n, the grand coalition is the unique equivalence class, and the payment of every agent should depend only on total profit $F^1 + F^2 + \cdots + F^n$. Thus, the only geometric mechanism that is an equivalence-class mechanism occurs when $\lambda = 1$, i.e. when agent n obtains all the profit.

The final result characterizes the class of time-monotonic and technology-monotonic mechanisms within the class of efficient and continuous mechanisms.

Theorem 3. *Consider the vector of times T. The following three properties are equivalent for the efficient and continuous mechanism φ:*

(i) φ *is time-monotonic at* T*,*

(ii) φ *is technology-monotonic at* T*,*

(iii) φ *is an equivalence-class mechanism that is monotonic.*

The proof of this result is in Appendix A. One consequence of this result is that technology-monotonicity and time-monotonicity are equivalent within the class of efficient and continuous mechanisms. The class of mechanisms characterized is large, as shown in Examples 3 and 4, it is widely dependent on the connectedness of projects, and is time-independent.

It is readily seen that if symmetry is further imposed in Theorem 3, then the class of mechanisms depends on whether or not P contains projects of cardinality greater than or equal to two. Indeed, if $P = \{\{1\}, \ldots, \{n\}\}$ then by Theorem 2 the only mechanisms are convex combinations of the equal sharing and altruistic mechanisms. On the other hand, if P contains all coalitions of cardinality l, for some $l \geq 2$, then all projects are connected and the only symmetric mechanism is the average profit mechanism.

4 Conclusion

We have introduced and characterized a large class of mechanisms that implement the efficient time allocation in profit-sharing problems. This class expands earlier results characterizing narrow classes of mechanisms that only share profits disregarding the individual contributions of agents. Furthermore, unlike earlier studies, our characterization shows how the generation of the profit changes the class of efficient and continuous mechanism, and such generation should be taken into account by a planner interested in aligning his incentives with those of agents participating in the mechanism.

Our characterization allowed us to uncover several novel mechanisms that are efficient. For instance, consider the convex combination of the equal sharing mechanism and the altruistic mechanism: it divides a fixed share of the profit of a project in equal parts among the agents who generated the profit and the remaining share in equal parts among the agents who did not generate the profit. This mechanism is efficient only when the projects have the same size. Another novel mechanism includes the generalized equal sharing mechanism, where agents are given weights (e.g., determined by their rank in the firm), which are used to determine their share of the profit at every project in which they belong, and any unallocated profit in a project is divided among the agents who did not belong to this project. The generalized equal sharing mechanism is always efficient regardless of the set of available projects. Interestingly, both mechanisms share the profit of a project with agents who did not generate it.

Several questions about the characteristics of our model remain open. Our work studies a general setting where the planner is uninformed about the production functions, and thus, every production function is possible. However, more work needs to be done to understand the robustness of our results when restricting to specific classes of production functions (e.g., by restricting the domain of functions to perfect-complements or perfect-substitutes). Another direction is to characterize the mechanisms and classes of production functions where all Nash equilibria are efficient.

Our model focuses on the case where agents' time is costless, and therefore, traditional issues of moral hazard do not occur. The extensions of our characterizations to the case where agents' time is costly remain open questions.

While our work focuses mainly on and is motivated by incentives, further studies may incorporate the fairness aspect of the problem. Monotonicity (on time and technology) properties narrowed the class of efficient and continuous mechanisms, but we left intact the study of more traditional fairness axioms, such as those used widely in the bankruptcy and surplus-sharing literature (Thomson (2003); Moulin (2002)).

Most of the profit/cost-sharing literature studies static problems, and little has been discussed about the axiomatic study of dynamic sharing problems, especially regarding resource generation, the arrival/exiting of agents, and redistribution. An exception, Juarez et al. (2018), started such a study by focusing on the axiomatic division of finite and sequential benefits in companies, and then characterizing a narrow set of sharing rules that allow agents to select the most efficient path when the planner has incomplete information about the profits being distributed. However, this does not address how the generation of the resource (which is exogenously given in such a model) affects the sharing rule.

Finally, we assumed that agents behave selfishly, and the Nash equilibrium is a good predictor of this behavior. The theoretical study of profit/cost-sharing under other-regarding preferences, as well as alternative equilibrium notions, remains widely unexplored.

Appendix A. Proofs

Proof of Theorem 1

Proof. First, we show that if a mechanism is separable, it is efficient and continuous.

Consider a separable mechanism and suppose that an agent, say agent i, deviates from an efficient strategy. This deviation by agent i does not increase the total profit of the projects to which he belongs (by efficiency), and it does not affect projects to which he does not belong. Thus, by separability of the mechanism, the payoff of agent i cannot increase from an efficient strategy.

Next, we show that if a mechanism is efficient and continuous, it is separable for any set of production functions.

Step 1: We show that if a mechanism is efficient and continuous, then the payoff of agent i does not depend on his time allocation. In other words,

$$\varphi_i\left(t_i, t_{-i}, F\right) = \varphi_i\left(t_{-i}, F\right),$$

for simplicity of notation, where t_i is the strategy of agent i and $t_{-i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ is the collection of all agents' strategies, except agent i.

First, consider $\bar{K} \in \mathcal{P}_i$ and define the production functions as follows:

$$\begin{aligned} f^K(t) &= c^K, \quad \text{for } K \neq \bar{K}; \\ f^{\bar{K}}(t) &= c^{\bar{K}} + \epsilon\left(t_i^{\bar{K}}\right); \end{aligned}$$

where $c^K \in \mathbb{R}_+$ is an arbitrary constant for each $K \in \mathcal{P}$.

Following the definition of an efficient strategy, agent i allocates his full resources to project $\bar{K}$, i.e. $t_i = E_i^{\bar{K}} \in D_i^{T_i}$, where $E_{i,K}^{\bar{K}} = T_i$ if $K = \bar{K}$, and $E_{i,K}^{\bar{K}} = 0$ if $K \in \mathcal{P}_i \setminus \{\bar{K}\}$.

Then, for all $\tilde{t}_i \in D_i^{T_i}$, we find that

$$\varphi_i\left(E_i^{\bar{K}}, t_{-i}, f(E_i^{\bar{K}}, t_{-i})\right) \geq \varphi_i\left(\tilde{t}_i, t_{-i}, f(\tilde{t}_i, t_{-i})\right).$$

Let $c = [c^K]_{K\in\mathcal{P}}$. As ϵ goes to 0, $f(t) \to c$. Therefore, as a result of the continuity of φ, we find that

$$\varphi_i\left(E_i^{\bar{K}}, t_{-i}, c\right) \geq \varphi_i(\tilde{t}_i, t_{-i}, c), \quad \text{for any } \tilde{t}_i \in D_i^{T_i}. \tag{1}$$

Alternatively, fix $\tilde{t}_i \in D_i$ and consider the production functions as follows:

$$\begin{aligned} \tilde{f}^K(t^K) &= c^K + \epsilon\left(\min\{\tilde{t}_i^K, t_i^K\}\right), \quad \text{for } K \in \mathcal{P}_i; \\ \tilde{f}^K(t^K) &= c^K, \quad \text{for } K \in \mathcal{P}_{-i}. \end{aligned}$$

Note that the efficient profile requires agent i to report $\tilde{t}_i$, whereas the time allocation of the rest of the agents is irrelevant. Since φ is efficient, we find that

$$\varphi_i\left(\tilde{t}_i, t_{-i}, \tilde{f}\left(\tilde{t}_i, t_{-i}\right)\right) \geq \varphi_i\left(E_i^{\bar{K}}, t_{-i}, \tilde{f}\left(E_i^{\bar{K}}, t_{-i}\right)\right).$$

As ϵ goes to 0, $f(t) \to c$. Therefore, as a result of the continuity of φ, we find that

$$\varphi_i\left(\tilde{t}_i, t_{-i}, c\right) \geq \varphi_i\left(E_i^{\bar{K}}, t_{-i}, c\right). \tag{2}$$

Hence, through the inequalities (1) and (2),

$$\varphi_i\left(\tilde{t}_i, t_{-i}, c\right) = \varphi_i\left(E_i^{\bar{K}}, t_{-i}, c\right).$$

Thus, the payoff of agent i is independent of his time allocation. Similarly, the payoffs of the other agents are also independent of their own time allocation.

Step 2: We show that agent i's share depends on the sum of the profits of the projects to which he belongs.

First, consider $\bar{K} \in \mathcal{P}_i$ and define the production functions for any $\beta \in \mathbb{R}_+$ as follows:

$$\begin{aligned} f^{\bar{K}}(t) &= c^{\bar{K}} + \beta t_i^{\bar{K}} + \gamma \sum_{j \in \bar{K} \setminus \{i\}} \beta t_j^{\bar{K}} \\ f^{K}(t) &= c^{K} + \gamma \left(\sum_{j \in K} \beta t_j^{K} \right), \quad \text{for any } K \in \mathcal{P} \setminus \{\bar{K}\} \end{aligned}$$

where $\gamma < 1$ and $c^K \in \mathbb{R}_+$ is an arbitrary constant for each $K \in \mathcal{P}$.

In terms of efficiency, for all $(\tilde{t}_i, t_{-i}) \in \mathbb{D}^T$, we find that

$$\varphi_i \left(E_i^{\bar{K}}, t_{-i}, f\left(E_i^{\bar{K}}, t_{-i} \right) \right) \geq \varphi_i \left(\tilde{t}_i, t_{-i}, f\left(\tilde{t}_i, t_{-i} \right) \right).$$

Therefore, as γ goes to 1,

$$f\left(E_i^{\bar{K}}, t_{-i} \right) \to \left(c^{\bar{K}} + \beta T_i + \sum_{j \in \bar{K} \setminus \{i\}} \beta t_j^{\bar{K}}, \left[c^K + \sum_{j \in K \setminus \{i\}} \beta t_j^K \right]_{K \in \mathcal{P}_i \setminus \bar{K}}, \left[c^K + \sum_{j \in K} \beta t_j^K \right]_{K \in \mathcal{P} \setminus \mathcal{P}_i} \right) = F^*,$$

and

$$f\left(\tilde{t}_i, t_{-i} \right) \to \left(\left[c^K + \beta \tilde{t}_i^K + \sum_{j \in K \setminus \{i\}} \beta t_j^K \right]_{K \in \mathcal{P}_i}, \left[c^K + \sum_{j \in K} \beta t_j^K \right]_{K \in \mathcal{P}_{-i}} \right) = G^*.$$

Thus, as a result of the continuity of φ_i,

$$\varphi_i \left(E_i^{\bar{K}}, t_{-i}, F^* \right) \geq \varphi_i \left(\tilde{t}_i, t_{-i}, G^* \right). \tag{3}$$

Therefore, transferring all of i's time to the project $\bar{K}$ does not decrease the share of agent i.

Alternatively, for a given $\tilde{t}_i \in D_i^{T_i}$, consider the following production functions:

$$\begin{aligned} \tilde{f}^K(t) &= c^K + \gamma \min\left\{ \tilde{t}_i^K, t_i^K \right\} + \sum_{j \in K} \beta t_j^K, \quad \text{for any } K \in \mathcal{P}_i; \\ \tilde{f}^K(t) &= c^K + \left(\sum_{j \in K} \beta t_i^K \right) \text{ where } K \in \mathcal{P}_{-i}. \end{aligned}$$

For $0 < \gamma < 1$, the optimal profile requires $t_i = \tilde{t}_i$. Comparing this with the suboptimal profile $E_i^{\bar{K}}$, and making γ converge to zero, we find that

$$\varphi_i\left(E_i^{\bar{K}}, t_{-i}, F^*\right) \leq \varphi_i\left(\tilde{t}_i, t_{-i}, G^*\right). \tag{4}$$

Therefore, by inequalities 3 and 4,

$$\varphi_i\left(E_i^{\bar{K}}, t_{-i}, F^*\right) = \varphi_i\left(\tilde{t}_i, t_{-i}, G^*\right) \text{ for all } (\tilde{t}_i, t_{-i}) \in \mathbb{D}^T$$

Note that F^* and G^* coincide in the projects $\mathcal{P}_{-i}$, and generate the same aggregate profit within $\mathcal{P}_i$. Indeed, G^* is an arbitrary transfer of βT_i units of profit from project $\bar{K}$ to projects in $\mathcal{P}_i \setminus \{\bar{K}\}$. Thus, since $\{c^K\}_{K\in\mathcal{P}}$ and β are arbitrary non-negative constants, the payoff for agent i is invariant to the reallocation of profits in the projects in which agent i is involved.

Hence, through Steps 1 and 2, φ_i depends on the aggregate profit of the projects to which i belongs, as well as others' time allocations and profits only. When there is no confusion, we denote the payment $\varphi_i(t, F)$ of agent i simply as $\varphi_i\left(\sum_{K\in\mathcal{P}_i} F^K, (F^B)_{B\in\mathcal{P}_{-i}}, t_{-i}\right)$.

Step 3: We show that a mechanism is a non-decreasing function on the total output of its own projects.

For any $\bar{K} \in \mathcal{P}$, such that $i \in \bar{K}$, consider the following production functions:

$$\begin{aligned} \tilde{f}^{\bar{K}}(t) &= c^{\bar{K}} + \delta t_i^{\bar{K}}; \\ \tilde{f}^{K}(t) &= c^{K}, \quad \text{for any } K \in \mathcal{P} \setminus \{\bar{K}\}. \end{aligned}$$

where $\delta > 0$ and $c^K \in \mathbb{R}_+$ is an arbitrary constant for each $K \in \mathcal{P}$.

Next, at the efficient profile, agent i contributes his full time allocation to the project $\bar{K}$, whereas the time allocation of the rest of the agents is irrelevant. Therefore, for any arbitrary profile t:

$$\varphi_i\left((c^{\bar{K}} + \delta T_i, c^{-\bar{K}}), \left(c^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) \geq \varphi_i\left((c^{\bar{K}} + \delta t_i^{\bar{K}}, c^{-\bar{K}}), \left(c^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right).$$

Thus, by step 2,

$$\varphi_i\left(\sum_{K\in\mathcal{P}_i} c^K + \delta T_i, \left(c^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right) \geq \varphi_i\left(\sum_{K\in\mathcal{P}_i} c^K + \delta t_i^{\bar{K}}, \left(c^B\right)_{B\in\mathcal{P}_{-i}}, t_{-i}\right).$$

Since $\delta > 0$, $\{c^K\}_{K\in\mathcal{P}_i}$ and $t_i^{\bar{K}} \leq T_i$ are arbitrary numbers, agent i's payoff is non-decreasing on the total output of his own projects. □

Proof of Theorem 2

Proof. Clearly, the second part of the theorem implies the first part. We prove in Steps 1 and 2 the second part. Step 3 proves the third part.

Consider the mechanism φ, which is efficient, continuous, anonymous, and time-independent.

Step 1: We show that φ is linear.

Note that according to Theorem 1, anonymity, and time independence, there exists a function g, such that φ_i can be written as

$$\varphi_i(F) = g\left(\sum_{K\in\mathcal{P}_i^C} F^K, (F^B)_{B\in\mathcal{P}_{-i}^C}\right) \tag{5}$$

for all $i \in N$ and $F \in \mathbb{F}$.

It is important to remark that anonymizing the output of $g\left(\sum_{K\in\mathcal{P}_i^C} F^K, (F^B)_{B\in\mathcal{P}_{-i}^C}\right)$ is the same as for any project profit permutation with equal cardinality in $\mathcal{P}_{-i}^C$.

In fact, if $\bar{M} = (M, M, ..., M)$ for any $M \in \mathbb{R}$, by anonymity and budget balance

$$\varphi_i(\bar{M}) = g(|\mathcal{P}_i^C|M, M, ..., M) = \frac{|\mathcal{P}^C|}{|N|}M, \quad \forall i \in N. \tag{6}$$

We proceed to show that g is linear. Let F be an arbitrary vector of profits in $\mathcal{P}^C$.

Without loss of generality, there are projects $R\cup\{1\}$ and $R\cup\{2\}$ in $\mathcal{P}^C$, where $|R|+1 \in T$ and $1 \notin R$, $2 \notin R$. We denote $\tilde{F}$ as the vector, such that

$$\tilde{F}^K = \begin{cases} F^{R\cup\{1\}}, & \text{if } K = R\cup\{2\} \\ F^{R\cup\{2\}}, & \text{if } K = R\cup\{1\} \\ F^K, & \text{otherwise.} \end{cases}$$

Note that $\sum_{K\in\mathcal{P}^C} F^K = \sum_{K\in\mathcal{P}^C} \tilde{F}^K$. Moreover, by applying anonymity, we have $\varphi_i(F) = \varphi_i(\tilde{F})$ for all $i \in N \setminus \{1,2\}$. Therefore,

$$\begin{aligned} \varphi_1(F) + \varphi_2(F) &= \sum_{K\in\mathcal{P}^C} F^K - \sum_{i\in N\setminus\{1,2\}} \varphi_i(F) \\ &= \sum_{K\in\mathcal{P}^C} F^K - \sum_{i\in N\setminus\{1,2\}} \varphi_i(\tilde{F}) \\ &= \varphi_1(\tilde{F}) + \varphi_2(\tilde{F}) \end{aligned}$$

by budget balance, in which case $\varphi_1(F) - \varphi_1(\tilde{F}) = \varphi_2(\tilde{F}) - \varphi_2(F)$.

Hence, if

$$A = \sum_{K \in \mathcal{P}_1^C} F^K, \quad x = F^{R\cup\{2\}}, \quad z_{-1} = (F^B)_{B \in \mathcal{P}_{-1}^C \setminus R\cup\{2\}},$$
$$B = \sum_{K \in \mathcal{P}_2^C} F^K, \quad y = F^{R\cup\{1\}}, \quad z_{-2} = (F^B)_{B \in \mathcal{P}_{-2}^C \setminus R\cup\{1\}}.$$

by (5) we have

$$g(A, x, z_{-1}) - g(A + x - y, y, z_{-1}) = g(B + y - x, x, z_{-2}) - g(B, y, z_{-2}).$$

Thus, for $\alpha = x - y$ and $C = B - \alpha$ we have

$$g(A, x, z_{-1}) - g(A + \alpha, x - \alpha, z_{-1}) = g(C, x, z_{-2}) - g(C + \alpha, x - \alpha, z_{-2}).$$

Particularly, observe that it is possible to give a vector F such that this expression hold for any $A, C, \alpha \in \mathbb{R}_+$ and $z = z_{-1} = z_{-2}$ in such a way that

$$g(A, x, z) - g(A + \alpha, x - \alpha, z) = g(C, x, z) - g(C + \alpha, x - \alpha, z).$$

This identity means that the change in the output of g when we transfer an amount of a project to the first entry depends purely on the amount transferred and the cardinality of the project and it does not depend of the amount in the first entry.

Thus, for each cardinality b, we define the function

$$h_b(\alpha) := g(A, x, z) - g(A + \alpha, x - \alpha, z)$$

that measures the change in the output of g when any $\alpha \in \mathbb{R}_+$ is transferred from any project with cardinality b to the first entry.

Note that $h_b(\alpha)$ is an additive function due to

$$\begin{aligned} h_b(\alpha) &= g(A, x, z) - g(A + \alpha, x - \alpha, z) \\ h_b(\beta) &= g(A + \alpha, x - \alpha, z) - g(A + \alpha + \beta, x - \alpha - \beta, z), \end{aligned}$$

and then

$$\begin{aligned} h_b(\alpha) + h_b(\beta) &= g(A, x, z) - g(A + \alpha + \beta, x - \alpha - \beta, z) \\ &= h_b(\alpha + \beta). \end{aligned}$$

Since g is continuous, $h_b : \mathbb{R} \to \mathbb{R}$ is continuous and additive, then there is $\lambda_b \in \mathbb{R}$ such that, $h_b(\alpha) = \lambda_b \alpha$.

Now, we list all projects in $\mathcal{P}^C_{-i}$ by $(B_1, B_2, \ldots, B_r)$ with cardinalities $(b_1, b_2, \ldots, b_r)$ respectively. If $\bar{F} = \frac{1}{|\mathcal{P}^C|} \sum_{K \in \mathcal{P}^C} F^K$,

$$
\begin{aligned}
h_{b_j}(F^{B_j} - \bar{F}) &= g\left(\sum_{K \in \mathcal{P}^C_i} F^K + \sum_{p=1}^{j-1} (F^{B_p} - \bar{F}), \bar{F}, \cdots, \bar{F}, F^{B_j}, \cdots, F^{B_r} \right) \\
&- g\left(\sum_{K \in \mathcal{P}^C_i} F^C + \sum_{p=1}^{j} (F^{B_p} - \bar{F}), \bar{F}, \bar{F}, \cdots, \bar{F}, F^{B_j+1}, \cdots, F^{B_r} \right)
\end{aligned}
$$

Therefore, we have

$$
\sum_{j=1}^{r} h_{B_j}(F^{B_j} - \bar{F}) = g\left(\sum_{K \in \mathcal{P}^C_i} F^K, F^{B_1}, \cdots, F^{B_r} \right) - g\left(|\mathcal{P}^C_i| \bar{F}, \bar{F}, \cdots, \bar{F} \right).
$$

Then, by applying 6, we have

$$
g\left(\sum_{K \in \mathcal{P}^C_i} F^K, F^{B_1}, \cdots, F^{B_r} \right) = \sum_{j=1}^{r} \lambda_{B_j}(F^{B_j} - \bar{F}) + \frac{|\mathcal{P}^C|}{|N|} \bar{F},
$$

and we conclude that g is linear, following which φ is linear.

Step 2: We show that

$$
\varphi_i(F) = \alpha \sum_{S \in \mathcal{P}^C_i} F^S + \sum_{R \in \mathcal{P}^C_{-i}} \frac{1 - |R|\alpha}{n - |R|} F^R.
$$

As illustrated, g and φ are linear, and so

$$
\varphi_i(F) = \alpha \sum_{S \in \mathcal{P}^C_i} F^S + \sum_{R \in \mathcal{P}^C_{-i}} \beta_R F^R.
$$

Let E_P be the vector of profits, so that $E^K_P = 1$ if $K = P$ and 0 otherwise. By anonymity $\varphi_i(E_P) = \alpha$ for all $i \in P$ and $\varphi_j(E_P) = \beta_P$ for all $j \notin P$.

Now, with the budget balance property, we have

$$
|P|\alpha + (n - |P|)\beta_P = 1.
$$

Next,

$$\beta_P = \frac{1 - |P|\alpha}{n - |P|} \quad \text{for any } P \in \mathcal{P}^C.$$

Therefore, the step follows immediately.

Step 3: We show that if $\mathcal{P} = \mathcal{P}^C \cup \{N\}$, then φ is the average profit mechanism.

If $\mathcal{P}^C = \emptyset$, then N is the only feasible project in $\mathcal{P}$. By time-independence, anonymity and budget balance

$$\varphi\left(F^N\right) = \frac{1}{|N|} F^N.$$

On the other hand, if $C \neq \emptyset$, there are projects $R \cup \{1\}$ and $R \cup \{2\}$ in $\mathcal{P}^C$, where $1 \notin R$, $2 \notin R$ and we can proceed as in step 2 to obtain

$$\varphi_i(F) = \alpha \sum_{S \in \mathcal{P}_i^C} F^S + \sum_{R \in \mathcal{P}_{-i}^C} \frac{1 - |R|\alpha}{n - |R|} F^R,$$

for some α.

Through anonymity, $\varphi_i(E_N) = \alpha$ for all $i \in N$. Thus, $\alpha|N| = 1$ by budget balance and then $\alpha = 1/|N|$ and

$$\varphi_i(F) = \frac{1}{|N|} \sum_{K \in \mathcal{P}^C} F^K.$$

□

Proof of Theorem 3

Proof. **Property (iii)** ⇒ **Property (i)** is obvious. We show in steps A1-A3 that **Property (i)** ⇒ **Property (iii).**

Step A1. φ_i is independent of time.

By applying Theorem 1, the payoff φ_i of agent i depends only on the time allocations of others. We now demonstrate that φ_i is independent of the others' time allocations t_{-i}. Consider the following constant production functions:

$$f^K(t^K) = \alpha^K, \quad \forall K \in \mathcal{P}.$$

Note that under these production functions, any allocation of time is an efficient Nash equilibrium. Suppose that the time for some agent $j \neq i$ increases from T_j to $\tilde{T}_j = T_j + \epsilon$ for $\epsilon > 0$.

Now, for any project K in which agent j belongs to, time-monotonicity implies that there is an efficient and continuous extension Φ of φ over $[T_j, T_j + \epsilon]$ such that

$$\Phi_i\left(\alpha; \bar{t}_{-\{i,j\}}, \bar{t}_j^K + \epsilon, \left(\bar{t}_j^P\right)_{P \in \mathcal{P}_j \setminus \{K\}}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \bar{t}_{-i}.$$

By taking the limit as ϵ tends to zero,

$$\varphi_i\left(\alpha; \bar{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \bar{t}_{-i}.$$

By exchanging the roles of t_{-i} and $\bar{t}_{-i}$, we find that

$$\varphi_i\left(\alpha; t_{-i}\right) \geq \varphi_i\left(\alpha; \bar{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \bar{t}_{-i}.$$

Thus,

$$\varphi_i\left(\alpha; t_{-i}\right) = \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Thus, φ is independent of time allocation. For simplicity, we write $\varphi(F)$ instead of $\varphi(t, F)$.

Step A2. Now, we show that the payoff of an agent is invariant to transfers of profit between connected projects.

Consider two non-disjoint projects, namely R and S. Let j be a player in $R \cap S$. Furthermore, consider the following set of production functions for any $\beta \in \mathbb{R}_+$:

$$\begin{aligned} f^R(t^R) &= \alpha^R + \beta t_j^R; \\ f^S(t^S) &= \alpha^S + \beta t_j^S; \\ f^K(t^K) &= \alpha^K, \quad \forall K \neq R, S. \end{aligned}$$

Note that under efficiency, agent j assigns any distribution of his time to projects R and S while the other agents assign their time arbitrarily.

Now, consider that the time for agent j increases, i.e. $\tilde{T}_j = T_j + \epsilon$.

By time-monotonicity, there is an efficient and continuous extension Φ of φ such that

$$\varphi_i\left(\alpha^{-S,R}, \alpha^S + \beta T_j, \alpha^R\right) \leq \Phi_i\left(\alpha^{-S,R}, \alpha^S, \alpha^R + \beta(T_j + \epsilon)\right),$$

for any player i.

The limit as ϵ tends to 0 implies

$$\varphi_i\left(\alpha^{-S,R},\alpha^S+\beta T_j,\alpha^R\right)\le\varphi_i\left(\alpha^{-S,R},\alpha^S,\alpha^R+\beta T_j\right).$$

By exchanging the assignment of time, we have

$$\varphi_i\left(\alpha^{-S,R},\alpha^S,\alpha^R+\beta T_j\right)\le\Phi_i\left(\alpha^{-S,R},\alpha^S+\beta(T_j+\epsilon),\alpha^R\right).$$

The limit as ϵ tends to 0 implies

$$\varphi_i\left(\alpha^{-S,R},\alpha^S,\alpha^R+\beta T_j\right)\le\varphi_i\left(\alpha^{-S,R},\alpha^S+\beta T_j,\alpha^R\right).$$

Then,

$$\varphi_i\left(\alpha^{-S,R},\alpha^S,\alpha^R+\beta T_j\right)=\varphi_i\left(\alpha^{-S,R},\alpha^S+\beta T_j,\alpha^R\right),$$

Hence, the payoff of any agent i is invariant to transfers of profit between non-disjoint projects. By repeating the argument for all pairs of non-disjoint projects in the sequence of connectedness, we complete the proof.

When there is no confusion, we denote $\varphi(t,F)$ simply as $\varphi(A)$, where $A\in\mathcal{P}/\sim$ and

$$A_{[P]}=\sum_{K\in[P]}F^K.$$

Step A3. We prove that $\varphi(A)$ is monotonic.

Let $A\in\mathbb{R}^{\mathcal{P}/\sim}$ and consider the vector $\tilde{A}=(\tilde{A}_{[j]},A_{-[j]})\in\mathbb{R}^{\mathcal{P}/\sim}$ such that $\tilde{A}_{[j]}>\tilde{A}_{[j]}$ for some $j\in N$. We will show that $\varphi(\tilde{A})\ge\varphi(A)$.

We denote $\mathcal{P}/\sim=\{[S_1],\ldots,[S_t]\}$, where $S_k\in\mathcal{P}$ is a representative project of the class $[S_k]$ for $k=1,\ldots,t$, and $[S_k]\neq[S_l]$ for all k and l. Without loss of generality, assume that $[S_1]=[j]$.

Define the following production functions:

$$\begin{aligned}
f^K(t^K) &= \frac{A_{[K]}}{T_j}t_j^K,\ \text{for } K=S_1,\\
f^K(t^K) &= A_{[K]},\ \text{for } K=S_k \text{ and } k=2,\ldots,t,\\
f^K(t^K) &= 0,\ \text{for all } K\notin\{S_1,\ldots,S_t\}.
\end{aligned}$$

Note that under efficiency, agent j assigns his time to project S_1. Such efficient vector generates profits in $\mathcal{P}/\sim$ equal to A. On the other hand, for an increase of time to $\tilde{T}_j=\frac{\tilde{A}_{[j]}}{A_{[j]}}T_j$, the efficient allocation generates profits in $\mathcal{P}/\sim$ equal to $\tilde{A}$. Therefore, by time-monotonicity, $\varphi(A) \leq \varphi(\tilde{A})$.

Property (iii) $\Rightarrow$ **Property (ii)** is obvious. We show in steps B1-B3 that **Property (ii)** $\Rightarrow$ **Property (iii).**

Step B1. φ_i is independent of time.

By applying Theorem 1, the payoff of agent i depends only on the time allocations of others:

$$\varphi_i(t, F) = \varphi_i\left(F; t_{-i}\right), \quad \forall i \in N.$$

We now establish that φ_i is independent of the others' time allocations t_{-i}.
Consider the following constant production functions:

$$f^K(t^K) = \alpha^K, \quad \forall K \in \mathcal{P}.$$

Note that under these production functions, any allocation of time is an efficient Nash equilibrium. Suppose that the technology in the project S has been improved.

$$\tilde{f}^S(t^S) = \alpha^S + \epsilon.$$

Then, through technology monotonicity,

$$\varphi_i\left(\left(\alpha^K\right)_{K\in\mathcal{P}\setminus S}, \alpha^S + \epsilon; \tilde{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By taking the limit as ϵ tends to zero,

$$\varphi_i\left(\alpha; \tilde{t}_{-i}\right) \geq \varphi_i\left(\alpha; t_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

By exchanging the roles of t_{-i} and $\tilde{t}_{-i}$ we find that

$$\varphi_i\left(\alpha; t_{-i}\right) \geq \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Thus,

$$\varphi_i\left(\alpha; t_{-i}\right) = \varphi_i\left(\alpha; \tilde{t}_{-i}\right), \quad \forall i, \quad t_{-i} \quad \text{and} \quad \tilde{t}_{-i}.$$

Therefore, φ is independent of t_{-i}. For simplicity, we write $\varphi(t, F)$ simply as $\varphi(F)$.

Step B2. In this step, we show that the payoff of an agent is invariant to transfers of profit between connected projects. Consider two non-disjoint projects, namely R and S. Let j be a player in $R \cap S$. Furthermore, consider the following set of production functions:

$$
\begin{aligned}
f^T(t^R) &= c^R + \beta t_j^R; \\
f^S(t^S) &= c^S + \beta t_j^S; \\
f^K(t^K) &= c^K, \quad \forall K \neq S, R,
\end{aligned}
$$

for any $\beta \in \mathbb{R}_+$.

Note that under efficiency, agent j assigns any distribution of his time to projects R and S while other agents assign their time arbitrarily.

Consider the following technology improvement of f^T:

$$
\begin{aligned}
f^T(t^R) &= c^R + \beta t_j^R + \epsilon; \\
f^S(t^S) &= c^S + \beta t_j^S; \\
f^K(t^K) &= c^K, \quad \forall K \neq S, R.
\end{aligned}
$$

Through technology monotonicity, the payoff for any player i satisfies the notion that

$$\varphi_i \left(c^{-S,R}, c^S + \beta T_j, c^R\right) \leq \varphi_i \left(c^{-S,R}, c^S, c^R + \beta T_j + \epsilon\right).$$

At the limit, when ϵ tends toward zero, we find that

$$\varphi_i \left(c^{-S,R}, c^S + \beta T_j, c^R\right) \leq \varphi_i \left(c^{-S,R}, c^S, c^R + \beta T_j\right).$$

Alternatively, consider the production functions:

$$
\begin{aligned}
f^T(t^R) &= c^R + \beta t_j^R; \\
f^S(t^S) &= c^S + \beta t_j^S + \epsilon; \\
f^K(t^K) &= c^K, \quad \forall K \neq S, R,
\end{aligned}
$$

By repeating the above argument we find that

$$\varphi_i \left(c^{-S,R}, c^R + \beta T_j, c^S\right) \leq \varphi_i \left(c^{-S,R}, c^R, c^S + \beta T_j + \epsilon\right).$$

As ϵ tends toward zero, this leads to

$$\varphi_i \left(c^{-S,R}, c^R + \beta T_j, c^S\right) \leq \varphi_i \left(c^{-S,R}, c^R, c^S + \beta T_j\right).$$

Thus,

$$\varphi_i \left(c^{-S,R}, c^S, c^R + \beta T_j \right) = \varphi_i \left(c^{-S,R}, c^S + \beta T_j, c^R \right),$$

Hence, the payoff for any agent i is invariant to transfers of profit between non-disjoint projects. Repeating the argument for all pairs of non-disjoint projects completes the proof.

When there is no confusion, we denote $\varphi(t, F)$ simply as $\varphi(A)$, where $A \in \mathcal{P}/\sim$ and

$$A_{[P]} = \sum_{K \in [P]} F^K.$$

Step B3. We prove that φ is monotonic.

Consider the following constant production functions:

$$f^K(t^K) = a^K, \quad \forall K \in \mathcal{P},$$

such that

$$A_{[P]} = \sum_{K \in [P]} a^K$$

Note that any allocation of time is an efficient Nash equilibrium.

Suppose that the technology in the project $S \in \mathcal{P}$ has been improved by an arbitrary $\epsilon > 0$:

$$\tilde{f}^S(t^S) = \alpha^S + \epsilon.$$

By technology monotonicity,

$$\varphi(A_{[S]}, A_{-[S]}) \leq \varphi(A_{[S]} + \epsilon, A_{-[S]}) \quad \forall \epsilon > 0 \quad \forall S \in \mathcal{P}.$$

□

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