# Group strategyproof cost sharing: The role of indifferences

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**Summary.** Every agent reports his willingness to pay for one unit of a good. A mechanism allocates goods and cost shares to some agents. We characterize the group strategyproof ($GSP$) mechanisms under two alternative continuity conditions interpreted as tie-breaking rules. With the maximalist rule ($MAX$) an indifferent agent is always served. With the minimalist rule ($MIN$) an indifferent agent does not get a unit of the good.

$GSP$ and $MAX$ characterize the cross-monotonic mechanisms. These mechanisms are appropriate whenever symmetry is required. On the other hand, $GSP$ and $MIN$ characterize the sequential mechanisms. These mechanisms are appropriate whenever there is scarcity of the good.

Our results are independent of an underlying cost function; they unify and strengthen earlier results for particular classes of cost functions.

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**JEL classification:** C72, D44, D71, D82.

# 1   Introduction

Units of a non-transferable and indivisible good (or service) are available at some non-negative cost. Agents are interested in consuming at most one unit of that good and are characterized by their valuation of it (which we call their utility). We look for deterministic mechanisms that elicit these utilities from the agents, allocate goods to some agents, and charge money only to the agents who are served such that no agent pays more than his utility.

These mechanisms have been widely explored in the cost sharing literature (see below). The canonical example is sharing the cost of providing some optional service to geographically dispersed agents (e.g., Internet), where the cost function is not necessarily symmetric. Another example is auctions where the seller has multiple copies of a good.

When agents have private information about their utility, the incentive compatibility of the mechanism, here interpreted as strategyproofness ($SP$), is an issue. The mechanisms that satisfy $SP$ are such that every agent is given the opportunity to buy a unit of the good at a price that depends exclusively on the reports of the other agents.

A familiar strengthening of $SP$ is group strategyproofness ($GSP$). This property rules out coordinated misreports of any group of agents. $GSP$ is particularly interesting in settings where the designer of the mechanism has little information about the types of agents participating in the economy, for instance, when the designer is dealing with agents in a large network like the Internet. In these settings, it is usually the case that agents have the ability to coordinate misreports and hence increase their net utility. $GSP$ is a robust property that rules out coordinated misreports under any possible information context. In particular, it works whether the information on individual characteristics is private or not.[1]

For an $SP$ mechanism, whether the agents who are offered a price equal to their valuation are served is of no consequence. Not so for $GSP$ mechanisms. $GSP$ is clearly violated if such an agent can be "bossy," i.e., affect the welfare of another agent without altering his own.[2] For instance, consider the mechanism that offers to the agents in $\{1, 2\}$, following the order $1 \succ 2$, the first unit at price $p$ and the second unit at price $p'$, $p' > p$. Assume that the first agent's utility for a unit of the good equals exactly $p$ and the second agent's utility is strictly larger than $p$, then $GSP$ requires that agent 1 not be served. Otherwise, agent 1 can help agent 2 by reporting a utility below $p$, whereby agent 2 is offered the cheaper price $p$.

This paper characterizes the $GSP$ mechanisms under two continuity conditions, interpreted as tie-breaking rules. With the maximalist tie-breaking rule ($MAX$), an agent who

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[1]On the other hand, $GSP$ does not rule out side payments between agents. Schummer[2000] has shown the very limited class of mechanisms that rule out coordination and transfers of money between the agents in different economies. Only the constant price mechanisms (where agents are offered units of goods at a constant price) rule out coordination and transfer of money between agents in our setting. See also Goldberg et al.[2005] and Juarez[2011b] for related results.

[2]In some contexts, $GSP$ is equivalent to the combination of $SP$ and non-bossiness: Papai[2000, 2001], Ehlers et al.[2003], Svensson et al.[2002]. In our context, a similar equivalence holds by imposing two alternative non-bossy conditions, see Mutuswami[2005].

is indifferent between getting or not getting a unit of the good will always get a unit of the good. With the minimalist rule ($MIN$), the indifferent agents never get a unit of the good.

The mechanisms that satisfy $GSP$ and $MAX$ are the cross-monotonic mechanisms (Theorem 1), where, unlike in the above example, the price offered to an agent weakly decreases as more agents are served. Specifically, for any subset of agents $S$ consider a vector of non-negative payments $x^S \in [0, \infty]^N$ that are zero for all agents not in $S$. A collection of payments is cross-monotonic if the payments are weakly decreasing as the coalition increases. Given a cross-monotonic collection of payments, we construct the mechanism as follows. For a report of utilities, we say a coalition $S$ is reachable if all agents in $S$ are willing to pay the prices given by $x^S$ for a unit of the good. By the cross-monotonicity of the set of payments, the union of two reachable coalitions at a utility profile is another reachable coalition. For a report of utilities, the mechanism allocates $S^*$ at cost $x^{S^*}$, where $S^*$ is the largest reachable coalition at that utility profile.

The mechanisms that satisfy $GSP$ and $MIN$ are a subset of the sequential mechanisms (Theorem 2). Loosely speaking, consider any binary tree of size $n$ such that exactly one agent is attached to every node and any path from the root to a terminal node goes through all agents exactly once. At every decision node, we also attach a non-negative price. Given this tree, we construct the mechanism as follows. First, we offer service to the root agent at the price attached to his node. We proceed on the right branch from the root if service is purchased and on the left branch if it is not. The key restriction on prices is that for any two nodes in the tree to which the same agent is attached such that the node on the right is realizable by indifferent agents from the node on the left,[3] the price on the node on the right is not smaller than the price on the node on the left.

Surprisingly, the (welfare-wise) intersection of sequential and cross-monotonic mechanisms is almost empty. It contains only the fixed cost mechanisms (Corollary 1), offering to each agent a price completely independent of the reports.

An important property of cross-monotonic mechanisms is to allow equal treatment of equals, which no other $GSP$ mechanism does (Proposition 2). On the negative side, when there are only $k$ units of the good available, $k < n$, cross-monotonic mechanisms must exclude $n-k$ agents from the mechanism, that is, they will never be served at any profile (see section 6.3). By contrast, not all sequential mechanisms exclude agents ex-ante. In fact, only the priority mechanisms, where agents are offered sequentially a unit of the good at a fixed price until someone accepts the offer, meet $GSP$, do not exclude any agent and allocate at most one unit of the good at any profile (Proposition 3).

We do not make an actual cost function part of the definition of a mechanism. That is, we place no constraint on the total cost shares collected from the agents who are served. Thus, our characterization results of $GSP$ mechanisms are entirely orthogonal to budget balance and other feasibility requirements (such as bounds on the budget surplus or deficit). Naturally, one of the first questions we ask about the class of mechanisms identified in Theorems 1 and 2 is: when can they be chosen so as to cover exactly a given cost function? In sections 6.4.1 and 6.4.2 we answer these questions under a weak symmetry assumption.

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[3]See the precise conditions in definition 9.

In this way, we recover most mechanisms identified in the earlier literature.

## 2   Related literature

There is some interesting literature on the design of $GSP$ mechanisms for assignment problems of heterogeneous goods when money is not available (Ehlers[2002], Ehlers et al.[2003], Papai [2000, 2001] and Svensson et al.[2002]). Unfortunately, this literature usually characterizes mechanisms with poor equity properties (e.g., dictatorial mechanisms). By contrast, the class of $GSP$ mechanisms when money is available is very rich (see below).

The design of $GSP$ cost sharing mechanisms for heterogeneous goods was first discussed by Moulin[1999] and Moulin and Shenker[2001]. When the cost function is submodular (concave), cross-monotonic mechanisms are characterized by $GSP$, budget balance, voluntary participation, non-negative transfers and strong consumer sovereignty.[4] Roughgarden et al.[2006a, 2006b], Pál et al.[2003] and Immorlica et al.[2008] consider cross-monotonic mechanisms when the cost function is not submodular. Roughgarden et al.[2006a] uses cross-monotonic mechanisms to approximate budget balance when the actual cost function is not submodular. Immorlica et al.[2008] show that new cross-monotonic mechanisms emerge when consumer sovereignty is relaxed.

The sequential mechanisms of our Theorem 2 are discussed by Moulin[1999], who imposes budget balance for a supermodular (convex) cost function. Theorem 1 there asserts wrongly that all $GSP$ mechanisms meeting budget balance, voluntary participation, non-negative transfers and strong consumer sovereignty charge successive marginal costs following an independent ordering of the agents. We correct this erroneous statement in example 8.

Roughgarden et al.[2009] uncover a very clever class of weakly $GSP$ mechanisms that are neither cross-monotonic nor sequential (see also Devanur et al.[2005]). This class contains sequential and cross-monotonic mechanisms, as well as hybrid mechanisms. They apply these mechanisms to the vertex cover and Steiner tree cost sharing problems to improve the efficiency of algorithms derived from cross-monotonic mechanisms. A closely related paper is the companion paper Juarez[2011b], which develops a model where indifferences are ruled out. For instance, agents report an irrational number and payments are rational. It turns out that the class of $GSP$ mechanisms becomes very large. In particular, it contains mechanisms very different from cross-monotonic and sequential mechanisms (and also those discussed by Roughgarden et al.[2009]). Juarez[2011b] provides three equivalent characterizations of the $GSP$ mechanism in this economy, two of which are generalizations of the cross-monotonic and sequential mechanisms discussed in this paper (see also Pountourakis et al.[2010]).

When a cost function is specified, an important question is to evaluate the trade-offs between efficiency and budget balance. Moulin and Shenker[2001] discuss this issue for budget balanced cross-monotonic mechanisms when the underlying cost function is submodular. In particular, they find that the cross-monotonic Shapley value mechanism, where the pay-

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[4]Strong consumer sovereignty says that every agent has reports such that he gets (or does not get) a unit of the good irrespective of other people's reports.

ment of a coalition equals its stand-alone cost, minimizes the worst absolute surplus loss.[5] Juarez[2011a] analyzes similar trade-offs for supermodular cost functions. Contrary to the submodular case, one can construct optimal sequential mechanisms that cut the efficiency loss by half with respect to the optimal budget balanced mechanism.

Finally, findings by Schummer[2000], Goldberg et al.[2005] and Juarez[2011b] on fixed cost mechanisms are closely related to our Corollary 1. They characterize these mechanisms under a strengthening of $GSP$, where agents, acting in coalition, can manipulate the mechanism by misreporting their utility and transferring money between them.

# 3    The model

For a vector $x \in \mathbb{R}^M$, we denote by $x_S$ the projection of $x$ over $S \subset M$. For the vectors $x, y \in \mathbb{R}^M$, we say $x \leq y$ if $x_i \leq y_i$ for all $i$. We say $x < y$ if $x_i < y_i$ for all $i$. Let $1_M$ be the unitarian vector in $\mathbb{R}^M$, that is, $1_M = (1, 1, \ldots, 1)$.

There is a finite number of agents $N = \{1, 2, \ldots, n\}$. Every agent has a utility (willingness to pay) for getting one unit of the good. Let $u \in \mathbb{R}_+^N$ be the vector of these utilities. Therefore, if agent $i$ gets a unit paying $x_i$, his net utility is $u_i - x_i$. If he does not get a unit of the good, his net utility is zero.

**Definition 1** *A **mechanism** $(G, \varphi)$ allocates to every vector of utilities $u$ a coalition of agents who get goods $G(u) \subseteq N$ and the cost shares (payments) $\varphi(u) \in \mathbb{R}_+^N$ such that:*

  *i. if $i \notin G(u)$ then $\varphi_i(u) = 0$*

  *ii. if $i \in G(u)$ then $\varphi_i(u) \leq u_i$.*

The definition above includes familiar constraints previously discussed in the literature. For instance, we restrict our attention to non-negative mechanisms, requiring all cost shares to be positive or zero. This is a common assumption when no transfers between agents are allowed and we do not want to subsidize any of them. The mechanisms will also meet the requirement of individual rationality, which implies that all agents enter the mechanism voluntarily. That is, if an agent is served, then he will never pay more than his utility (condition $ii$). On the other hand, because we study non-negative mechanisms, individual rationality implies that the agents who are not served should pay nothing (condition $i$).

The net utility of agent $i$ in the mechanism $(G, \varphi)$, denoted by $NU_i$, is $NU_i(u) = \delta_i(G(u))(u_i - \varphi_i(u))$.[6] That is, if $i \in G(u)$, then $NU_i(u) = u_i - \varphi_i(u)$; and if $i \notin G(u)$, then $NU_i(u) = 0$. Let $NU(u)$ be the vector of such net utilities. Note that two different mechanisms may be welfare-wise equivalent, that is, their net utilities at any profile are equal.

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[5]See also Juarez[2008] for a comparison of average cost and random priority using this measure. Moulin[2008] uses a similar measure to compare the serial, incremental and average cost methods.

[6]$\delta$ is the classic Dirac delta function, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

We want to characterize the mechanisms that are group strategyproof. That is, we want to rule out coordinated misreports by any group of agents. If a group of agents misrepresent their preferences with at least one agent in the group strictly profiting, then another agent in the group will lose.

**Definition 2 (Group strategyproofness)** *A mechanism $(G, \varphi)$ is **group strategyproof** (GSP) if for all $T \subset N$, and all utility profiles $u$ and $u'$ such that $u'_{N \setminus T} = u_{N \setminus T}$, if there is an agent $i \in T$ such that $(u_i - \varphi_i(u'))\delta_i(G(u')) > NU_i(u)$, then there exists another agent $j \in T$ such that $(u_j - \varphi_j(u'))\delta_j(G(u')) < NU_j(u)$.*

We define next our two systematic continuity conditions. Similar continuity conditions have been used in other models, for instance, Deb and Razzolini[1999]. These are tractability conditions that allow us to get closed-form mechanisms. Nevertheless, these conditions can be easily interpreted; see below.

**Definition 3**   • ***Upper continuity (Maximalist tie-breaking rule (MAX)).*** *A mechanism $(G, \varphi)$ satisfies $MAX$ if for any $i$, $u_{-i} \in \mathbb{R}_+^{N \setminus i}$, and $u_i^1 \geq u_i^2 \geq \cdots \to \gamma^*$ such that $i \in G(u_i^k, u_{-i})$ and $\varphi_i(u_i^k, u_{-i}) = \gamma^*$ for all $k$, then $i \in G(\gamma^*, u_{-i})$.*

• ***Lower continuity (Minimalist tie-breaking rule (MIN)).*** *A mechanism $(G, \varphi)$ satisfies $MIN$ if for any $i$, $u_{-i} \in \mathbb{R}_+^{N \setminus i}$, and $u_i^1 \geq u_i^2 \geq \cdots \to \gamma^*$ such that $i \in G(u_i^k, u_{-i})$ and $\varphi_i(u_i^k, u_{-i}) = \gamma^*$ for all $k$, then $i \notin G(\gamma^*, u_{-i})$.*

In the space of strategyproof ($SP$) mechanisms, upper and lower continuity can be interpreted as tie-breaking rules. Upper continuity ($MAX$) serves the agents who are indifferent between getting or not getting a unit of the good, whereas lower continuity ($MIN$) does not serve the indifferent agents. To see this, consider an $SP$ mechanism. Then, there exist arbitrary pricing functions $f_i : \mathbb{R}_+^{N \setminus i} \to [0, \infty]$ for $i = 1, \ldots, n$, such that at the utility profile $u$, agent $i$ is offered a unit of the good at price $f_i(u_{-i})$. That is, if $u_i > f_i(u_{-i})$, then $i$ is served at price $f_i(u_{-i})$; if $u_i < f_i(u_{-i})$, then $i$ is not served and pays nothing; and if $u_i = f_i(u_{-i})$, then $i$ may or may not get a unit of the good at this price. Under $MAX$, if $u_i = f_i(u_{-i})$, then the agent gets a unit of the good at price $f_i(u_{-i})$. On the other hand, under $MIN$, if $u_i = f_i(u_{-i})$, then $i$ does not get a unit of the good and pays nothing.

$GSP$ mechanisms greatly differ depending on whether $MAX$ or $MIN$ is satisfied. For instance, consider a $GSP$ mechanism generated by the pricing functions $f_1, \ldots, f_n$ and a utility profile $u$ such that agent $i$ is an indifferent agent, $u_i = f_i(u_{-i})$, and agent $j$ gets positive net utility, $u_j > f_j(u_{-j})$. Then, under $MAX$, agent $j$ does not benefit when the utility of agent $i$ decreases. To see this, suppose that $\tilde{u}_i < u_i$ and $f_j(\tilde{u}_i, u_{-ij}) < f_j(u_i, u_{-ij})$. Then, when the true utility profile is $u$, by $MAX$ agent $i$ gets a unit of the good at a price $f_i(u_{-i})$ that is exactly equal to his utility, and agent $j$ gets a unit of the good at a price equal to $f_j(u_i, u_{-ij})$. Therefore, agent $i$ can help agent $j$ by misreporting $\tilde{u}_i$, since at the profile $(\tilde{u}_i, u_{-i})$, agent $i$ does not get any good and agent $j$ gets a lower price $f_j(\tilde{u}_i, u_{-ij})$ for a unit of the good. In contrast, under $MIN$, agent $j$ does not benefit when the utility of agent $i$ increases.

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More generally, the first step in the proof of Theorem 1 shows that under $MAX$ the prices offered to the agents weakly decrease as the utility profile increases. Hence, under $MAX$, agents weakly benefit when indifferent agents get a unit of the good. This contrasts with $MIN$, where agents are weakly better off when indifferent agents do not get a unit of the good.

# 4    Cross-monotonic mechanisms and $MAX$

**Definition 4** *A **cross-monotonic set of cost shares** (payments) $\chi^N = \{x^S \in \mathbb{R}_+^N \mid S \subseteq N\}$ is such that:*

> *i. $x_{N \setminus S}^S = 0$ for all $S \subseteq N$, and*

> *ii. if $S \subseteq T$, then $x_i^S \geq x_i^T$ for all $i \in S$*

In a cross-monotonic set of cost shares, there is exactly one set of cost shares for every coalition $S$. We interpret $x^S$ as the payment when the agents in $S$, and only they, are served.

The key feature of a cross-monotonic set of cost shares is that they do not increase as the coalition increases. This implies that for every utility profile $u$, the **set of reachable coalitions,** $F(u) = \{S \in 2^N \mid x^S \leq u\}$, has a maximum element with respect to the inclusion $\subseteq$. To see this, note that if $S, T \in F(u)$, then by cross-monotonicity, $S \cup T \in F(u)$.

**Definition 5** *A mechanism $(G, \varphi)$ is **cross-monotonic** if there exists a cross-monotonic set of cost shares $\chi^N$ such that for all $u \in \mathbb{R}_+^N : G(u)$ is the maximum reachable coalition at $u$ and $\varphi(u) = x^{G(u)}$.*

**Theorem 1** *A mechanism satisfies GSP and $MAX$ if and only if it is cross-monotonic.*

The proof is in the appendix.

In an economy without indifferences, cross-monotonic mechanisms are also characterized by $GSP$ and monotonicity in size, that is, if $u \leq \tilde{u}$, then $G(u) \subseteq G(\tilde{u})$. See Juarez[2011b] for details.

Given a cross-monotonic set of cost shares $\chi^N$, we can also implement the truthful outcome of the cross-monotonic mechanism by playing the following demand game proposed by Moulin[1999]. We offer agents in $N$ units of the good at price $x^N$. If all of them accept the offer, then everyone is served at price $x^N$. If only agents in $S$ accept, then we remove agents in $N \setminus S$ from the game and offer agents in $S$ units of the good at price $x^S$. Continue similarly until all of the agents in a coalition have accepted or every agent in $N$ has been removed from the game.

**Example 1 (Cross-monotonic mechanisms for $n = 1, 2$)** *The one-agent mechanisms can be described by a constant $x \in [0, \infty]$. The agent gets a unit and pays $x$ if his utility is greater than or equal to $x$. He does not get a unit and pays nothing otherwise.*
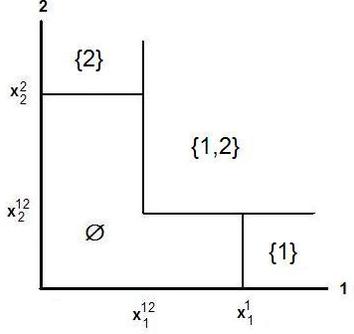
Figure 1: Generic form of a 2-agent cross-monotonic mechanism.

The two-agent mechanisms should be generated by a cross-monotonic set of cost shares. Thus, $0 \leq x_1^{\{1,2\}} \leq x_1^{\{1\}}$ and $0 \leq x_2^{\{1,2\}} \leq x_2^{\{2\}}$ (see figure 1).

By $MAX$, the level set of $\{1,2\}$ is closed. The borders between the level sets of $\{1\}$ and $\emptyset$, and $\{2\}$ and $\emptyset$, should belong to $\{1\}$ and $\{2\}$, respectively.

**Example 2** *Immorlica et al.[2008] propose an example where exactly one agent pays a positive amount when a coalition of agents is served. This example relaxes a key strong consumer sovereignty condition on Moulin's[1999] result and therefore is not captured by Moulin's mechanisms. However, it is captured by our class of cross-monotonic mechanisms. For a submodular cost function $C : 2^N \to \mathbb{R}_+$, order the agents arbitrarily, say, $i_1 \succ i_2 \succ \cdots \succ i_n$. Offer the agents, following this order, a unit of the good at the cost of himself and the agents after him. The mechanism ends when someone accepts the offer or when we have made an offer to every agent. That is, agent $i_1$ will be offered a unit of the good at price $C(i_1, \ldots, i_n)$. If he accepts the offer, the mechanism ends there and everyone is served at price zero except by agent $i_1$, who pays $C(i_1, \ldots, i_n)$. If $i_1$ rejects the offer, then we offer agent $i_2$ a unit of the good at price $C(i_2, \ldots, i_n)$, and so on. The cross-monotonic set of cost shares that implement this mechanism are $x_{i^*}^S = C(D_{i^*})$ and $x_j^S = 0$ for all $j \neq i^*$, where $i^*$ is the maximal element in $S$ and $D_i^*$ is the set that contains $i^*$ and all agents dominated by $i^*$ with $\succ$.*

## 5  Sequential mechanisms and $MIN$

**Definition 6** *A **sequential tree** is a binary tree of length $n$ such that:*

    *i. at every node there is exactly one agent in $N$ and a price in $[0, \infty]$,*

    *ii. every path from the root to a terminal node contains all agents in $N$ exactly once.*

**Definition 7 (Sequential mechanisms)** *A mechanism is a **sequential mechanism** if there exists a sequential tree that implements the outcome of the mechanism as follows for every utility profile:*

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*The agent in the root of the tree is offered a unit of the good at the price of his node. If his utility is strictly greater than the price offered, then he is allocated a unit of the good at this price and we go right on the tree. If his utility is less than or equal to the price offered, then we do not allocate him a unit and we go left on the tree. We continue similarly with the following agent until the end of the tree is reached.*
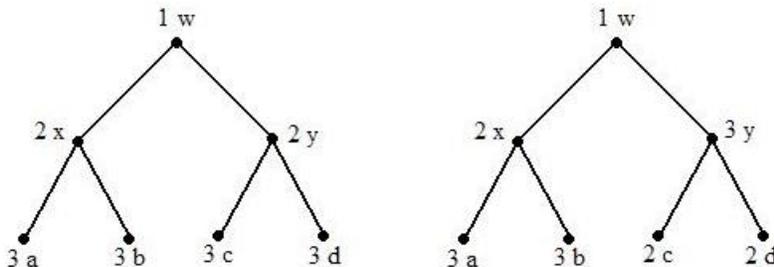


Figure 2: Sequential trees for three agents. (a) Agents follow order 1,2,3. (b) Agents 2 and 3 follow different orders depending on whether agent 1 goes right or left.

**Example 3** *In figure 2 we show the only two possible (up to renaming the agents) sequential trees for the agents in $N = \{1, 2, 3\}$. Every node contains a number and a letter. The number represents the agent at this node. The letters represent a price in $[0, \infty]$.*

*Consider the sequential tree of figure 2(a) and the mechanism $(G, \varphi)$ that it implements. If the utility profile $u$ is such that $u_1 > w$, $u_2 > y$ and $u_3 \leq d$, then the outcome is $G(u) = \{1, 2\}$ and $\varphi(u) = (w, y, 0)$.*

*On the other hand, if $\tilde{u}$ is such that $\tilde{u}_1 \leq w, \tilde{u}_2 > x$ and $\tilde{u}_3 \leq b$, then $G(\tilde{u}) = \{2\}$ and $\varphi(\tilde{u}) = (0, x, 0)$.*

Sequential mechanisms are not always group strategyproof. For instance, consider the mechanism generated by the sequential tree of figure 2(a). If $y < x$, then when the true utility profile is such that $u_1 = w$ and $u_2 > y$, agent 1 can help agent 2 by reporting a utility greater than $w$, whereby agent 2 is offered a unit at a cheaper price.[7] Definition 10 gives the exact conditions under which a sequential tree will generate a $GSP$ mechanism.

Given a sequential tree, consider any path in the tree and a non-terminal node $\zeta$ in this path. We say $\zeta$ is **losing (winning)** on this path if the edge in the path that follows $\zeta$ is a left (right) edge. For instance, the path $[1w, 2y, 3c]$ in figure 2(a) contains one winning node and one losing node. $1w$ is winning and $2y$ is losing.

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[7]However, these mechanisms are **weakly group strategyproof**, that is, if a coalition of agents successfully misreports, then at least one agent in this coalition is indifferent. The class of weakly group strategyproof mechanisms is very rich but very difficult to characterize. It contains all the mechanisms discussed in this paper. See Juarez[2011b] for more details.

One useful path is from the root of the tree to a node. We denote this path by $P(\zeta)$ starting at node $\zeta$. For instance, in figure 2(a), $P(3c) = [1w, 2y, 3c]$, $P(3d) = [1w, 2y, 3d]$ and $P(2x) = [1w, 2x]$.

Note that the intersection of two paths from the root of the tree is also a path from the root of the tree. We use $\sqcap$ to denote it. For instance, in figure 2(a), $P(3c) \sqcap P(3d) = [1w, 2y]$. Note that this intersection may also lead to the degenerate path that contains only the root of the tree, for instance, $P(2x) \sqcap P(2y) = [1w]$.

**Definition 8** *Let $\zeta$ and $\zeta'$ be two nodes in a sequential tree. We say that node $\zeta$ is **on the left** of $\zeta'$ (or, alternatively, $\zeta'$ is **on the right** of $\zeta$) if the terminal node of $P(\zeta) \sqcap P(\zeta')$ is losing on $P(\zeta)$ and winning on $P(\zeta')$.*

For instance, in figure 2(a), $P(3c) = [1w, 2y, 3c]$, $P(3d) = [1w, 2y, 3d]$. Since $2y$ is losing in $[1w, 2y, 3c]$ and winning in $[1w, 2y, 3d]$, then $3c$ is on the left of $3d$.

Given a node $\zeta$ and an agent $i \in P(\zeta)$, we denote by $x_i^\zeta$ the price of agent $i$ in the path $P(\zeta)$.

**Definition 9 (Realizability by indifferent agents)** *Consider a sequential tree and two nodes $\zeta$ and $\zeta'$ in the tree such that $\zeta$ is on the left of $\zeta'$. We say $\zeta'$ is **realizable by indifferent agents** from $\zeta$ if there is a utility profile $u$ such that:*

a. *$u$ visits $\zeta$,*

b. *there is a group of indifferent agents $S$ at $u$, that is, $u_S = x_S^\zeta$, such that for some $\tilde{u}_S > u_S$ the profile $(\tilde{u}_S, u_{-S})$ visits the node $\zeta'$,*

c. *the agents in $S$ are not worse off at $(\tilde{u}_S, u_{-S})$, that is, $x_S^{\zeta'} \le x_S^\zeta$.*

Two nodes are realizable by indifferent agents if there is a group of agents who can increase their report without increasing their payments and visit the second node. For instance, consider the tree in figure 2(a). Node $3d$ is realizable by indifferent agents from $3b$ whenever $x$ and $y$ are finite. To see this, consider the utility profile $u = (w, u_2, u_3)$ such that $u_2 > \max\{x, y\}$. Clearly, $u$ visits $3b$. If $\tilde{u}_1 > w$, then the profile $(\tilde{u}_1, u_2, u_3)$ visits $3d$.

On the other hand, if $y \le x$, then the node $3c$ cannot be realized by indifferent agents from $3b$. This is because at any utility profile that realizes $3b$, the utility $u_2$ of agent 2 is such that $u_2 > x \ge y$. Therefore, agent 2 will always be served independent of the utility of agent 1.

We are especially interested in the realizability by indifferent agents of nodes with a common agent.

**Definition 10** *A sequential tree is **valid** (satisfies **validity**) if for any nodes $\zeta$ and $\zeta'$ with a common agent $k$ such that $\zeta'$ is realizable by indifferent agents from $\zeta$, $x_k^\zeta \le x_k^{\zeta'}$.*

*We say a sequential mechanism is **valid** (satisfies **validity**) if it is implemented by a valid sequential tree.*

A sufficient condition for validity is that for any two nodes that contain the same agent, the price of the node on the left is not smaller than the price of the node on the right. This condition is necessary when there are at most three agents (see examples 4 and 5). Example 6 shows that this is not necessary when there are more than three agents. We now characterize the collection of sequential trees that are valid.

**Proposition 1** *A sequential tree is valid if and only if for any two nodes $\zeta$ and $\zeta'$ with a common agent $k$ such that $\zeta$ is on the left of $\zeta'$ : If $x_k^\zeta > x_k^{\zeta'}$, then there exist nodes $\tilde{\zeta} \in P(\zeta)$ and $\bar{\zeta} \in P(\zeta')$ with a common agent $i$ and:*

*(a) $\tilde{\zeta}$ is losing in $P(\zeta)$, $\bar{\zeta}$ is winning in $P(\zeta')$ and $x_i^\zeta < x_i^{\zeta'}$, or*

*(b) $\tilde{\zeta}$ is winning in $P(\zeta)$ and $\bar{\zeta}$ is losing in $P(\zeta')$ and $x_i^\zeta \geq x_i^{\zeta'}$.*

The proof is in the appendix.

Validity is a necessary condition of a sequential mechanism that is $GSP$. Indeed, consider two nodes $\zeta$ and $\zeta'$ as in definition 10 above and assume that $x_k^\zeta > x_k^{\zeta'}$. We see that the indifferent agents can help agent $k$ by moving from node $\zeta$ to node $\zeta'$ at some utility profile. Since $\zeta'$ is realizable by indifferent agents from $\zeta$, then there is a utility profile $u$ that visits $\zeta$ and a group of indifferent agents $S$ such that $u_S = x_S^\zeta$, $x_S^\zeta \geq x_S^{\zeta'}$, and for some $\tilde{u}_S > u_S$ $(\tilde{u}_S, u_{-S})$ visits $\zeta'$. Assume without loss of generality that $u_k > x_k^{\zeta'}$. Then the indifferent agents in $S$ can help $k$ when the true utility profile is $u$ by misrepresenting their preferences to $\tilde{u}_S$. Agent $k$ will get a good at the cheaper price $x_k^{\zeta'}$, whereas the indifferent agents in $S$ are offered units of the good at the cheaper price $x_S^{\zeta'}$ because $x_S^{\zeta'} \leq x_S^\zeta$.

We now state the second main theorem of the paper. It characterizes the class of $GSP$ and $MIN$ mechanisms. This class contains only the valid sequential mechanisms.

**Theorem 2** *A mechanism is $GSP$ and $MIN$ if and only if it is a valid sequential mechanism.*

**Proof.**

We leave for the appendix the necessity of the proof. Valid sequential mechanisms trivially meet $MIN$. We prove by contradiction that these mechanisms meet $GSP$.

Assume that coalition $S$ profitably misreports $\tilde{u}_S$ at the true profile $u$. Let $k \in S$ be an agent who strictly increases his net utility by misreporting. Let $\zeta$ and $\zeta'$ be the nodes that contain agent $k$ in the paths that generate $G(u)$ and $G(\tilde{u}_S, u_{-S})$, respectively.

First note that $\zeta$ is on the left of $\zeta'$. To see this, let $i^*$ be the agent in the terminal node of $P(\zeta) \sqcap P(\zeta')$. Then, in order to move from $P(\zeta)$ to $P(\zeta')$, agent $i^*$ misreports. If $i^*$ is served in $P(\zeta)$, then by $MIN$ his net utility is positive, so he will never agree to move to $P(\zeta')$ because he is not served there.

Since agent $k$ strictly increases his net utility, then $x_k^\zeta > x_k^{\zeta'}$. By Proposition 1, condition (a) or (b) is satisfied. Assume condition (a) is satisfied. That is, there exist nodes $\tilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent $i$ such that $\tilde{\zeta}$ is losing in $P(\zeta)$, $\bar{\zeta}$ is winning in $P(\zeta')$ and

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$x_i^\zeta < x_i^{\zeta'}$. Since $\widetilde{\zeta}$ is losing in $P(\zeta)$, then $u_i \leq x_i^\zeta < x_i^{\zeta'}$. Thus, for the path $P(\zeta')$ to realize, $i \in S$ and $\widetilde{u}_i > x_i^{\zeta'}$. Hence, the net utility of agent $i$ is negative when he misreports because $u_i < x_i^{\zeta'}$. This is a contradiction.

On the other hand, assume condition (b) of Proposition 1 is satisfied. That is, there exist nodes $\widetilde{\zeta}$ and $\bar{\zeta}$ that contain the same agent $i$ such that $\widetilde{\zeta}$ is winning in $P(\zeta)$, $\bar{\zeta}$ is losing in $P(\zeta')$ and $x_i^\zeta \geq x_i^{\zeta'}$. Given that $\widetilde{\zeta}$ is winning in $P(\zeta)$, $u_i > x_i^\zeta \geq x_i^{\zeta'}$. Thus, for the path $P(\zeta')$ to realize, $i \in S$ and $\widetilde{u}_i \leq x_i^{\zeta'}$. Hence, the net utility of agent $i$ strictly decreases from $u_i - x_i^\zeta$ to zero when he misreports. This is a contradiction.

∎

**Example 4 (Valid sequential mechanisms for $n = 1, 2$)** *The one-agent mechanisms are easy to describe. Given $x_1 \in [0, \infty]$, agent 1 gets a unit of the good at price $x_1$ if and only if $u_1 > x_1$.*

*A two-agent mechanism such that 2 has priority over 1 is shown in figure 3. Agent 2 gets a unit of the good at price $x^2$ if and only if $u_2 > x^2$. If 2 gets a unit of the good, then agent 1 gets a unit of the good at price $d^1$ if $u_1 > d^1$. On the other hand, if agent 2 did not get a unit of the good, then agent 1 gets a unit of the good at price $d^2$ if $u_1 > d^2$. By validity of the tree $d^2 \leq d^1$.*
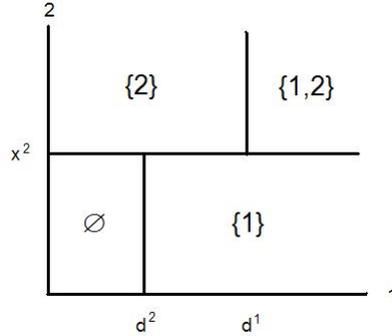


Figure 3: Generic form of a 2-agent valid sequential mechanism.

The following example shows that any valid sequential mechanism for three agents with finite prices can be represented by a sequential tree such that for any two nodes with a common agent, the price on the node on the left is not larger than the price on the node on the right. This property is true only for three or fewer agents. Example 6 provides a four-agent example where this representation does not hold.

**Example 5 (Valid sequential mechanisms for $n = 3$)** *Assume there are three agents. Figure 2 shows sequential trees for three agents. Every node contains an agent from $\{1, 2, 3\}$ and a non-negative price.*

*On figure 2(a), a valid sequential tree (assuming finite prices) implies: $x \leq y$, $a \leq b \leq d$ and $a \leq c \leq d$. Also, if $x < y$ then $b \leq c$.*

*To see this, consider nodes $2x$ and $2y$. Since they are consecutive nodes, their paths to the root of the tree differ only in $2x$ and $2y$, respectively. Then, conditions (a) and (b) of Proposition 1 cannot be satisfied. Hence, $x \leq y$.*

*Similarly, $a \leq b$ and $c \leq d$ are satisfied by comparing nodes $3a$ and $3b$, and $3c$ and $3d$ respectively.*

*On the other hand, by comparing nodes $3a$ and $3c$, conditions (a) and (b) of Proposition 1 are not satisfied because $2x$ and $2y$ are both losing. Hence $a \leq c$. Similarly $b \leq d$.*

*Now consider the nodes $3b$ and $3c$. If $x < y$, then condition (a) of Proposition 1 is not satisfied because $2y$ is not winning. Condition (b) of Proposition 1 is not satisfied because $x < y$. Therefore, it cannot be that $b > c$. Hence, if $x < y$, then $a \leq b \leq c \leq d$.*

*Now, assume $x = y$. From the argument given above, $a \leq b \leq d$ and $a \leq c \leq d$.*

*If $b \leq c$, then for every two nodes with the same agent, the price on the node on the left is smaller than or equal to the price on the node on the right.*

*On the other hand, if $b > c$, then because agents 1 and 2 have priority, we can exchange their order on the tree. This will look like figure 4. With this order, for every two nodes with the same agent, the price on the node on the left is smaller than or equal to the price on the node on the right.*
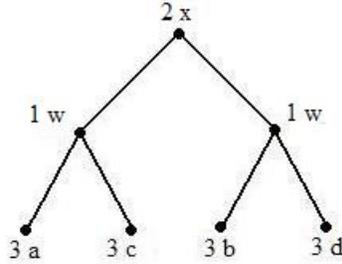


Figure 4: Three-agent sequential tree such that the positions of agents 1 and 2 can be switched without affecting the final outcome.

*Now consider figure 2(b). The validity of the tree (assuming finite prices) requires that $a \leq b \leq y$ and $x \leq c \leq d$. That is, for every two nodes with the same agent, the price on the node on the left is smaller than or equal to the price on the node on the right.*

*To see this, by comparing nodes $3a$ and $3b$, and $2c$ and $2d$, we get (similarly to the case above) that $a \leq b$ and $c \leq d$, respectively.*

*Now we compare nodes $3b$ and $3y$. There is no common agent in their path to the root; thus, conditions (a) and (b) cannot be satisfied. Hence, $b \leq y$. That is, $a \leq b \leq y$.*

*Similarly, by comparing nodes $2x$ and $2c$, $x \leq c$. Hence, $x \leq c \leq d$.*
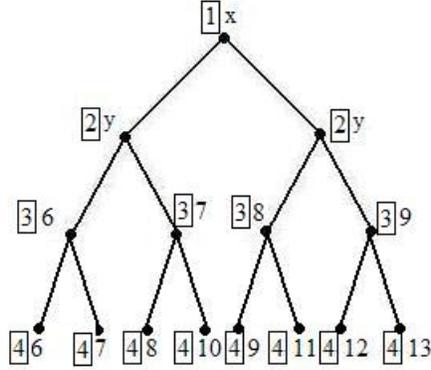
Figure 5: Four-agent valid sequential tree such that for every two nodes with the same agent, the price of the node on the left is not smaller than the price of the node on the right. This violation is found in the nodes for agent 4 with prices 10 and 9, since the node with price 10 is on the left of the node with price 9.

**Example 6** *Consider the mechanism generated by the sequential tree of figure 5 (agents are in the rectangles). For every two nodes with the same agent, the price on the node on the left is not greater than the price on the node on the right, except for nodes (4 10) and (4 9). At these nodes, their paths to the root contain the common agent 2. This agent meets condition (b). Therefore, this tree is valid.*

*However, the price on the node (4 10) is greater than the price on the node (4 9).*

*Since agents 1 and 2 have priority, we can also exchange their positions and leave agent 2 in the root. If this is the case, node (3 8) is on the left of (3 7).*

# 6 Comparison between cross-monotonic and sequential mechanisms

In this part, we study group strategyproof mechanisms when there are alternative constraints in the economy. In section 6.1, we study the class of mechanisms that are welfare-wise equivalent to a cross-monotonic and to a valid sequential mechanism. In section 6.2, we study GSP under a familiar equity constraint, namely equal treatment of equals. In section 6.3, we study the class of GSP mechanisms when there is only one unit of good available. Finally, in section 6.4, we study GSP mechanisms that are feasible for different shapes of cost functions.

## 6.1 The intersection of cross-monotonic and sequential mechanisms

There is a small class of mechanisms that are welfare equivalent to both a sequential and a cross-monotonic mechanism.[8]

**Definition 11** *A mechanism $(G, \varphi)$ is a **fixed cost mechanism** if there exist $x_1, \ldots, x_n \in [0, \infty]$, such that for every utility profile $u$ and every agent $i$:*

   *i. if $u_i > x_i$, then $i \in G(u)$,*

   *ii. if $u_i < x_i$, then $i \notin G(u)$,*

   *iii. if $i \in G(u)$, then $\varphi_i(u) = x_i$*

   *iv. if $i \notin G(u)$, then $\varphi_i(u) = 0$*

A fixed cost mechanism offers agent $i$ a unit of the good at price $x_i$. Indifferences are broken arbitrarily. That is, for the utility profile $u$, agent $i$ is guaranteed a unit at price $x_i$ if $u_i > x_i$. Agent $i$ does not get a unit if $u_i < x_i$. At $u_i = x_i$ he may or may not get a unit.

**Corollary 1** *A mechanism is welfare equivalent to a cross-monotonic and a valid sequential mechanism if and only if it is a fixed cost mechanism.*

The proof is in the appendix.

This result shows that the behavior of indifferences has a big impact on the class of $GSP$ mechanism. But one can argue that indifferences are rare events, so that a better model is one where the domain of utilities and the class of mechanisms preclude indifferences. In such a domain, the class of $GSP$ mechanisms will contain many more mechanisms than the sequential and cross-monotonic mechanisms. Juarez[2011b] analyzes such a domain and characterizes the corresponding $GSP$ mechanisms.

## 6.2 Equal treatment of equal agents

**Definition 12** *We say a mechanism $(G, \varphi)$ satisfies **equal treatment of equals (ETE)** if for any $u$ such that $u_i = u_j$, $i \in G(u)$ then $j \in G(u)$ and $\varphi_i(u) = \varphi_j(u)$.*

**Proposition 2** *A mechanism meets GSP and ETE if and only if it is welfare equivalent to a cross-monotonic mechanism with equal cost shares.[9]*

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[8]The intersection of sequential and cross-monotonic mechanisms contains only the trivial mechanism where no agent is served at any profile. This is because if agent $i$ is served at some profile $u$ paying $\infty > p_i \geq 0$, then at the profile $(p_i, u_{-i})$ agent $i$ would be served under a cross-monotonic mechanism but not so under a sequential mechanism. Therefore, no agent can be served at any profile.

[9]A **cross-monotonic mechanism with equal cost shares** is generated by a cross-monotonic set of cost shares that allocate the same payments to the agents who are being served at every set of cost shares. That is, the cost shares $x^S$ of the agents in $S$ are such that $x_i^S = x_j^S$ for all $i, j \in S$.

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The proof is in the appendix.[10]

This result is especially compelling when dividing costs that are symmetric. The cross-monotonic mechanisms with equal cost shares are the only $GSP$ mechanisms meeting the basic equity requirement of $ETE$. This proposition rules out sequential mechanisms and also those $GSP$ mechanisms discussed by Juarez[2011b] and Roughgarden et al.[2009].

The downside of this proposition is that many interesting applications have cost functions that are not symmetric (see, for instance, Roughgarden et al.[2009]), where $ETE$ does not make sense.

## 6.3   Limited number of goods

When a social planner or seller has (can produce) less than $n$ units of a good, it is impossible to meet simultaneously $ETE$ and $GSP$.[11] This is easy to check by looking at the utility profiles of the form $(x, \ldots, x)$, $x > 0$. By $ETE$, $G(x, \ldots, x) = \emptyset$ for all $x$. Hence, by Proposition 2 above and taking into account that the smallest cost share in a cross-monotonic mechanism is achieved when serving $N$, the mechanism should not allocate any units at all.

Moreover, when there is a scarcity of the good, cross-monotonic mechanisms exclude ex-ante some agents from the mechanism.[12] That is, if only $k$ units of the good are available, $k < n$, then any cross-monotonic mechanism is such that $n - k$ agents are not served at any profile. To see this, note that coalition $N$ never gets service; therefore, the cost shares of $N$ should have at least one coordinate equal to $\infty$. Thus, the agent $i$ with such a coordinate never participates in the game because his smallest cost share is achieved when serving $N$. We remove this agent from the game and proceed similarly with the remaining coalition $N \setminus i$, until we have removed at least $n - k$ agents.

On the other hand, there are many sequential mechanisms that do not ex-ante exclude any agent. For instance, if there are $k$ units of the good available, any valid sequential tree where every branch has at most $k$ finite prices will generate a $GSP$ mechanism that allocates at most $k$ units of the good at any utility profile. If $k \geq 3$, more complex $GSP$ mechanisms that are combinations of sequential and cross-monotonic mechanisms can be constructed (see Juarez[2011b] and Roughgarden et al.[2009]).

**Definition 13** *A mechanism $(G, \varphi)$ is a **priority mechanism** if there exist an ordering of the agents $i_1, \ldots, i_n$ and arbitrary prices $x^1, x^2, \ldots, x^n \in [0, \infty)$ such that for the utility profile $u$, if $u_{i_1} \leq x^1, u_{i_2} \leq x^2, \ldots u_{i_{k-1}} \leq x^{k-1}$ and $u_{i_k} > x^k$, then $i_k \in G(u)$ and $\varphi_{i_k}(u) = x^k$.*

A priority mechanism is such that agent $i_1$ is offered a unit of the good at price $x^1$. If $u_{i_1} > x^1$, then agent $i_1$ is served at price $x^1$ and the mechanism stops there. On the other

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[10]Anonymity in welfare (see Ashlagi et al.[2012]) requires that agents with identical utilities get the same net utilities. Since the mechanisms studied by this paper allocate zero payments to the agents who do not get a unit of the good, then $ETE$ is equivalent to anonymity in welfare at the utility profiles where agents get positive net utility. Proposition 2 also holds when replacing $ETE$ by anonymity in welfare.

[11]Except by the trivial mechanism that does not serve anyone at any profile.

[12]We say a mechanism **does not exclude ex-ante any agent** if for every agent $i$ there is a utility profile $u^i$ such that this agent is served.

hand, if $u_{i_1} \leq x^1$, then agent $i_2$ is offered a unit of the good at price $x^2$. If $u_{i_2} > x^2$, then agent $i_2$ is served at price $x^2$ and the mechanism stops there. On the other hand, if $u_{i_2} \leq x^2$, then agent $i_3$ is offered a unit of the good at price $x^3$, etc. That is, in a priority mechanism, only the first agent (following the priority ordering) who has utility larger than his respective price is served at that price.

Note that priority mechanisms are valid sequential mechanisms implemented by the valid sequential tree such that agents are ordered linearly following the order $i_1, \ldots, i_n$; only the left most branch of the tree has prices equal to $(x^1, x^2, \ldots, x^n)$ and any other node has a price equal to $\infty$.

**Proposition 3** *Suppose a mechanism is GSP, allocates at most one unit of the good at any utility profile and does not exclude ex-ante any agent, then the mechanism is welfare equivalent to a priority mechanism.*

The proof is in the appendix.

Note that this proposition is independent of the tie-breaking rule. In particular, it shows that when there is only one unit of the good available, a subclass of the valid sequential mechanisms are the only $GSP$ mechanisms that do not exclude ex-ante any agent.

The priority mechanisms are compelling when randomizations are disallowed. Giving priority to the agents is natural in settings with limited units of the good. For instance, when allocating a scarce drug to sick people, priority is often given to the sickest people first.

If randomizations are allowed, many more interesting mechanisms emerge. For instance, the mechanism that allocates each object to the agents for free with probability $\frac{1}{n}$ is $GSP$ and always allocates the object. This mechanism also satisfies $ETE$.

## 6.4 Feasible cost sharing mechanisms

A **cost function** is a non-negative function $C : 2^N \to \mathbb{R}_+$ such that $C(S) \leq C(T)$ for $S \subseteq T$. It specifies the cost of serving every coalition of agents.

We say a mechanism $(G, \varphi)$ is **feasible** for the cost function $C$ if $\sum_{i \in G(u)} \varphi_i(G(u)) \geq C(G(u))$ for all utility profiles $u$. A feasible mechanism collects at least the cost of serving the agents $G(u)$ at every utility profile $u$.

A mechanism is **budget-balanced** for the cost function $C$ if the cost is exactly collected at every utility profile $u$. That is, $\sum_{i \in G(u)} \varphi_i(G(u)) = C(G(u))$ for all utility profiles $u$.

Given an arbitrary cost function $C$, there exist sequential and cross-monotonic mechanisms that are feasible for $C$. Indeed, for any mechanism that serves agents at non-zero cost, we can always find a large enough homothetic expansion of the set of payments that make the corresponding mechanism feasible for the cost function. For instance, for the cross-monotonic set of cost shares, $\chi$, consider the cross-monotonic set of cost shares $\lambda \cdot \chi = \{\lambda \cdot x^S | x^S \in \chi\}$ for some $\lambda > 0$. If $x^S \neq 0$ for all $S$, we can always find a large $\lambda$ that makes the cross-monotonic mechanism generated by $\lambda \cdot \chi$ feasible for the cost function $C$. Nevertheless, such a mechanism could be very wasteful (charge the agents too much).

The recent literature deals with the role of wastefulness. In particular, the companion paper Juarez[2011a] characterizes optimal feasible mechanisms (using the worst absolute surplus loss measure) for an arbitrary *symmetric* cost function. When the cost function has decreasing average cost, Theorem 1 in Juarez[2011a] shows that the optimal $GSP$ mechanism would be the cross-monotonic mechanism where $x_i^S = AC(S)$ for all $i \in S$.[13] On the other hand, the optimal mechanism for a cost function with increasing average cost would be a sequential mechanism.

Other measures that approximate budget balance have been discussed in the literature. For instance, a mechanism is $\beta-$**budget balanced** for the cost function $C$ if $\frac{C(G(u))}{\beta} \leq \sum_{i \in G(u)} \varphi_i(G(u)) \leq C(G(u))$ for all utility profiles $u$. For different shapes of cost functions, this relaxation of budget balance can greatly improve the efficiency of the GSP mechanisms (see, for instance, Roughgarden et al.[2006a, 2009]).

### 6.4.1 Cross-monotonic mechanisms

Moulin[1999] shows that in the space of submodular cost functions, any mechanism that is budget-balanced, $GSP$ and satisfies a strong consumer sovereignty condition should be implemented as a cross-monotonic mechanism for a set of cross-monotonic and budget-balanced cost shares. The result proposed by Theorem 1 is more general. We show that cross-monotonic mechanisms emerge simply from the combination of $GSP$ and $MAX$. However, as shown in example 7, this does not imply that the cost function defined by $C(S) = \sum_{i \in S} x_i^S$ is submodular. Hence, we capture Moulin's mechanisms and a few more.

**Example 7** *Consider any cost function $C : 2^N \to \mathbb{R}_+$ such that its average cost function $AC$, $AC(S) = \frac{C(S)}{|S|}$, is not increasing as the coalition increases, that is, $AC(S) \leq AC(T)$ for all $T \subset S \subseteq N$.*

*$x_i^S = AC(S)$ if $i \in S$, $x_i^S = 0$ if $i \notin S$, defines a cross-monotonic set of cost shares that covers the cost exactly.*

*It is easy to see that the monotonicity of $AC$ does not imply the submodularity of $C$. Hence, there are cross-monotonic sets of cost shares whose associated cost function is not submodular.*

*Conversely, the cost function generated by a cross-monotonic mechanism with equal cost shares is such that the average cost function $AC$ is not increasing as the coalition increases.*

The problem of finding the cost function generated by an arbitrary (non-symmetric) cross-monotonic set of cost shares is a difficult problem. Sprumont[1990] and Norde et al.[2002] provide characterizations of these cost functions in simple cases.

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[13]This is similar to the main result of Moulin and Shenker[2001], except that Juarez[2011a] allows more general mechanisms that can generate a budget surplus, whereas Moulin and Shenker[2001] cover only the class of budget-balanced mechanisms.

### 6.4.2 Sequential mechanisms

Sequential mechanisms are related to the incremental cost mechanisms of Moulin[1999]. That is, consider a supermodular (convex) cost function and a sequential tree. Start with the agent $i_1$ in the root and offer him a unit of the good at price $C(i_1)$. If he buys, continue with the agent $i_2$ on the right of the tree and offer him a unit of the good at price $C(i_1, i_2) - C(i_1)$. If $i_1$ did not buy, then offer the agent on the left of the tree, $k_2$, a unit of the good at price $C(k_2)$. Proceed similarly with the following agents until you reach the end of the tree.

Theorem 1 in Moulin[1999] suggests that incremental cost mechanisms are $GSP$ mechanisms when the cost function is supermodular. However, this is not true, as shown in the next example.

**Example 8** *Consider the supermodular cost function:*

$$C(i) = 1, C(1,2) = 3, C(1,3) = 5, C(2,3) = 6, C(1,2,3) = 15.$$

*By choosing the ordering $1 \succ 2 \succ 3$, the cost shares are as follows:*
*$x^{\{1,2,3\}} = (1,2,12), x^{\{1,2\}} = (1,2,0), x^{\{1,3\}} = (1,0,4), x^{\{2,3\}} = (0,1,5), x^{\{i\}} = 1_i$.*

*When the utility profile is $u = (1, 1.5, 4.5)$ there are two options depending on whether 1 decides to get or not get a unit. If agent 1 gets a unit, then 2 does not get a unit and 3 gets a unit. Thus, $\{1,3\}$ is served and the cost shares are $(1,0,4)$. If agent 1 does not get a unit, then 3 does not get a unit. Thus, $\{2\}$ is served and the cost shares are $(0,1,0)$. Given that 1 is indifferent between getting and not getting a unit, he may help 2 or 3. Thus, the mechanism cannot be $GSP$.*

What is important from Moulin[1999] is that incremental cost mechanisms may not be fully $GSP$, but they are $GSP$ except when agents are indifferent between getting and not getting a unit of the good. Thus, the mistake is minor.

Whenever the supermodular cost function and the ordering of the agents give a sequential mechanism that is valid, it must be captured by a sequential mechanism discussed in section 5.

On the other hand, given a valid sequential mechanism, the associated budget balance cost function (the cost of $S$ defined as the sum of the payments when $S$ is served) may not be supermodular. Therefore, valid sequential mechanisms capture even more mechanisms than those generated by the incremental cost mechanisms.

Given the difficulty of describing the class of valid sequential mechanisms, it is nearly impossible to describe the class of cost functions generated by arbitrary valid sequential mechanisms. The exception to this difficulty comes when we restrict our attention to valid sequential mechanisms under equal sharing. The typical cost functions (up to renaming the agents) can be described by

$$C(S) = |S|(\max_{k \in S} a_k) \text{ where } a_1 \geq a_2 \geq \cdots \geq a_n \geq 0.$$

Note that even these types of cost functions might not be supermodular. Indeed, we can easily find values such that $C(1,3) + C(2,3) = 2a_1 + 2a_2 > 3a_1 + a_3 = C(1,2,3) + C(3)$.

# 7 Conclusions

This paper characterizes the $GSP$ mechanisms under two alternative continuity conditions. On the one hand, cross-monotonic mechanisms are characterized by $GSP$ and $MAX$. These mechanisms are very useful when symmetry is required. However, they are very inefficient when there is a scarcity of the good.

On the other hand, sequential mechanisms are characterized by $GSP$ and $MIN$. These mechanisms are appropriate when there is a scarcity of the good, for instance, when there is only one unit of the good available. Unfortunately, deterministic sequential mechanisms fail standard equity tests such as equal treatment of equals.

The combination of $GSP$ and equal treatment of equals leads to a subset of the cross-monotonic mechanisms. On the other hand, when there is only one unit of the good available, the only $GSP$ mechanisms are a subset of the sequential mechanisms. The welfare-wise intersection of cross-monotonic and sequential mechanisms leads to the fixed cost mechanisms, where agents are offered a unit of the good at a price that is independent of the reports of the other agents.

Group strategyproof mechanisms without any of the two alternative continuity conditions can be easily constructed; for instance, some priority compositions of sequential and cross-monotonic mechanisms are $GSP$ (see Juarez[2011b] or Roughgarden[2009]). Similarly to Juarez[2011b], Pountourakis et al.[2010] provide abstract conditions on the set of cost shares that characterize $GSP$ mechanisms in a limited economy without indifferences (where a given coalition will have at most one vector of payments). This shows the difficulty of providing a tractable description of the entire class of $GSP$ mechanisms in a more traditional economy such as the one studied in this paper.

# Proofs

**Proof of Theorem 1.**

Cross-monotonic mechanisms clearly meet $MAX$. See Moulin[1999] for the proof that cross-monotonic mechanisms meet $GSP$.

**Any mechanism that is $MAX$ and $GSP$ is cross-monotonic.**

Let $(G, \varphi)$ be a mechanism that meets these properties. Recall that $f_i(u_{-i})$ is the price agent $i$ should pay to get a unit of the good when the utilities of the remaining agents are $u_{-i}$. Also, $NU_i(u)$ denotes the net utility of agent $i$ at the profile $u$.

The proof of this part is divided into four steps. Steps 1 and 2 are very similar to steps 1 and 2 in the proof of Theorem 2. However, the step 1 in this proof is more involved because $MAX$ does not imply that an agent is served if and only if his net utility is positive.

**Step 0.**[Monotonicity] $f_j(\widetilde{u}_i, u_{-ij}) \leq f_j(u_i, u_{-ij})$ for all $\widetilde{u}_i > u_i$.

Proof.

We prove this by contradiction. Suppose $f_j(\widetilde{u}_i, u_{-ij}) > f_j(u_i, u_{-ij})$. Let $v_j$ be such that $f_j(\widetilde{u}_i, u_{-ij}) > v_j > f_j(u_i, u_{-ij})$.

**Case 1.** $f_i(v_j, u_{-ij}) > \widetilde{u}_i$.

By $SP$, agent $i$ is not served at the profiles $(\widetilde{u}_i, v_j, u_{-ij})$ and $(u_i, v_j, u_{-ij})$ because

$$f_i(v_j, u_{-ij}) > \widetilde{u}_i > u_i.$$

Hence when the true utility profile is $(\widetilde{u}_i, v_j, u_{-ij})$, agent $i$ can help $j$ by misreporting $u_i$. This contradicts $GSP$.

**Case 2.** $f_i(v_j, u_{-ij}) \leq u_i$.

By $SP$ and $MAX$, agent $i$ is served at the profiles $(\widetilde{u}_i, v_j, u_{-ij})$ and $(u_i, v_j, u_{-ij})$ because $f_i(v_j, u_{-ij}) \leq u_i < \widetilde{u}_i$. Hence, similarly to case 1, when the true utility profile is $(\widetilde{u}_i, v_j, u_{-ij})$, agent $i$ can help $j$ by misreporting $u_i$. This also contradicts $GSP$.

**Case 3.** $u_i < f_i(v_j, u_{-ij}) \leq \widetilde{u}_i$.

Let $w_i = f_i(v_j, u_{-ij})$. By $SP$ and $MAX$, agent $i$ is being served at price $w_i$ at the profiles $(\widetilde{u}_i, v_j, u_{-ij})$ and $(w_i, v_j, u_{-ij})$. Thus, by $GSP$ $f_j(w_i, u_{-ij}) \geq v_j$. To see this, assume $f_j(w_i, u_{-ij}) < v_j$. Then, when the true profile is $(\widetilde{u}_i, v_j, u_{-ij})$, agent $i$ helps $j$ by misreporting $w_i$ because agent $j$ is not served at the profile $(\widetilde{u}_i, v_j, u_{-ij})$, whereas he would be served at the profile $(w_i, v_j, u_{-ij})$. This contradicts $GSP$. Therefore, $f_j(w_i, u_{-ij}) \geq v_j$.

Hence, at the true profile $(w_i, v_j, u_{-ij})$, agents $i$ and $j$ get zero net utility because $f_j(w_i, u_{-ij}) \geq v_j$ and $w_i = f_i(v_j, u_{-ij})$. Thus, agent $i$ helps $j$ by reporting $u_i$ : A1gent $i$ is not served at the misreporting because $u_i < f_i(v_j, u_{-ij})$; however, agent $j$ is better off because $v_j > f_j(u_i, u_{-ij})$. This contradicts $GSP$.

**Step 1.** If $G(u) = S^*$ and $\varphi(u) = \varphi^*$, then for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} \geq \varphi_{S^*}$ and $\widetilde{u}_{N \setminus S^*} \leq u_{N \setminus S^*}$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

Proof.

We prove step 1 in steps 1.1 and 1.2.

**Step 1.1.** Let $i \in S^*$ and $\widetilde{u}_i \geq f_i(u_{-i}) = \varphi_i^*$. We will prove that $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

First, note that by $SP$ and $MAX$, $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi_i^*$.

Second, note that $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then when the true profile is $u$, agent $i$ helps $j$ by reporting $\widetilde{u}_i$. This contradicts $GSP$. Similarly, if $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$, then agent $i$ helps $j$ by misreporting $\widetilde{u}_i$ when the true utility profile is $u$.

Third, note that if $j \in S^* \setminus i$ and $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) > 0$, then $j \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_j(\widetilde{u}_i, u_{-i}) = \varphi_j^*$.

Finally, to get a contradiction, assume that $G(\widetilde{u}_i, u_{-i}) \neq S^*$. Then, there is an agent $j$ such that $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$ and either (A.1.) $j \in S^*$ but $j \notin G(\widetilde{u}_i, u_{-i})$ or (A.2.) $j \notin S^*$ but $j \in G(\widetilde{u}_i, u_{-i})$. We show next that these situations cannot occur.

**Case A.1.** $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$. Assume $j \in S^*$ but $j \notin G(\widetilde{u}_i, u_{-i})$.

Since $j \notin G(\widetilde{u}_i, u_{-i})$, by $SP$ and $MAX$ $f_j(\widetilde{u}_i, u_{-ij}) > u_j = f_j(u_{-j})$. Thus, by step 0, $u_i > \widetilde{u}_i \geq \varphi_i^*$.

Let $v_j$ be such that $v_j > u_j$. By step 0,

$$f_i(v_j, u_{-ij}) \leq f_i(u_j, u_{-ij}) = \varphi_i^* \leq \widetilde{u}_i < u_i.$$

Therefore, when the true profile is $(\widetilde{u}_i, v_j, u_{-ij})$, agent $i$ can help $j$ by misreporting $u_i$: Agent $i$ is served in both profiles at price $f_i(v_j, u_{-ij})$; however, agent $j$ is offered a unit at the cheaper price $f_j(u_{-j})$ when $i$ misreports. This contradicts $GSP$.

**Case A.2.** $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u) = 0$. Assume $j \notin S^*$ but $j \in G(\widetilde{u}_i, u_{-i})$.

By $SP$ and $MAX$, $f_j(\widetilde{u}_i, u_{-ij}) = u_j > f_j(u_{-j})$. So, we are in exactly the same situation as case A.1 by switching the role of $\widetilde{u}_i$ and $u_i$. Therefore, this case cannot occur.

By repeatedly applying step 1.1 to every agent in $S^*$, we have that $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

**Step 1.2.** Let $j \notin S^*$ be such that $\widetilde{u}_j < u_j$. Then $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

Since $\widetilde{u}_j < u_j < f_j(\widetilde{u}_{S^*}, u_{-S^* \cup j})$, then by $SP$ $j \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. Similarly to step 1.1, by $GSP$ $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$.

Assume that $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) \neq S^*$. Clearly, if $NU_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) > 0$ for some $k \neq j$, then $k \in S^*$, $k \in G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$ and $\varphi_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi_k^*$.

Thus, there is $k$ such that $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$ and either (B.1) $k \in S^*$ but $k \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$; or (B.2) $k \notin S^*$ but $k \in G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$. We show next these cases cannot occur.

**Case B.1.** $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$. Assume $k \in S^*$ and $k \notin G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$.

By $SP$ and $MAX$,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > \widetilde{u}_k = f_k(u_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}). \tag{1}$$

Let $v_k$ be such that $v_k > \widetilde{u}_k$. By monotonicity (step 0):

$$f_j(v_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}).$$

First, we assume that

$$f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \geq f_j(v_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) > u_j > \widetilde{u}_j.$$

Then, when the true profile is $(v_k, \widetilde{u}_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$, agent $j$ can help agent $k$ by misreporting $u_j$ : Agent $j$ does not get a unit in either profile; however, by equation 1, agent $k$ gets a unit at the cheaper price $f_k(u_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$ when $j$ misreports $u_j$. This contradicts $GSP$.

On the other hand, we now assume

$$f_j(v_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq u_j < f_j(\widetilde{u}_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}). \tag{2}$$

Let $v_j$ be such that $v_j > u_j$. By step 0,

$$f_k(v_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) \leq f_k(u_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j}) = \widetilde{u}_k < v_k. \tag{3}$$

Thus, when the true profile is $(\widetilde{u}_k, v_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$, agent $k$ helps $j$ by misreporting $v_k$ : By equation 3, agent $k$ is served at a price $f_k(v_j, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$ in either profile; however, by equation 2 agent $j$ is served at the cheaper price $f_j(v_k, \widetilde{u}_{S^* \setminus k}, u_{-S^* \cup j})$ when $k$ misreports. This contradicts $GSP$.

Hence, if $k \in S^*$, then $k \in G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$ and $\varphi_k(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi_k(\widetilde{u}_{S^*}, u_{-S^*})$.

**Case B.2.** $NU_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*}) = 0$. Assume $k \notin S^*$ and $k \in G(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j})$.
By $SP$ and $MAX$,

$$f_k(\widetilde{u}_j, \widetilde{u}_{S^*}, u_{-S^* \cup j,k}) = u_k < f_k(u_j, \widetilde{u}_{S^*}, u_{-S^* \cup j,k}).$$

However, this contradicts monotonicity (step 0) because $\widetilde{u}_j < u_j$.

By applying step 1.2 for every agent in $N \setminus S^*$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

**Step 2.** If $G(u) = G(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.
Proof.
Let $S^* = G(u) = G(\widetilde{u})$, $v_S = \max(\widetilde{u}_S, u_S)$ and $v_{N \setminus S} = \min(\widetilde{u}_{N \setminus S}, u_{N \setminus S})$ (where max and min are taken coordinate by coordinate).
By step 1, comparing $v$ and $u$, $G(v) = S^*$ and $\varphi(v) = \varphi(u)$. Similarly, comparing $v$ and $\widetilde{u}$, $\varphi(v) = \varphi(\widetilde{u})$. Hence $\varphi(u) = \varphi(\widetilde{u})$.

**Step 3.**

In this final step we prove the theorem by induction on the number of agents. The base of induction is the case $n = 1$. The mechanisms are easy to construct. Given $x \in [0, \infty]$, if $u_1 \geq x$, then $(G, \varphi)(u_1) = (\{1\}, x)$. On the other hand, if $u_1 < x$, then $(G, \varphi)(u_1) = (\emptyset, 0)$. These mechanisms are clearly cross-monotonic.

For the induction hypothesis, assume that any $GSP$ and $MAX$ mechanism for $k$ agents, $k < n$, is cross-monotonic. We prove this for the $n-$agent case. The proof is divided in a long Case 1 and a short Case 2.

Let $(G, \varphi)$ be a $GSP$ and $MAX$ mechanism defined for the agents $N = \{1, \ldots, n\}$.

**Case 1.** Assume that there is a utility profile $u^*$ such that $G(u^*) = N$.

Let $x^N = \varphi(u^*)$. By step 1, for all $\widetilde{u} \geq x^N$, $G(\widetilde{u}) = N$ and $\varphi(\widetilde{u}) = x^N$.

For every agent $j \in N$, consider the set of utility profiles such that $u_j = 0$, that is

$$U^j = \{u \in \mathbb{R}^N_+ \mid u_j = 0\}.$$

Let $(G^j, \varphi^j)$ denote the restriction of $(G, \varphi)$ to $U^j$. That is, $G^j(v) = G(v, 0) \cap (N \setminus j)$ and $\varphi^j(v) = \varphi(v, 0)_{N \setminus j}$ for all $v \in \mathbb{R}^{N \setminus j}_+$. By the induction hypothesis, $(G^j, \varphi^j)$ is a cross-monotonic mechanism in $U^j$. Therefore, there exists a cross-monotonic set of cost shares that define $(G^j, \varphi^j)$. Let $\widetilde{\rho}^j$ be such cross-monotonic set of cost shares.

Let $S^* = \max\{G^j(u) | u \in U^j\}$ where the maximum is taken with the inclusion $\subseteq$. By cross-monotonicity of $(G^j, \varphi^j)$, this maximum is reached at some utility profile in $U^j$. Notice that in any mechanism with finite cost shares, $S^* = N \setminus j$. However, if there are some cost shares with infinite values, it might be that $S^* \subsetneq N \setminus j$.

Also, notice that two different sets of cost shares can generate the same mechanism. For instance, in the case of two agents, the trivial mechanism ($S^j(u) = \emptyset$ for all $u^j$) can be simultaneously defined by the cross-monotonic set of cost shares $\{x^\emptyset = (0,0), x^1 = (\infty, 0), x^2 = (0, \infty), x^{12} = (\infty, \infty)\}$ or by the set of cost shares $\{x^\emptyset = (0,0), x^1 = (\infty, 0), x^2 = (0, \infty), x^{12} = (10, \infty)\}$. In order to simplify the proof, we will use the former rather than the latter.

That is, let $\bar{\rho}^j = \{y^T \in R^{N \setminus j}_+ | T \in 2^{N \setminus j}\}$ be the set of cost shares defined as follows for a coalition $T \in 2^{N \setminus j}$:

i. $y^T = x^T$ if $T \subseteq S^*$ and $x^T \in \widetilde{\rho}^j$.

ii. $y^T_i = \infty$ if $i \in T \setminus S^*$ and $T \nsubseteq S^*$.

iii. $y^T_i = x^{S^*}_i$ if $i \in S^* \cap T$ and $T \nsubseteq S^*$; where $x^{S^*} \in \widetilde{\rho}^j$.

Clearly, if $S^* = N \setminus j$, then $\bar{\rho}^j = \widetilde{\rho}^j$. If $S^* \neq N \setminus j$, then it might be that $\bar{\rho}^j \neq \widetilde{\rho}^j$, as we saw in the example above. Nevertheless, we see below that $\bar{\rho}^j$ will generate the mechanism $(G^j, \varphi^j)$.

First, we show that $\bar{\rho}^j$ is a cross-monotonic set of cost shares. Indeed, consider $L \subset M \subseteq N \setminus j$ and $k \in L$. If $k \notin S^*$, then by part $ii$ : $y^M_k = y^L_k = \infty$. Now assume that $k \in S^*$. If $L \subset M \subseteq S^*$, then by part $i$ : $y^M_k = x^M_k \leq x^L_k = y^L_k$ where $x^M, x^L \in \widetilde{\rho}^j$. If $M \nsubseteq S^*$, then by part $iii$ : $y^M_k = x^{S^*}_k \leq y^L_k$.

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Next, we show that $\bar{\rho}^j$ generates the mechanism $(S^j, \varphi^j)$. Indeed, let $v$ be a utility profile for $N \setminus j$ agents. Then $G^j(v) \subseteq S^*$ by definition of $S^*$. If $T \nsubseteq S^*$, then $T$ is not reachable at $v$ because $y_k^T = \infty$ for $k \in T \setminus S^*$. Since $G^j(v)$ is the maximum reachable coalition for the utility profile $v$ using cost shares in $\widetilde{\rho}^j$ and $\bar{\rho}^j$ coincides with $\widetilde{\rho}^j$ for any subset in $2^{S^*}$, then $G^j(v)$ is the maximum reachable coalition for the utility profile $v$ using cost shares in $\bar{\rho}^j$.

Let $\rho^j$ be the expansion of $\bar{\rho}^j$ to $U^j$ by adding a $j-$th coordinate equal to zero (that is, the expansion from the space $\mathbb{R}^{N \setminus j}$ to $\mathbb{R}^N$).

Recall that by the induction hypothesis, for every $j \in N$ we are able to construct the cross-monotonic set of cost $\rho^j$ in $U^j$ that generates the mechanism $(G, \varphi)$ restricted to $U^j$ (i.e., $\rho^j$ generates the mechanism $(G^j, \varphi^j)$).

Now, we define a cross-monotonic set of cost shares that will generate the mechanism $(G, \varphi)$ in the whole space of utilities $\mathbb{R}_+^N$. In order to do this, we will use the cross-monotonic set of cost shares $\rho^j$ defined in every $U^j$ for $j \in N$.

Let $\rho^* = \{\tilde{x}^T \in \mathbb{R}^N | T \subseteq N\}$ denote a set of cost shares defined as follows for a coalition $T \subsetneq N$:

    i. $\tilde{x}^T = \max_{\{x^T \in \rho^j | j \in N \setminus T\}} x^T$, where max is taken coordinate by coordinate.

    ii. The cost share of coalition $N$ is simply $x^N$ (which exists by the assumptions of case 1 and is unique by step 2).

**Step 3.1** $\rho^*$ is a cross-monotonic set of cost shares.
Proof.
Let $S \subset T \subsetneq N$ and $k \in S$. Note that $x_k^S \geq x_k^T$ holds for any $i \in N \setminus T$, $x^S, x^T \in \rho^i$ by cross-monotonicity on $\rho^i$. Therefore:

$$\tilde{x}_k^T = \max_{\{x^T \in \rho^j | j \in N \setminus T\}} x_k^T \leq \max_{\{x^T \in \rho^j | j \in N \setminus T\}} x_k^S$$

since $N \setminus T \subset N \setminus S$:

$$\max_{\{x^T \in \rho^j | j \in N \setminus T\}} x_k^S \leq \max_{\{x^T \in \rho^j | j \in N \setminus S\}} x_k^S = \tilde{x}_k^S$$

Therefore: $\tilde{x}_k^T \leq \tilde{x}_k^S$ for $S \subset T \subsetneq N$ and $k \in S$.

Now, we check that $x_i^N \leq \tilde{x}_i^{N \setminus j}$ for all $j \in N$, $i \in N \setminus j$ and $\tilde{x}_i^{N \setminus j} \in \rho^*$. First notice $\tilde{x}^{N \setminus j} = x^{N \setminus j} \in \rho^j$. Let $S^*$ defined as above be the maximum coalition that is served by $(G^j, \varphi^j)$.[14] By the choice of $\bar{\rho}^j$, if $i \in N \setminus (j \cup S^*)$, then $x_i^{N \setminus j} = \infty > x_i^N$. On the other hand, if $i \in S^*$, then $x_i^{N \setminus j} = x_i^{S^*} \in \rho^j$. Let $u \in U^j$ be such that $G^j(u) = S^*$ ($u$ exists by definition of $S^*$). If $i \in S^*$, then $\varphi_i^j(u) = \varphi_i^j(u) = x_i^{S^*}$ by the definition of $(G^j, \varphi^j)$. Let $\widetilde{u} = (x_i^N, \max(x_{-i}^N, u_{-i}))$. Since $\widetilde{u} \geq x^N$, then $G(\widetilde{u}) = N$ holds by step 1. Thus, $x_i^N = f_i(\widetilde{u}_{-i})$. On the other hand, by step 0, $x_i^N = f_i(\widetilde{u}_{-i}) \leq f_i(u_{-i})$. Since $f_i(u_{-i}) = \varphi_i(u) = x_i^{S^*}$, then $x_i^N \leq x_i^{S^*}$.

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[14]See the formal definition on the fourth paragraph after the start of case 1.

**Step 3.2** The cost shares from $\rho^*$ will coincide with the payments given by the mechanism $(G, \varphi)$. That is, $\varphi(u) = \tilde{x}^{G(u)} \in \rho^*$ for all $u$.

Proof.

First, by assumption there is $u^*$ such that $G(u^*) = N$ and we defined above $x^N = \varphi(u^*) \in \rho^*$ By step 1, $G(u) = N$ if and only if $u \geq x^N$. Moreover, by step 2, if $G(u) = N$, then $\varphi(u) = x^N \in \rho^*$.

Second, consider a coalition $\bar{S} \neq N$ and a utility profile such that $G(u) = \bar{S} \neq N$. We claim that $\varphi(u) = \tilde{x}^{\bar{S}} \in \rho^*$. To see this, by step 1 for any $j \in N \setminus \bar{S}$, $\varphi(u) = \varphi(0, u_{-j}) = x^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$. Thus, for any $i, j \in N \setminus \bar{S}$, $x^{\bar{S}} = \varphi(u) = y^{\bar{S}}$ where $x^{\bar{S}} \in \rho^j$ and $y^{\bar{S}} \in \rho^i$. Hence, $\tilde{x}^{\bar{S}} = \max_{\{x^{\bar{S}} \in \rho^j | j \in N \setminus T\}} x^{\bar{S}} = x^{\bar{S}} = \varphi(u)$.

**Step 3.3** The mechanism $(G, \varphi)$ is cross-monotonic.

Proof.

Since the payments in $\rho^*$ are cross-monotonic and the payments in the mechanism $(G, \varphi)$ coincides with them, then an agent $i$ cannot be served if his utility is smaller than $x_i^N$. Hence, the mechanism $(G, \varphi)$ satisfies:

- If $u \geq x^N$, then $G(u) = N$ and $\varphi(u) = x^N$.

- If for some $i$, $u_i < x_i^N$, then $i \notin G(u)$. Thus, by step 1 $(G, \varphi)(u) = (G, \varphi)(0, u_{-i}) = (G^i(u_{N \setminus i}), \tilde{x}^{G^i(u_{N \setminus i})})$.

Now we check that $(G, \varphi)$ is the cross-monotonic mechanism generated by $\rho^*$. If $u \geq x^N$, then $G(u) = N$ and obviously $N$ is the largest reachable coalition in $\rho^*$.

Assume $u$ is such that $u_i < x_i^N$ for some agent $i$. Let $S^* = G(u) = G^i(u_{N \setminus i})$. Let $F^*(u)$ be the set of reachable coalitions at $u$ for the cross-monotonic set of payments $\rho^*$ and $F^j(u)$ be the set of reachable coalitions at $u$ for the cross-monotonic set of payments $\rho^j$. Since $\tilde{x}^T \geq x^T$ for any $T \subseteq N$, $\tilde{x}^T \in \rho^*$ and $x^T \in \rho^j$, then $\max F^*(u) \subseteq \max F^j(u)$. By cross-monotonicity of the mechanism $(G^j, \varphi^j) : \max F^j(u) = S^*$. Therefore, $\max F^*(u) \subseteq S^*$.

By step 3.2, $\varphi(u) = \tilde{x}^{S^*}$, therefore $u \geq \tilde{x}^{S^*}$, thus $S^* \in F^*(u)$, therefore $S^* \subseteq \max F^*(u)$. Hence, $\max F^*(u) = S^*$.

**Case 2.** Assume that there is no $u^*$ such that $G(u^*) = N$.

We will show that there is $j \in N$ such that $j \notin G(\tilde{u})$ for all $\tilde{u}$. We prove this by contradiction. Assume that for any $j$ there is $u^j$ such that $j \in G(u^j)$. Let $v = \max(u^1, \ldots, u^n)$ where max is taken coordinate by coordinate. By step 0, at $v$ every agent $j$ is offered a unit of the good at a price not greater than $u_j^j$; thus, $j \in G(v)$ for all $j \in N$. This is a contradiction.

Since there is an agent who is not served at any profile, say, agent $j^*$, then by step 1 $(G, \varphi)(u) = (G, \varphi)(u_{-j^*}, 0)$ for all $u$. Hence, by the induction hypothesis the mechanism is cross-monotonic.

**Proof of Proposition 1.**

$\Leftarrow$)

We prove it by contradiction. We start with a sequential tree and consider two nodes $\zeta$ and $\zeta'$ with a common agent $k$ such that $\zeta'$ is realizable by indifferent agents from $\zeta$. Suppose that $x_k^\zeta > x_k^{\zeta'}$.

Then, there exist nodes $\tilde{\zeta} \in P(\zeta)$ and $\bar{\zeta} \in P(\zeta')$ with a common agent $i$ such that condition (a) or (b) is satisfied.

Consider the utility profile $u$ that visits $\zeta$, the group of indifferent agents $S$ and $\tilde{u}_S > u_S$ such that $(\tilde{u}_S, u_{-S})$ visits $\zeta'$.

Suppose that condition (a) is satisfied. First, notice $i \notin S$ because $x_i^\zeta < x_i^{\zeta'}$. Hence $(\tilde{u}_S, u_{-S})$ cannot visit $\zeta'$ because $u_i \leq x_i^\zeta < x_i^{\zeta'}$ but $\bar{\zeta}$ is winning in $P(\zeta')$. This is a contradiction.

Suppose condition (b) is satisfied. Since $\tilde{\zeta}$ is winning in $P(\zeta)$ and $x_i^\zeta \geq x_i^{\zeta'}$ then $u_i > x_i^\zeta \geq x_i^{\zeta'}$. Therefore $(\tilde{u}_S, u_{-S})$ cannot visit $\zeta'$ because $\bar{\zeta}$ is losing in $P(\zeta')$ but $u_i > x_i^{\zeta'}$. This is a contradiction.

Hence $x_k^\zeta \leq x_k^{\zeta'}$.

$\Rightarrow)$

We prove it by contradiction. We start with a sequential tree and consider two nodes $\zeta$ and $\zeta'$ with a common agent $k$ such that $\zeta$ is on the left of $\zeta'$. Also assume that $x_k^\zeta > x_k^{\zeta'}$.

Suppose that any two nodes $\tilde{\zeta} \in P(\zeta)$ and $\bar{\zeta} \in P(\zeta')$ with a common agent $i$ do not meet any of the conditions (a) or (b) of Proposition 1. Then, such nodes should meet any of the following condition:

1. $\tilde{\zeta}$ is losing in $P(\zeta)$, $\bar{\zeta}$ is winning in $P(\zeta')$ and $x_i^\zeta \geq x_i^{\zeta'}$.

2. $\tilde{\zeta}$ is winning in $P(\zeta)$ and $\bar{\zeta}$ is losing in $P(\zeta')$ and $x_i^\zeta < x_i^{\zeta'}$.

3. $\tilde{\zeta}$ and $\bar{\zeta}$ are losing in $P(\zeta)$ and $P(\zeta')$.

4. $\tilde{\zeta}$ and $\bar{\zeta}$ are winning in $P(\zeta)$ and $P(\zeta')$.

Let $i^*$ be the agent in the terminal node of $P(\zeta) \sqcap P(\zeta')$. Fix a utility profile $u$ such that:

a. $u_{i^*}$ equals the price of his node.

b. $u_k$ such that $x_k^\zeta > u_k > x_k^{\zeta'}$

c. $u_i = x_i^\zeta$ if condition 1 holds.

d. $u_i = \frac{x_i^\zeta + x_i^{\zeta'}}{2}$ if condition 2 holds.

e. $u_i = 0$ if condition 3 holds.

f. $u_i$ such that $u_i > \max(x_i^\zeta, x_i^{\zeta'})$ if condition 4 holds.

*g.* If $j$ is a winning agent in a node that belongs to $P(\zeta)$ but does not belong to $P(\zeta')$, or if $j$ is a winning agent in a node that belongs to $P(\zeta')$ but does not belong to $P(\zeta)$, then $u_j$ is bigger than the price of its node.

*h.* If $j$ is a losing agent in a node that belongs to $P(\zeta)$ but does not belong to $P(\zeta')$, or if $j$ is a losing agent in a node that belongs to $P(\zeta')$ but does not belong to $P(\zeta)$, then $u_j = 0$.

*i.* Any other agent has zero utility.

First, note that the profile $u$ realizes the path $P(\zeta)$.

If an agent is losing in $P(\zeta)$, then either his utility equals zero, or condition 1 is satisfied, or he is $i^*$. If his utility equals zero, by $MIN$ he is not served. If condition 1 is satisfied, then $u_i = x_i^{\zeta'} \leq x_i^{\zeta}$ so he is not served. If he is $i^*$, then his utility equals the price of his node, so he is not served.

On the other hand, if an agent is winning in $P(\zeta)$, then his utility is greater than the price of his node. To see this, if condition 2 is satisfied, then by part $d$ he is served. If condition 4 is satisfied, then by part $f$ he is served. The remaining winning agents are served by part $g$.

Let $T$ be the agents who meet condition 1 and $S = T \cup \{i^*\}$. Notice that $u_S = x_S^{\zeta}$ and $x_S^{\zeta'} \leq x_S^{\zeta}$.

Let $\widetilde{u}_S$ be such that $\widetilde{u}_j > u_j$ if $j \in T \cup \{i^*\}$. Then the utility profile $(\widetilde{u}_S, u_{-S})$ realizes the path $P(\zeta')$.

To see this, an agent $j$, $j \neq i^*$, whose node is in $(P(\zeta) \sqcap P(\zeta'))$ did not change his utility, so $(P(\zeta) \sqcap P(\zeta'))$ is realized.

If $i$ meets condition 2, then $u_i = \frac{x_i^{\zeta} + x_i^{\zeta'}}{2} < x_i^{\zeta'}$, so he is not served.

If $i$ meets condition 3, then by $e$ his utility equals zero; thus, he is not served.

If $i$ meets condition 4, then by $f$ his utility is bigger than $x_i^{\zeta'}$; thus, he is served.

If $i \in T \cup i^*$, then he is winning in $P(\zeta')$ because $\widetilde{u}_i > u_i \geq x_i^{\zeta} \geq x_i^{\zeta'}$.

Therefore, $\zeta'$ is realizable by indifferent agents from $\zeta$. Since the sequential tree is valid, then $x_k^{\zeta} \leq x_k^{\zeta'}$. This contradicts the initial assumption.

## Proof of Theorem 2.

**Any $GSP$ and $MIN$ mechanism is a valid sequential mechanism.**

Let $(G, \varphi)$ denote a mechanism that meets $GSP$ and $MIN$. Steps 1, 2, 3 and 4 are four preliminary properties of $(G, \varphi)$. Steps 5 and 6 prove that $(G, \varphi)$ is a sequential mechanism.

**Step 1.** If $G(u) = S^*$ and $\varphi(u) = \varphi^*$, then for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} > \varphi_{S^*}$ and $\widetilde{u}_{N \setminus S^*} \leq u_{N \setminus S^*}$, $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.
Proof.
First, note that by $MIN$, an agent gets positive net utility if and only if he is served.

Let $i \in S^*$. Then $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$. To see this, if $i \notin G(\widetilde{u}_i, u_{-i})$ or $\varphi_i(\widetilde{u}_i, u_{-i}) > \varphi_i^*$, then agent $i$ misreports $u_i$ when the true profile is $(\widetilde{u}_i, u_{-i})$, which contradicts $SP$. On the other hand, if $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) < \varphi_i^*$, then agent $i$ misreports $\widetilde{u}_i$ when the true profile is $u$, which also contradicts $SP$. Therefore, $i \in G(\widetilde{u}_i, u_{-i})$ and $\varphi_i(\widetilde{u}_i, u_{-i}) = \varphi_i^*$.

Let $j \neq i$. If $NU_j(\widetilde{u}_i, u_{-i}) > NU_j(u)$, then agent $i$ helps $j$ by misreporting $\widetilde{u}_i$ when the true profile is $u$. This contradicts $GSP$. The case $NU_j(\widetilde{u}_i, u_{-i}) < NU_j(u)$ is analogous. Thus, $NU_j(\widetilde{u}_i, u_{-i}) = NU_j(u)$ for all $j \neq i$. Therefore, by $MIN$ $G(\widetilde{u}_i, u_{-i}) = S^*$ and $\varphi(\widetilde{u}_i, u_{-i}) = \varphi^*$.

By applying the previous argument to each agent in $S^*$, we have that $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$ and $\varphi(\widetilde{u}_{S^*}, u_{-S^*}) = \varphi^*$.

Let $j \notin S^*$. Then $G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$. First note that $j \notin G(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j})$; otherwise, by the voluntary participation in the mechanism (condition $ii$):

$$\varphi_j(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) < \widetilde{u}_j \leq u_j.$$

Thus, agent $j$ misreports $\widetilde{u}_j$ when the true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$. This contradicts $SP$.

On the other hand, if $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) < NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for some $k \neq j$, then agent $j$ helps $k$ by reporting $\widetilde{u}_j$ when the true profile is $(\widetilde{u}_{S^*}, u_{-S^*})$; this contradicts $GSP$. Similarly, by $GSP$ $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) > NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ cannot occur. Thus, $NU_k(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = NU_k(\widetilde{u}_{S^*}, u_{-S^*})$ for all $k \neq j$. Hence, by $MIN$ $G(\widetilde{u}_{S^* \cup j}, u_{-(S^* \cup j)}) = S^*$ and $\varphi(\widetilde{u}_{S^* \cup j}, u_{-S^* \cup j}) = \varphi^*$.

By repeatedly applying the previous argument to every agent in $N \setminus S^*$, we have that $G(\widetilde{u}) = S^*$ and $\varphi(\widetilde{u}) = \varphi^*$.

**Step 2.** If $G(u) = G(\widetilde{u})$ then $\varphi(u) = \varphi(\widetilde{u})$.
Proof.
Let $S^* = G(u) = G(\widetilde{u})$, $v_S = \max(\widetilde{u}_S, u_S)$ and $v_{N \setminus S} = \min(\widetilde{u}_{N \setminus S}, u_{N \setminus S})$ (where max and min are taken coordinate by coordinate).

By step 1, comparing $v$ and $u$, $G(v) = S^*$ and $\varphi(v) = \varphi(u)$. Similarly, comparing $v$ and $\widetilde{u}$, $\varphi(v) = \varphi(\widetilde{u})$.

By step 2, there exists at most one vector of payments for every coalition. Let $x^{S^*}$ be the payment of coalition $S^*$ when $S^*$ is served at some profile.

**Step 3.** Let $u$ be such that $G(u) = S^*$ and $\varphi(u) = \varphi^*$. Then for every $i \in S^*$ and $u_i^* \leq \varphi_i^*$, $S^* \backslash i \subseteq G(u_i^*, u_{-i})$ and $\varphi_{S^* \backslash i}(u_i^*, u_{-i}) \leq \varphi_{S^* \backslash i}^*$.
Proof.
First note that for every $j \in S^* \setminus i$, $j \in G(\varphi_i^*, u_{-i})$ and $\varphi_j(\varphi_i^*, u_{-i}) \leq \varphi_j^*$. Indeed, by $MIN$ the net utility of agent $j$ at $u$ is positive. If $j \notin G(\varphi_i^*, u_{-i})$ or $\varphi_j(\varphi_i^*, u_{-i}) > \varphi_j^*$ then agent $i$ can help $j$ by misreporting $u_i$ when the true profile is $(\varphi_i^*, u_{-i})$ : By $MIN$, agent $i$ is not being served at the profile $(\varphi_i^*, u_{-i})$; thus, he is indifferent between misreporting $u_i$ and getting a unit at price $\varphi_i^*$, or truly reporting $\varphi_i^*$ and not getting a unit, whereby agent $j$ is better off at $u$. This contradicts $GSP$.

Finally, since $i \notin G(\varphi_i^*, u_{-i})$ and by step 1, $G(u_i^*, u_{-i}) = G(\varphi_i^*, u_{-i})$ and $\varphi(u_i^*, u_{-i}) = \varphi(\varphi_i^*, u_{-i})$ for all $u_i^* \leq \varphi_i^*$.

**Step 4.** If $G(u) = S^*$, then for any $T \subset S^*$, there exists $\widetilde{u}$ such that $G(\widetilde{u}) = T$ and $x_T^T \leq x_T^{S^*}$.

Proof.

Let $\bar{u} = (u_{S^*}, 0_{-S^*})$. By step 1, $G(\bar{u}) = S^*$ and $\varphi(\bar{u}) = x^{S^*}$. Let $i \in S^*$. By $MIN$ $i \notin G(x_i^{S^*}, \bar{u}_{-i})$. By step 3, $S^* \setminus i \subset G(x_i^{S^*}, \bar{u}_{-i})$. Since the utilities of agents outside $S^*$ are zero, then by $MIN$ $S^* \setminus i = G(x_i^{S^*}, \bar{u}_{-i})$. Thus, by step 3, $x_{S \setminus i^*}^{S \setminus i^*} \leq x_{S \setminus i^*}^S$. Finally, to check the claim we repeatedly apply the above argument to every agent in $S^* \setminus T$.

**Step 5.** Assume there is $u^*$ such that $G(u^*) = N$. Then, there is an agent $i \in N$ who is offered a unit of the good at a price $x^i$ that is independent of the utilities of the other agents. That is, for any utility profile, if $u_i \leq x^i$, then $i \notin G(u)$; and if $u_i > x^i$, then $i \in G(u)$ and $\varphi(u) = x^i$. In this case, we say agent $i$ has priority.

We prove this by induction in the size of $N$.

If $N = \{1\}$, then the $GSP$ and $MIN$ mechanisms are clearly fixed cost mechanisms. That is, there is a fixed price $x \in [0, \infty]$ such that if $u_1 > x$, then 1 is served at price $x$. If $u_1 \leq x$, then he is not served.

For the induction hypothesis, assume that for any $GSP$ and $MIN$ mechanism for $n-1$ agents there is an agent who has priority. Let $(G, \varphi)$ be a mechanism for the agents in $N = \{1, \ldots, n\}$.

For every $j$, consider the utility profiles where agent $j$ has zero utility, that is,

$$U^j = \{u \in \mathbb{R}_+^N \mid u_j = 0\}.$$

By $MIN$, agent $j$ is not being served at any profile of $U^j$. Thus, the restriction of $(G, \varphi)$ to $U^j$ defines a $MIN$ and $GSP$ mechanism for the agents in $N \setminus j$. Let $\rho^j = \{x^S \mid S \not\ni i\}$ be the set of payments at this mechanism. Note that because $N$ is being served, then by step 4 every coalition $S \subset N$ is being served at a profile in $U^j$. In particular, $\rho^j$ contains a payment for every group of agents that does not contain agent $j$. Also, note that by step 2, if $x^T \in \rho^j$ and $\tilde{x}^T \in \rho^k$, then $x^T = \tilde{x}^T$.

Finally, by step 4 payments are non-decreasing as the coalition increases. That is, if $S \subset T$, then $x_S^S \leq x_S^T$.

By the induction hypothesis, on $\rho^1$ there is an agent $i_1$ who has priority. The monotonicity of the payments implies $x_{i_1}^{N \setminus 1} = x_{i_1}^{i_1}$. Similarly, there is an agent who has priority on $\rho^{i_1}$. Call this agent $i_2$; thus, $x_{i_2}^{N \setminus i_1} = x_{i_2}^{i_2}$. We continue this procedure until we reach a cycle. Without loss of generality, we assume the cycle is $i_1, i_2, \ldots, i_k$. This means $i_{j+1}$ has priority on $\rho^{i_j}$ for $j = 1, \ldots, k-1$, and $i_1$ has priority on $\rho^{i_k}$.

**Case 1.** The cycle has a size of less than $n$, that is, $k < n$.

Fix $v_{N \setminus \{i_1, i_2, \ldots, i_k\}}$ such that $v_{N \setminus \{i_1, i_2, \ldots, i_k\}} > x_{N \setminus \{i_1, i_2, \ldots, i_k\}}^N$.

Consider the profiles

$$U = \{u \in \mathbb{R}_+^N \mid u_{N \setminus \{i_1, i_2, \ldots, i_k\}} = v_{N \setminus \{i_1, i_2, \ldots, i_k\}}\}.$$

Note that for every $u \in U$, $N \setminus \{i_1, i_2, \ldots, i_k\} \subset G(u)$. Indeed, consider $(\widetilde{u}_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$ such that $\widetilde{u}_{\{i_1, i_2, \ldots, i_k\}} > x^N_{i_1, i_2, \ldots, i_k}$. By step 1, $G(\widetilde{u}_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}}) = N$. By steps 1 and 3, $N \setminus \{i_1\} \subseteq G(u_{i_1}, \widetilde{u}_{\{i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$. Similarly, $N \setminus \{i_1, i_2\} \subseteq G(u_{i_1, i_2}, \widetilde{u}_{\{i_3, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$. Continuing this way, $N \setminus \{i_1, i_2, \ldots, i_k\} \subseteq G(u_{\{i_1, i_2, \ldots, i_k\}}, u_{-\{i_1, i_2, \ldots, i_k\}})$.

By step 4, for every coalition $T$ such that $N \setminus \{i_1, i_2, \ldots, i_k\} \subset T$, there is $\widetilde{u} \in U$ such that $G(\widetilde{u}) = T$. This is clear because coalition $N$ is being served at some profile of $U$, so we can reduce (one agent at a time) the utility of the agents not in $T$ to zero.

Clearly, the mechanism restricted to $U$ defines a $GSP$ mechanism for the agents in $\{i_1, i_2, \ldots, i_k\}$. By the induction hypothesis, there is an agent who has priority, say $i_1$. Thus, $x_{i_1}^{N \setminus \{i_2, \ldots, i_k\}} = x_{i_1}^N$. On the other hand, because $i_1$ has priority on $\rho^{i_k}$, $x_{i_1}^{i_1} = x_{i_1}^{N \setminus \{i_2, \ldots, i_k\}}$. Therefore, $x_{i_1}^N = x_{i_1}^{i_1}$. Hence by the monotonicity of the payments $x_{i_1}^T = x_{i_1}^S$ for all $S, T \subseteq N$ such that $i_1 \in S$ and $i_1 \in T$.

Finally, we prove agent $i_1$ has priority. Assume there is $u$ such that $u_{i_1} > x_{i_1}^{i_1}$ but $i_1 \notin G(u)$. Consider the profile $(u_{i_1}, \tilde{u}_{-i_1})$ where $\tilde{u}_{-i_1} > \max(x_{-i_1}^N, u_{-i_1})$ and max is taken coordinate by coordinate. By step 1, $G(u_{i_1}, \tilde{u}_{-i_1}) = N$. By steps 1 and 3, $N \setminus \{i_2\} \subseteq G(u_{i_1}, u_{i_2}, \tilde{u}_{-i_1, i_2})$. Similarly, by steps 1 and 3, $N \setminus \{i_2, i_3\} \subseteq G(u_{i_1}, u_{i_2}, u_{i_3}, \tilde{u}_{-i_1, i_2, i_3})$. Continuing this way, $\{i_1\} \subseteq G(u)$. This is a contradiction.

On the converse, if $w$ is such that $w_{i_1} < x_{i_1}^{i_1}$, then clearly $i_1 \notin G(w)$ because $x_{i_1}^{i_1}$ is the smallest payment of agent $i_1$ in the mechanism $(G, \varphi)$. Hence, $i_1$ has priority on $(G, \varphi)$.

**Case 2.** The cycle has a size of $n$, that is, $k = n$.

Without loss of generality, assume that agent 2 has priority over $N \setminus 1$, agent 3 has priority over $N \setminus 2, \ldots$, etc. Thus,

$$x_2^2 = x_2^{N \setminus 1}, \ldots, x_3^3 = x_3^{N \setminus 2}, \ldots, x_1^1 = x_1^{N \setminus n}. \tag{4}$$

Also, to get a contradiction, assume that there is no agent who has priority. That is,

$$x_2^{N \setminus 1} < x_2^N, x_3^{N \setminus 2} < x_3^N, \ldots, x_1^{N \setminus n} < x_1^N.$$

Let $u^*$ be such that $G(u^*) = N$. By $MIN$, $u^* > x^N$.

Since $G(u^*) = N$, then $N \setminus 1 = G(x_1^N, u_{-1}^*)$. Indeed, by step 3, $N \setminus 1 \subseteq G(x_1^N, u_{-1}^*)$ and by $SP$ and $MIN$ $1 \notin G(x_1^N, u_{-1}^*)$. Thus, $N \setminus 1 = G(x_1^N, u_{-1}^*)$. In particular, $2 \in G(x_1^N, u_{-1}^*)$ and he pays $x_2^2$ because he has priority in $\rho^1$ $(x_2^2 = x_2^{N \setminus 1})$.

Similarly, by step 3, $N \setminus \{1, 3\} \subset G(x_{1,3}^N, u_{-1,3}^*)$, in particular $2 \in G(x_{1,3}^N, u_{-1,3}^*)$, and he pays no more than $x_2^2$. By repeating the same argument for every agent different than 2: $2 \in G(x_{-2}^N, u_2^*)$ and he pays no more than $x_2^2$. Therefore, by $SP$: $2 \in G(x_{-2}^N, x_2^N) = G(x^N)$ because $u_2^* > x_2^N > x_2^2$.

Finally, since everything is symmetric, $G(x^N) = N$. This contradicts $MIN$.

**Step 6.** Assume that there is no $u$ such that $G(u) = N$. If the mechanism is not trivial ($G(u) \neq \emptyset$ for some $u$), there is an agent who has finite priority. That is, there is an agent $i^*$ and a payment $x^*$, $0 \leq x^* < \infty$, such that $i^* \in G(u)$ for all $u$ such that $u_{i^*} > x^*$.

First, note that there is a group of agents $S^*$ who have priority. That is, for all $\widetilde{u}$ such that $\widetilde{u}_{S^*} \geq x_{S^*}^{S^*}$, $G(\widetilde{u}) = S^*$. To see this, consider $\widetilde{u}$ such that $\widetilde{u} > x^T$ for all possible payments $x^T$ such that $x^T = \varphi(v)$ for some $v$ (we know by step 2 that there is at most one vector of payments for every coalition; thus, it is feasible to choose such $\widetilde{u}$). Let $S^*$ be such that $G(\widetilde{u}) = S^*$. Note that for any $i \notin S^*$, $G(\widetilde{u}_{-i}, v_i) = S^*$ for all $v_i$. Indeed, if $v_i \leq \widetilde{u}_i$, then by step 1 $i \notin G(\widetilde{u}_{-i}, v_i)$. On the other hand, if $v_i > \widetilde{u}_i$, then $i \notin G(\widetilde{u}_{-i}, v_i)$. This is easy to see by contradiction. Assume that $i \in G(\widetilde{u}_{-i}, v_i)$. Then, by the choice of $\widetilde{u}$, $\varphi_i(\widetilde{u}_{-i}, v_i) < \widetilde{u}_i < v_i$. Therefore, by step 1, $i \in G(\widetilde{u})$, which is a contradiction.

Hence, $G(\widetilde{u}_{-i}, v_i) = S^*$ for all $v_i$. Thus, by changing the utilities of the agents in $N \setminus S^*$ one at a time, $G(\widetilde{u}_{S^*}, u_{-S^*}) = S^*$. Hence, by step 1, $G(\bar{u}_{S^*}, u_{-S^*}) = S^*$ for all $\bar{u}_{S^*} \geq x_{S^*}^{S^*}$ and all $u_{-S^*}$.

We now prove step 6 by induction. For $n = 1$, if $G(u) \neq 1$ for all $u$, then clearly the mechanism is trivial ($G(u) = \emptyset$ for all $u$). So the claim is true.

For the induction hypothesis, assume the claim is true for any mechanism of $n - 1$ agents. We prove it for any mechanism of $n$ agents.

Let $S^*$ be defined as above and $j \notin S^*$. Consider the restriction of the mechanism to $U^j = \{u \in \mathbb{R}_+^N \mid u_j = 0\}$. Then this restriction is a $GSP$ and $MIN$ mechanism for the agents in $N \setminus j$. By induction and step 5, there is an agent $i^*$ who has (finite) priority for the agents $N \setminus j$. Clearly, $i^* \in S^*$; otherwise, his payment is dependent on the agents in $S^*$.

We now prove by contradiction that for any profile $u_{-i^*}$, $i^*$ has priority. Assume there is $u$ such that $f_{i^*}(u_{-i^*}) \neq x_{i^*}^{S^*}$, where $f_{i^*}(u_{-i^*})$ is the price for a unit of the good that the mechanism offers to agent $i^*$ when the utilities of the other agents are $u_{-i^*}$ (recall this function exists because the mechanism meets $SP$). Let $u_{i^*} = \widetilde{u}_{i^*}$, a utility bigger than all possible payments for agent $i^*$, in particular, $u_{i^*} > x_{i^*}^{S^*}$. First, note that $j \in G(u)$; otherwise, by step 1 $G(u) = G(0, u_{-j})$ and $\varphi(u) = \varphi(0, u_{-j})$. Thus, $i^*$ is served at $u$ at a price equal to $x_{i^*}^{S^*}$, which contradicts our assumptions. Hence, $j \in G(u)$. By step 3, $f_{i^*}(u_{-i^*}) > x_{i^*}^{S^*}$.

Let $k \in S^* \setminus i^*$ and $\bar{u}_k > \max(u_k, x_k^{S^*})$, then $f_{i^*}(\bar{u}_k, u_{-k,i^*}) \geq x_{i^*}^{S^*}$. Indeed, if $k \in G(u)$, then by step 1 $G(\bar{u}_k, u_{-k}) = G(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x_{i^*}^{S^*}$. On the other hand, if $k \notin G(u)$ and $k \notin G(\bar{u}_k, u_{-k})$, then by step 1 $G(\bar{u}_k, u_{-k}) = G(u)$ and $f_{i^*}(\bar{u}_k, u_{-k,i^*}) = f_{i^*}(u_k, u_{-k,i^*}) > x_{i^*}^{S^*}$. Finally, if $k \notin G(u)$ and $k \in G(\bar{u}_k, u_{-k})$, then by step 3 $f_{i^*}(\bar{u}_k, u_{-i^*,k}) \geq f_{i^*}(u_k, u_{-i^*,k}) > x_{i^*}^{S^*}$.

By repeatedly applying the above argument to every agent in $S^* \setminus i^*$, we conclude that $f_{i^*}(u_{-S^*}, \bar{u}_{S^* \setminus i^*}) > x_{i^*}^{S^*}$ for some $(u_{i^*}, \bar{u}_{S^* \setminus i^*}) \geq x_{S^*}^{S^*}$. This contradicts the priority of coalition $S^*$.

Steps 5 and 6 showed that for any $GSP$ and $MIN$ mechanism there exists an agent whose payment is independent of the other agents' utilities. By induction, this clearly implies that the mechanism is sequential. We showed in the main text (see the paragraph after definition 10) that any $GSP$ sequential mechanism is valid.

**Proof of Corollary 1.**

If the mechanism is welfare equivalent to a $GSP$ and $MAX$, then it is welfare equivalent to a cross-monotonic mechanism; therefore, the payment of every agent $i$ does not increase when the coalition increases.

Similarly, if the mechanism meets $GSP$ and $MIN$, then by step 4 in the proof of Theorem 2, the payment of every agent $i$ does not decrease when the coalition increases.

Therefore, the cost share of every agent $i$ should be fixed.


**Proof of Proposition 2.**

Recall that $1_N = (1, \ldots, 1) \in \mathbb{R}_+^N$. For a non-negative number $x$, let $x \cdot 1_N = (x, \ldots, x) \in \mathbb{R}_+^N$.

By $ETE$, $G(x \cdot 1_N) = N$ or $G(x \cdot 1_N) = \emptyset$ for all $x \geq 0$, since all agents should either be served or not served at a symmetric utility profile.

**Case 1.** Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.

Proof.
Step 1.1. If $NU_k(u) = 0$ for all $u \in \mathbb{R}_+^N$ and $k \in N$, then $G(u) = \emptyset$ for all $u \in \mathbb{R}_+^N$.
Proof. If $NU_k(u) = 0$ but $G(u) = S \neq \emptyset$ for some utility profile $u$, then $\varphi_i(u) = u_i$ for all $i \in S$. Thus, by SP, for $k \in S$ and $v_k > u_k : k \in G(v_k, u_{-k})$ and $\varphi_k(v_k, u_{-k}) = u_k$; thus, $NU_k(v_k, u_{-k}) > 0$.
Step 1.2. Assume $G(x \cdot 1_N) = \emptyset$ for all $x > 0$, then $NU(u) = 0$ for all $u \in \mathbb{R}_+^N$.
Proof. Assume there is an agent $k$ such that $NU_k(u) > 0$ at some utility profile $u$. Let $u^{max} = \max(u_1, \ldots, u_n) \cdot 1_N$. Then, $G(u^{max}) = \emptyset$. Thus, when the true profile is $u^{max}$, agents in $N$ help $k$ by misreporting $u$ : Agent $k$ is strictly better off because he is getting a unit at a price below $u_k$, while any other agent $j$ may or may not get a unit at a price less than or equal to $u_j$. This contradicts $GSP$.
Steps 1.1 and 1.2 combined prove case 1.

**Case 2.** There exists $x^* \geq 0$ such that $G(x^* \cdot 1_N) = N$.

Proof.
By $ETE$, there exists $y^* \geq 0$ such that $\varphi_i(x^* \cdot 1_N) = y^*$ for all $i$.

**Step 2.1.** For all $u > y^* \cdot 1_N$, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.
Proof.
First, assume that $x^* > y^*$. Let $v = x^* \cdot 1_N$. By $SP$, $1 \in G(v_{-1}, u_1)$ and $\varphi_1(v_{-1}, u_1) = y^*$. Thus, by $GSP$, $G(v_{-1}, u_1) = N$ and $\varphi_i(v_{-1}, u_1) = y^*$ for all $i$. Changing the profiles one agent at a time $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$
Now, assume that $y^* = x^*$. Consider $\widetilde{x}$ such that $\widetilde{x} > x^*$. Let $\widetilde{v} = \widetilde{x} \cdot 1_N$. By ETE, $G(\widetilde{v}) = \emptyset$ or $G(\widetilde{v}) = N$. If $G(\widetilde{v}) = \emptyset$, then when the true profile is $\widetilde{v}$, coalition $N$ can improve

by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile. On the other hand, if $G(\widetilde{v}) = N$, then $\varphi(\widetilde{v}) = y^*$. Indeed, $\varphi(\widetilde{v}) \geq y^*$ because $\widetilde{x} > x^* = y^*$. If $\varphi(\widetilde{v}) > y^*$, then when the true profile is $\widetilde{v}$, all agents can improve by misreporting $x^* \cdot 1_N$, since all agents are served at price $y^* = x^*$ at that profile.

Therefore $G(\widetilde{x} \cdot 1_N) = N$, $\varphi(\widetilde{x} \cdot 1_N) = y^*$ and $\widetilde{x} > y^*$. By the initial case, $G(u) = N$ and $\varphi(u) = y^* \cdot 1_N$.

We finish the proof of case 2 by induction in the number of agents. We assume that any $GSP$ and $ETE$ mechanism for less than $n$ agents is welfare equivalent to a cross-monotonic mechanism that satisfies equal sharing (that is, all agents being served pay the same). We will prove it for a mechanism for $n$ agents. We will divide the proof into steps 3 and 4 (and several multi-steps and cases).

**Step 3.** If an agent is served, then he will not pay less than $y^*$ at any utility profile. That is, if $i \in G(u^*)$ for some $u^* \in \mathbb{R}_+^*$, then $\varphi_i(u^*) \geq y^*$.

Proof.

We will prove this step in cases 3.1 and 3.2.

**Case 3.1** $G(u^*) \neq N$.

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent $i$.

Without loss of generality, also assume that $j \notin G(u^*)$ and $\varphi_i(u^*) < u_i^*$, so agent $i$ gets a positive net utility at $u^*$.

Consider the profile $\widetilde{u} = (0, u_{-j}^*)$. Then by $GSP$, $i \in G(\widetilde{u})$ and $\varphi_i(\widetilde{u}) = \varphi_i(u^*)$. Otherwise, $j$ would help $i$ at the profile that gives $i$ higher utility.

Let $U^j = \{u \in \mathbb{R}^N | u_j = 0\}$ be the set of utility profiles where agent $j$ has utility zero. By induction, the restriction of the mechanism to $U^j$ is welfare equivalent to a cross-monotonic mechanism for $N \setminus j$ agents that satisfies equal sharing.

Since $\widetilde{u} \in U^j$ and $i \in G(\widetilde{u})$ and the mechanism restricted to $U^j$ is cross-monotonic with equal sharing, then we can find a utility profile $w \in U^j$ such that $w \geq \widetilde{u}$ and $G(w) = N \setminus j$ and $\varphi_i(w) \leq \varphi_i(\widetilde{u}) = \varphi_i(u^*) < y^*$.

Let $x^{N \setminus j} = \varphi(w)$. Clearly $x_k^{N \setminus j} = x_i^{N \setminus j} < y^*$ for all $k, i \in N \setminus j$, and $k \neq i$.

Let $\epsilon > 0$ be such that $y^* - \epsilon > x_i^{N \setminus j}$ and let $u = ((y^* + \epsilon) \cdot 1_N)$. Then by step 2.1, $G(u) = N$ and $\varphi(u) = x^N$. By $SP$, $i \notin G(y^* - \epsilon, u_{-j})$. Thus, by $GSP$ $G(y^* - \epsilon, u_{-j}) = N \setminus j$ and $\varphi(y^* - \epsilon, u_{-j}) = x^{N \setminus j}$.

Since $u_k > x_k^{N \setminus j}$ for all $k \in N \setminus j$, then by $GSP$ $G((y^* - \epsilon) \cdot 1_N) = N \setminus j$ and $\varphi((y^* - \epsilon) \cdot 1_N) = x^{N \setminus j}$. This contradicts $ETE$.

**Case 3.2.** $G(u^*) = N$

In order to derive a contradiction, we assume that $\varphi_i(u^*) < y^*$ for some agent $i$.

Assume without loss of generality that $u_i^* > \varphi_i(u^*)$ (otherwise, at the new profile we can increase the utility of agent $i$ and continue serving $N$ or a proper subset of $N$ (case 3.1 above)).

If agent $j \in G(u^*)$ is such that $\varphi_j(u^*) = u_j^*$, then at the profile $(0, u_{-j}^*)$ agent $j$ is not served, that is $j \notin G(0, u_{-j}^*)$. Moreover, $i \in G(0, u_{-j}^*)$ and $\varphi_i(0, u_{-j}^*) = \varphi_i(u^*)$ (otherwise,

$j$ helps $i$). Therefore, $\varphi_i(0, u^*_{-j}) = \varphi_i(u^*) < y^*$ and $G(0, u^*_{-j}) \neq N$; thus, we can apply the case 3.1 to the profile $(0, u^*_{-j})$.

On the other hand, if $\varphi_k(u^*) < u^*_k$ for all $k \in N$, then consider the utility profile $v^* = \max\{u^*_1, u^*_2, \ldots, u^*_n\} \cdot 1_N$. By $GSP$ (replacing one agent at a time): $G(v^*) = N$ and $\varphi(v^*) = \varphi(u^*)$. By $ETE$, $G(v^*) = N$ and $\varphi_j(v^*) = \varphi_i(u^*) < y^*$ for all $i, j \in N$. Therefore, when the true profile is $x^* \cdot 1_N$ (recap $\varphi(x^* \cdot 1_N) = y^* \cdot 1_N$), coalition $N$ can improve by reporting $v^*$. This contradicts GSP.

**Step 4.** The mechanism is welfare equivalent to a cross-monotonic mechanism with equal sharing.

Proof.

First, we construct the cost shares. For the agents in $N$, their cost shares will be $x^N = y^* \cdot 1_N$.

The cost shares of coalition $S$ equal $x^S$, where $x^S$ is the cost shares of coalition $S$ at $U^j$ for some $j \in N$, $S \subseteq N \setminus j$. These cost shares exist because the mechanism restricted to $U^j$ is welfare equivalent to a cross-monotonic mechanism. Moreover, it is well defined because if $G(u) = S$ for some $u \in U^j$, and $G(\tilde{u}) = S$ for some $\tilde{u} \in U^k$, where $S \subseteq N \setminus \{j, k\}$, then $\varphi(u) = \varphi(\tilde{u})$. To see this, by the induction hypothesis the mechanism restricted to $U^j$ and $U^k$ is cross-monotonic with equal sharing; therefore, $\varphi_j(u) = \varphi_l(u)$ and $\varphi_j(\tilde{u}) = \varphi_l(\tilde{u})$ for $j, l \in S$. If $\varphi(u) < \varphi(\tilde{u})$, then when the true profile is $\tilde{u}$, agents in $N$ can help $S$ by misreporting $u$. Similarly, if $\varphi(u) > \varphi(\tilde{u})$, then when the true profile is $u$, agents in $N$ can help $S$ by misreporting $\tilde{u}$. Hence, $\varphi(u) = \varphi(\tilde{u})$.

The cost shares are cross-monotonic. Indeed, for every $j \in N$, these cost shares are cross-monotonic in $U^j$. Also, $x^N_i = y^* \leq x^S_i$ for every $i \in S \subset N \setminus j$ by step 3.

Next, we show that the mechanism coincides (welfare-wise) with the cross-monotonic mechanism generated by the cost shares above.

Let $u$ be a utility profile.

**Step 4.1.** If $u_i \geq y^*$ for all $i \in N$, then the mechanism is welfare equivalent to the mechanism such that $G(u) = N$ and $\varphi_i(u) = y^*$.

If $u_i > y^*$ for all $i \in N$, then by step 2.1, $G(u) = N$ and $\varphi_i(u) = y^*$ for all $i \in N$.

If $u_i > y^*$ for all $i \in S$ and $u_j = y^*$ for all $j \in N \setminus S$, then by step 3 no agent will pay less than $y^*$. If an agent $k \in S$ is paying more than $y^*$ at $u$, then coalition $N$ can help $k$ by misreporting $x^*$, since all agents pay exactly $y^*$ at that profile.

**Step 4.2.** If $u_i < y^*$ for some $i$, then $NU(u) = NU(0, u_{-i})$.

By step 3, $i \notin G(u)$. By $SP$, $i \notin G(0, u_{-i})$. Thus, $NU_i(u) = NU_i(0, u_{-i})$. Moreover, by $GSP$ $NU_j(0, u_{-i}) = NU_j(u)$ for all $j \neq i$. To see this, if $NU_j(0, u_{-i}) > NU_j(u)$ then when the true profile is $u$, agent $i$ helps $j$ by misreporting $0$. Similarly, if $NU_j(0, u_{-i}) < NU_j(u)$, then when the true profile is $(0, u_{-i})$, agent $i$ helps $j$ by misreporting $u_i$. Therefore, $NU_j(0, u_{-i}) = NU_j(u)$.

Note that the allocation at every profile $u$ is welfare equivalent to serving the maximum reachable coalition at $u$ using the above cross-monotonic set of cost shares.

If $u$ satisfies the conditions of step 4.1, then the maximum reachable coalition given the cost shares is $N$. By step 4.1, the mechanism is welfare equivalent to serving $N$.

Now, assume $u$ satisfies the conditions of step 4.2. Let $S^*$ be the maximum reachable coalition for the cost shares above at the utility profile $u$. Clearly, $S^* \neq N$ because $u_i < y^* = x_i^N$ for some $i \in N$. Let $j \in N \setminus S^*$. Therefore, $x^{S^*} \in U^j$ and $x^{S^*}$ is the cost share of coalition $S^*$ in $U^j$. By the induction hypothesis, $(G(u), \varphi(u))$ is welfare equivalent to serving the maximum reachable coalition for the cost shares in $U^j$. Therefore, $(G(u), \varphi(u))$ is welfare equivalent to serving $S^*$ at prices $x^{S^*}$.

**Proof of Proposition 3.**

We prove the proposition by contradiction. Assume without loss of generality that every agent in $N$ is served in at least one profile and that there is no agent who has priority. Then for every agent $i$ there exist profiles $u^i$ and $\widetilde{u}^i$ such that $i \in G(u^i)$, $i \notin G(\widetilde{u}^i)$, $u_i^i, \widetilde{u}_i^i > \bar{x}_i$ where $\bar{x}_i = \varphi_i(u^i)$.

Let $v > \max_{k \in N}(u^k, \widetilde{u}^k)$ where max is taken coordinate by coordinate over all utility profiles $u^k$, $\widetilde{u}^k$.

By $GSP$, $G(v) \neq \emptyset$; otherwise, coalition $N$ misreports $u^1$ when the true profile is $v$. Assume that $G(v) = i^*$. By $GSP$, $\varphi_{i^*}(v) = \bar{x}_{i^*}$; otherwise, coalition $N$ misreports $u^{i^*}$ when the true profile is $v$ or vice versa.

By $SP$, $k \notin G(\widetilde{u}_k^{i^*}, v_{-k})$ for all $k \neq i^*$. Thus, by $GSP$, $G(\widetilde{u}_k^{i^*}, v_{-k}) = i^*$ and $\varphi_{i^*}(\widetilde{u}_k^{i^*}, v_{-k}) = \bar{x}_{i^*}$. Changing the profiles one agent at a time, $G(\widetilde{u}_{-i^*}^{i^*}, v_{i^*}) = i^*$ and $\varphi_{i^*}(\widetilde{u}_{-i^*}^{i^*}, v_{i^*}) = \bar{x}_{i^*}$.

Since $\widetilde{u}_{i^*} > \bar{x}_{i^*}$ then by strategyproof $G(\widetilde{u}^{i^*}) = i^*$. This is a contradiction.

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