Collusion-proof cost sharing mechanisms (Draft)

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Summary. An auction type mechanism elicits the valuations for getting a good (service) from the agents, allocates some goods (service) to some agents and charge money only to the served agents.

We characterize the mechanisms that are immune to three alternative types of collusion of the agents.

Under full collusion, agents can coordinate misreports and transfer money between them. We show that only the fixed cost mechanisms, that offer units of good to the agents at a fixed price are immune to full collusion.

On the other hand, if the utility profile is random and transfers of money between the agents are not available, agents may sign a contract and coordinate misreports ex-ante at every state of the world. We show that only sequential mechanisms are immune to this type of collusion.

Finally, if the utility profile is deterministic and transfers of money between the agents are not available, a large class of mechanisms are immune to coordinated misreports of the agents. We characterize this class and show that it contains most of the mechanisms previously discussed in early literature of group strategyproof cost sharing.

Keywords: *Cost sharing, Mechanism design, Strategyproofness, non-bossiness.*

1 Introduction

Units of a nontransferable, indivisible and homogeneous good (or service) are available at some non-negative cost. Agents are interested to consume at most one unit of that good and are characterized by their valuation for it (which we call their utility). We look for mechanisms that elicit these utilities from the agents, allocate some goods to some agents and charge some money only to the agents who are served. We are interested to characterize the group strategyproof

(GSP) mechanisms. These mechanisms play a key role in cost sharing literature and computer science applications.[1]

A tie-breaking rule defines the behavior of an agent who is offered a price equal to his valuation. Juarez[2007] provides a characterization of GSP mechanisms that meet one of two alternative tie-breaking rules.[2] These characterization leads two orthogonal classes of mechanisms. The first class are the cross monotonic mechanism, resembling an increasing auction. The second class are the sequential mechanisms, which offer sequentially to the agents units of good at a fixed price.

Juarez[2007] corollary 1 shows that the (welfarewise) intersection of GSP mechanisms that meet simultaneously the two tie-breaking rules contains only the fixed cost mechanisms, offering to each agent a price completely independent of the reports of the other agents. Therefore, mechanisms that meet GSP are very responsive to the way ties are broken. Only the fixed cost mechanisms are not responsive to indifferences.

To illustrate the limitations of mechanisms with indifferences, consider a mechanism for three agents 1, 2, and 3. Assume agent 1 is offered a unit of good at price p. If agent 1 gets a unit of good, then only agent 2 is offered a unit of good at price p'. On the other hand, if agent 1 does not get a unit of good, then only agent 3 is offered a unit of good at price p'. This mechanism does not belong to the class of GSP mechanisms independently of the way ties are broken. To see this, consider a utility profile where agent 1 has utility exactly equal to p and agents 2 and 3 have utility strictly bigger than p'. If agent 1 is served at this profile, then he can help agent 3 by misreporting a utility less than p. On the other hand, of agent 1 is not served at this profile, then he can help agent 2 by misreporting a utility bigger than p.

However, it can be argued that in practice ties are rare events, and that a model in which the preference domain is such that ties never occur is just as plausible as one where they play a critical role. We develop such a model where the set of payments is different to the set of utilities. In the example above, if the set of payments for agent 1 is $P_1 = \{p\}$ and the set of utilities is $U_1 = \mathbb{R}_+ \setminus p$, that is the utility of agent 1 is never equal to p, then the mechanism is GSP.

Similarly as above, we consider an economy where utilities of the agents are different to payments and characterize the GSP mechanisms. We show this class is large. It contains mechanisms very different from the cross-monotonic and sequential mechanisms discussed by Juarez[2007] (and even those discussed by Roughgarden[2007], see section 3.5). We provide several representative examples of this class in section 3. We give three equivalent characterizations of the GSP mechanism, two of these characterizations are generalization of the cross-monotonic and sequential mechanisms (see proposition 5, 6 and 7).

Finally, this job contributes to the more challenging question of characterizing the weakly group strategyproof[3] ($WGSP$) mechanisms in the general

[1] See Juarez[2007] for related literature.

[2] With the first tie-breaking rule the agents who are indifferent are always served. With the second tie-breaking rule, the indifferent agents are never served.

[3] That is, any misreport of a group of agents cannot strictly increase the net utility of all

economy where the space of payments equal the space of utilities. Our characterization is a complete characterization of the $WGSP$ mechanisms where the payment of the coalition being served is constant.[4] However, the class of $WGSP$ mechanisms is very large. In particular, it contains mechanisms such that the payment of a coalition being served is not constant. For instance, consider a mechanism for two agents, 1 and 2, where agent 1 is offered a unit of good at price p independent of the report of agent 2. On the other hand, agent 2 is offered a unit at price $f(u_1)$ that depends exclusively on the report of agent u_1, for any function $f : \mathbb{R}_+ \to [0, \infty]$. This mechanism is $WGSP$ and is not GSP. However, we rule out this type of mechanisms by imposing a weak bossy condition: if at two profiles (u_i, u_{-i}) and $(\tilde{u}_i, u_{-i})$ agent i has the same supply at the same price, then any other agent is indifferent in the two profiles (u_i, u_{-i}) and $(\tilde{u}_i, u_{-i})$.

2 The model

For a vector x, $x \in \mathbb{R}^M$, we denote by $x_{[S]}$ the projection of x over $S \subset M$. x_S represents the sum of the $S-$coordinates of x, $x_S = \sum_{i \in S} x_i$. When there is no confusion we denote the projection $x_{[S]}$ simply as x_S. Let 1_M the unitarian vector in $\mathbb{R}^M$, that is $1_M = (1, 1, \ldots, 1)$.

There is a finite number of agents $N = \{1, 2, \ldots, n\}$. Agent i has a utility (willingness to pay) for getting one unit of good in U_i, where $U_i \subset \mathbb{R}_{++}$. We also assume that U_i is dense in $\mathbb{R}_{++}$ with the usual Euclidean topology. $U = U_1 \times \cdots \times U_n$ the space of possible utilities. Let P_i the set of payments for agent i, such that $P_i \subset [0, \infty] \setminus U_i$. Let $P = P_1 \times \cdots \times P_n$. Let $u \in U$ a utility profile. If agent i gets a unit paying x_i, $x_i \in P_i$, his net utility is $u_i - x_i$. If he does not get a unit his net utility is zero. We denote by (U, P) this economy.

Definition 1 *A mechanism (S, φ) allocates to every vector of utilities $u \in U$ a coalition of agents who gets goods $S(u) \subset N$ and the cost shares (payments) $\varphi(u) \in P$.*

Therefore, the net utility of agent i in the mechanism, denoted by NU_i, is $NU_i(u) = \delta_i(S(u))(u_i - \varphi(u))$.[5] Let $NU(u)$ the vector of such net utilities. Notice two mechanism may be equivalent in the welfare sense, that is their net utilities at any profile are equal, but the mechanisms may be different.

Similarly to Juarez[2007], we restrict our attention to mechanisms that satisfy two familiar normative properties.

- **Nonnegative Transfers (NNT):** $\varphi(u) \in \mathbb{R}^N_+$.
- **Individual Rationality (Voluntary participation (VP)):** $\varphi_i(u) \leq u_i \delta_i(S(u))$.

agents who misreport.

[4] This comes immediately from the non-bossiness.

[5] δ is the classic delta function, $\delta_i(T) = 1$ if $i \in T$, and 0 otherwise.

Nonnegative transfers requires all cost shares to be positive or zero. This is a common assumption when no transfers between agents are allowed and we do not want to subsidize any of them.

On the other hand, individual rationality implies that all agents enter the mechanism voluntarily. That is, the ex-post net utility of the agents is never smaller than their ex-ante net utility. Because we are assuming nonnegative transfer, individual rationality implies the agents with zero utility should pay nothing. However, they may get a unit for free. This is a basic equity condition protecting individual rights.

We want to characterize the mechanisms that are group strategyproof. That is, any misreport of a group of agents do not decrease their net utility and strictly increase one o them.

- **Group strategyproof (GSP):** For all $S \subset N$, and all utility profiles u and u', $u, u' \in U$ such that $u'_{N \setminus S} = u_{N \setminus S}$, it cannot be that $NU_i(u) \leq (u_i - \varphi_i(u'))\delta_i(S(u'))$ for all $i \in S$ and strict for at least one agent.

3 *GSP* mechanisms

3.1 Sequential Mechanisms

Definition 2 *A sequential tree is a binary tree of length n such that:*

i. Every node in the tree contains an agent i, $i \in N$, and a price p_i, $p_i \in P_i$.

ii. Every path from the root to a terminal node contains all agents in N exactly once.

Definition 3 (Sequential mechanisms) *Given a sequential tree we construct a sequential mechanisms as follows:*

We offer the agent in the root of the tree a unit of good at the price of his node. If his utility is bigger than the offered price, then we allocate him a unit at this price and go right on the tree. If his utility is smaller than the offered price, then we do not allocate him a unit and go left on the tree. We continue similarly with the following agent until we reach the end of the tree.

Sequential mechanism are GSP. This is easy to see by induction in the number of agents. It is clearly true for any one agent mechanism. Assume this is also true for any sequential mechanism of $n-1$ agents, and consider a sequential mechanism for n agents. Let i the agent in the root of the sequential tree, and notice his offered price is independent of the utility of the other agents. If i is being served, he will never misreport a utility below his offered price. Similarly, if i is not being served, he will never misreport a utility above his offered price. Hence in a profitably misreport, we can assume without loss of generality that agent i does not misreport. Fixing the utility of agent i, the mechanism restricted to the agents in $N \setminus i$ is sequential, hence they cannot misreport.

A nice feature of sequential mechanisms in an economy without indifferences is that they are always GSP without imposing any restriction on prices. In particular, we do not have to impose the feasibility condition of Juarez[2007] in an economy with indifferences.

The most interesting feature of these mechanisms is the simplicity of implementation. For any cost function (independent of its shape!) we can find many mechanisms that cover this cost exactly. The prolific mechanism that implements it is the incremental marginal cost mechanism (Moulin[1999]), which offer units of good sequentially, following the order or the tree, at price equal to marginal cost. That is, for any arbitrary sequential tree without prices (prices are defined next), we start with agent i_1 in root of the tree and offer him a unit of good at price $C(i_1)$. If i_1 decides to buy, then we offer agents i_2 on the right of tree a unit of good at price $C(i_1, i_2) - C(i_1)$. If i_1 did not buy, then we continue with agent i_3 on left of tree and offer him a unit of good at price $C(i_3)$, etc.

In particular, for any cost function, the social planner has a lot of freedom to choose the mechanism from. As many as the number of different trees. For instance for $n = 3$ he can choose between 12 mechanisms. For $n = 4$ he can choose between 546 mechanisms.

Also, notice this analysis is independent of the shape of the cost function. In particular, if the cost function is supermodular, we get the incremental mechanisms defined by Moulin[1999]. If it is submodular, they capture many crossmotononic mechanisms. Finally, if the cost function is neither submodular or supermodular, we capture mechanisms that have not been discussed in the literature before.

3.1.1 A characterization of sequential mechanisms

Suppose agents are risk neutral neutral, and the utility profile is random. That is, at the lottery of utilities, $L = [\pi_1, \ldots, \pi_k; u^1, \ldots, u^k]$, state k occurs with probability p_k and brings a utility profile u^k to the agents.

A deterministic mechanism ξ is defined as above. It allocates to the lottery L, the lottery:

$$\xi(L) = [\pi_1, \ldots, \pi_k; \xi(u^1), \ldots, \xi(u^k)].$$

The net utility of an agent at the outcome $\xi(L)$ is simply the expected net utility.

A misreport of a group of agents is a misreport of this group of agents at every possible state. A mechanisms is random group strategyproof (RGSP) if any misreport that strictly increases the net utility of one agent in the group, also strictly decreases the net utility of another agent in the group.

Proposition 1 *A mechanisms is sequential if and only if it is RGSP.*

3.2 Cross-monotonic mechanisms

Definition 4 *A cross-monotonic set of cost shares (payments) assigns to every coalition $S \subseteq N$ a vector $x^S \in P$ such that $x^S_{[N \setminus S]} = 0$ and moreover*

$$\textit{If } S \subseteq T \textit{ then } x^S_{[S]} \geq x^T_{[S]}.$$

We interpret x^S as the payment when the agents in S, and only them, are served. Therefore, by NNT and VP it should be zero for the agents outside S.

The key feature is that payments should not increase as coalition increases. This implies that for every utility profile u the set of feasible coalitions, $F(u) = \{S \in 2^N \mid x^S \leq u\}$, has a maximum element with respect to the inclusion $\subseteq$. To see this, notice if $S, T \in F(u)$ then by cross-monotonicity $S \cup T \in F(u)$.

Definition 5 *Given a cross-monotonic set of cost shares χ^N, we define a cross-monotonic mechanism (S, φ) as follows. For every utility profile u, $S(u)$ is the maximum feasible coalition at u and $\varphi(u) = x^{S(u)}$.*

Juarez[2007] characterizes cross-monotonic mechanisms by GSP and a maximalist tie-breaking rule, namely that agents who are indifferent between getting or not getting a unit of good, will always be served. In the case of economies without tie-breaking rules, there are no indifferences, thus this characterization does not apply.

Proposition 2 *A mechanism (S, φ) is cross-monotonic if and only if it is GSP and population monotonic, that is if $u \leq \tilde{u}$ then $S(u) \subseteq S(\tilde{u})$.*

3.3 Fixed cost mechanisms

Definition 6 *Given $(x_1, \ldots, x_n) \in P$, the corresponding fixed cost mechanism offers to agent i a unit of good at price x_i. That is, for the utility profile u, agent i is guaranteed a unit at price x_i if $u_i > x_i$. Agent i does not get a unit if $u_i < x_i$.*

Juarez[2007] characterizes fixed cost mechanisms as the intersection of sequential mechanisms and cross-monotonic mechanisms. That is, the mechanisms where ties can be broken arbitrarily without affecting GSP.

Define a mechanism ξ as above, however assume agents can transfer money between them.

Proposition 3 *A mechanism is GSP in the economy with transfers of money if and only if it is fixed cost.*

Corollary 1 *A mechanism is GSP in the economy with transfers of money and goods if and only if it is a fixed cost mechanism with the same price offered to all agents.*

3.4 Priority composition

Consider a GSP mechanism (S, φ) defined for the agents in M, $M \subseteq N$.

For every $T \in 2^M$, consider an arbitrary GSP mechanism, (S^T, φ^T), defined for the agents in $N \setminus M$. Thus we have 2^M mechanisms.

Given these mechanisms, we construct a mechanism for the agents in N by giving priority to the agents in M. Agents in M first share according to (S, φ). Depending on their outcome, the agents in $N \setminus M$ share with the mechanism corresponding to that outcome.

Thus the mechanism $(\tilde{S}, \tilde{\varphi})$ defined for agent in N is defined as follows:

$$\tilde{S}(u) = T \cup S^T(u_{[N\setminus M]}) \text{ and } \tilde{\varphi}(u) = [\varphi(u_{[M]}), \varphi^T(u_{[N\setminus M]})] \text{ where } T = S(u_{[M]})$$

It is easy to check that the priority composition of these mechanisms is GSP.

A particular case of this composition is given in the sequential mechanisms. They are characterized by being priority composition of sequential mechanisms for smaller number of agents.

On the other hand, we can also compose sequential and cross-monotonic mechanisms. They will give a nice class of mechanisms whose associated cost function is neither submodular nor supermodular. Clearly, priority compositions of sequential and cross-monotonic mechanisms approximate any cost function.

3.5 Acyclic mechanisms, Roughgarden et.al.[22]

For every S, let $\tau(S)$ a linear ordering[6] of the agents in S. Thus agent $i \in S$ has order $\tau(i, S)$. Let $D(i, S)$ the agents strictly dominated by i under $\tau(S)$. Let $I(i, S)$ the agents on the same position to i (indifferent agents) under $\tau(S)$. We denote by τ the collection of these linear orderings.

Definition 7 *We say the collection of cost shares*

$$X = \{x^S \in P \mid S \subseteq N, x^S_{[N\setminus S]} = 0\}$$

is feasible for the ordering τ if the next two condition hold;

- $x_i^S = x_i^{S\setminus T}$ *for all $T \subseteq D(i, S)$ and $S \subseteq N$.*
- $x_i^S \leq x_i^{S\setminus T}$ *for all $T \subseteq D(i, S) \cup I(i, S)$ and $S \subseteq N$.*

Definition 8 *Given an ordering τ and a feasible collection of payments X for τ, implement an acyclic mechanism as follows:*

Start with agents in N. Following the order of $\tau(N)$, offer them units of good at prices given by x^N until someone accept refuses the offer or everyone accepted (if two agents are in same position, they get simultaneous offers). If all the agents in N accepted the offer, then everyone gets service at prices given

[6] That is, a complete and transitive relation.

by x^N. On the other hand, if some agents in N refused the offer, let L the set of agents who refused the offer. Remove these agents from the game, and start again with coalition $N \setminus L$.

Roughgarden et.al.[22] proves acyclic mechanisms are weakly GSP in an economy with indifferences. In our context without indifferences, this clearly implies acyclic mechanisms are GSP.

Notice this class of mechanisms contains all cross-monotonic mechanisms. They are described by an ordering $\tau(S)$ and a feasible collection of payments that ranks all the agents in the same position.

On the other hand, it also captures all sequential mechanisms. For a sequential tree, consider the branch from the root the a terminal node whose rightist agents are exactly S. Let $\tau(S)$ the order or the agents in S generated by this branch. The cost shares of S are given by the tree.

Unfortunately, there are many interesting mechanisms that are not acyclic. For instance sequential-dictator mechanisms or a cycle mechanism (see below).

3.6 Sequential-dictator mechanisms

Definition 9 *Given a sequential tree without payments, and a collection of cost shares $X = \{x^S \in P \mid S \subseteq N, x^S_{[N \setminus S]} = 0\}$, we implement the sequential dictator mechanism by offering to the agents, following the order of the tree, their best available cost share.*

That is, for the utility profile u, consider the set of feasible cost shares $F(u) = \{x^S \in X \mid u \geq x^S\}$. Let agent i_1 in root of the tree pick his best available set of cost shares (the set with his smallest payment). If i_1 picked something, continue similarly with the following agent on right of the tree and set of cost shares $F^1(u)$, $F^1(u) \subset F(u)$, the set of cost shares for which agent i_1 is indifferent. On the other hand, of agent i_1 did not pick anything, that is he is not included in every coalition of $F(u)$, then continue similarly with following agent on his left and cost shares $F(u)$.

Definition 10 *We say the set of cost shares X is feasible for $\succ$ if the next condition hold:*

If $T \succ S$ such that $S \cap T \neq \emptyset$ and $x^T_L > x^S_L$ for $L \subset S \cap T$, then one of the following conditions holds:

a. There is $M \in 2^N$ such that $M \succ T$ and $M \succ S$ and $x^M \leq max(x^T, x^S)$.

b. There is $Q \in 2^N$ such that $Q \succ S$ and $x^Q_S \leq x^S_S$, $x^Q_{T \setminus S} \leq x^T_{T \setminus S}$ and $x^Q_{(S \cup T)^c} = 0$.

A sequential dictator mechanisms is feasible if the set of cost shares X is feasible for the order $\succ$ generated by the sequential tree.

Feasible sequential dictator mechanisms capture all sequential and cross-monotonic mechanisms discussed above.

To check cross-monotonic mechanisms are sequential dictator, consider a set of cross-monotonic cost shares and **any** arbitrary order of the agents. Clearly property above is satisfied because if $T \succ S$ and $x_L^T > x_L^S$, then $M = T \cup S$ meets condition (a).

On the other hand, we now check sequential mechanism are dictator mechanisms. Consider the order of the coalitions given by tree. Assume $T \succ S$ such that $S \cap T \neq \emptyset$ and $x_L^T > x_L^S$ for some $L \subset S \cap T$. Because $T \succ S$, coalition S is on the left of T. Then, consider the node where the paths if S and T intersect. At this node, i_t, we have that $i_t \in T \setminus S$. Consider the coalition generated by the rightist agents before i_t, call it D. Thus $D \subset S \cap T$. Let $Q = D \cup i_t$. Then $Q \subset T$ and $T \succ Q$ but $Q \succ S$ (because i_t is ahead of S). $x_D^Q = x_S^Q$ because the paths of Q and S coincide until i_t. $x_{i_t}^Q = x_{i_t}^T$. Hence condition *iii* is satisfied.

Example 1 (A sequential-dictator mechanism which is not acyclic) *Consider three agents and the set of cost shares:*

$$(2,2,2),(3,2,0),(2,0,3),(0,1,3),(1,0,0),(0,2,0),(0,0,3)$$

Notice we can implement this sequential-dictator mechanism with the tree such that 3 is in the root. If 3 is rightist then the priority is $3 \succ 1 \succ 2$. *If 3 is leftist, then the priority is* $2 \succ 1$.

On the other hand, this mechanism is not acyclic. We check it by contradiction. Notice agents 1 and 2 cannot be in the first position of $\tau(\{1,2,3\})$ *because they have smallest payments at coalitions* $\{1\}$ *and* $\{2,3\}$ *respectively (this is precluded by conditions on* τ*). Hence 3 is on the first position alone. This is also a contradiction because then the payment of agent 3 is fixed at any coalition that contains him.*

3.7 A cycle mechanism

Consider the following set of cost shares for $n \geq 3$:

$x^N = (2,2,\ldots,2)$, $x^{N\setminus 1} = (0,1,2,2,\ldots,2)$, $x^{N\setminus 2} = (2,0,1,2,\ldots,2)$, $x^{N\setminus 3} = (2,2,0,1,2,\ldots,2)$, $\ldots$, $x^{N\setminus n} = (1,2,2,\ldots,2,0)$. $x_D^D = \infty \cdot 1_D$ for all D such that $\mid D \mid < n-1$.

To implement this mechanism, for the utility profile u, consider the set of feasible cost shares $F(u)$. If u is such that $u_i > 2$ for all $i \in N$ then $S(u) = N$.

If u is such that $u_i > 2$ for all $i \in N \setminus k$ and $u_k < 2$, then $S(u) = N \setminus k$.

If u is such that $u_i > 2$ for all $i \in N \setminus \{k,l\}$ and $u_{[k,l]} < (2,2)$, then if k and l are not consecutive, then $S(u) = \emptyset$. If $k = l+1$ then $S(u) = x^{N\setminus k}$.

That is, if N is served, then everyone will pay 2. On the other hand, if 1 is not being served, then 2 benefits (he only pays 1). Similarly, 3 benefits if 2 is not served, ..., 1 benefits if n is not served.

We check below that this mechanism is GSP.

This mechanism is not acyclic or sequential dictator. To see this, it is not sequential dictator because at the utility profile $u >> x^N$, the mechanisms is allocating N. But agents $i+1$ is better off with the profile $x^{N\setminus i}$.

Finally, we can create more cycle mechanism (in general more GSP mechanisms) by rescaling every agent's utility space. That is, given an arbitrary mechanism (S, φ) and increasing functions $f_i : [0, \infty] \to [0, \infty]$ such that $f_i(0) = 0$, for $i = 1, \ldots, n$. Define the mechanism $(\tilde{S}, \tilde{\varphi})$ as: $\tilde{S}(f(u)) = S(u)$ and $\tilde{\varphi}(f(u)) = f(\varphi(u))$ for all u. Where $f(u) = (f_1(u_1), \ldots, f_n(u_n))$. It is easy to prove that if (S, φ) is GSP then $(\tilde{S}, \tilde{\varphi})$ is GSP.

4 Characterizations of *GSP* mechanisms

When the number of agents is small, the class of mechanisms is easy to describe.

Proposition 4
- *Any GSP mechanism for two agents is either cross-monotonic or sequential.*
- *Any GSP mechanism for three agents is either an acyclic mechanism, sequential-dictator or cycle mechanism.*

When there are more then three agents, the mechanisms are more complicated. First notice that such mechanism should be such that whenever a coalition of agents is served, its payments are independent of the utility profile.

Lemma 1 *Suppose a mechanism (S, φ) is GSP. Then for any profiles $u, \tilde{u}$ such that $S(u) = S(\tilde{u})$, $\varphi(u) = \varphi(\tilde{u})$.*

Therefore, given the mechanism (S, φ), we can talk of its set of payments $\{x^T \mid T \subset N, x^T \in \mathbb{Q}^N\}$. Where x^T is the payment when coalition T is served. In the rest of this section, we provide three alternative conditions under which the set of payments generate a GSP mechanisms.

4.1 Generalized cross-monotonic mechanisms

Definition 11 *We say the set of cost shares*

$$\{x^S \in P \mid S \subset N, x_i^S = 0 \text{ for } i \notin S\}$$

is feasible if there is a strict relation[7] $\succ$ *in* 2^N *that satisfies the next three properties:*

i. *If $T \supset S$ then $T \succ S$.*

ii. *For any $\Lambda \subseteq 2^N$ there exist $M \in 2^N$ such that $x^M \leq \max_{T \in \Lambda} x^T$ and $M \succeq T$ for all $T \in \Lambda$.*

iii. *If $T \succ S$ such that $S \cap T \neq \emptyset$ and $x_L^T > x_L^S$ for $L \subset S \cap T$, $L \neq \emptyset$, then one of the following conditions holds:*

[7] As seen below, this relation may not be complete or transitive.

a. There is $M \in 2^N$ such that $M \succ T$ and $M \succ S$ and $x^M \leq max(x^T, x^S)$ (in particular notice $M \subset T \cup S$ when payments are strictly positive).

b. There is $Q \in 2^N$ such that $Q \succ S$ and $x^Q_S \leq x^S_S$, $x^Q_{T \setminus S} \leq x^T_{T \setminus S}$ and $x^Q_{(S \cup T)^c} = 0$.

Definition 12 *Given a feasible set of cost shares we define a generalized cross-monotonic mechanism as follow. For the vector u of utility profiles, consider the set of feasible coalitions, that is $F(u) = \{S \mid u >> x^S\}$. By condition ii, this set contains a maximal coalition S^* and this maximal coalition is feasible at u. Let the mechanism be such S^* is served at u with payments x^{S^*}.*

Condition i is the standard condition of monotonicity. If two coalitions are feasible and one is bigger than the other, then the bigger coalition should be served.

Condition ii says that any feasible set of payments has a maximal element. This condition is required for the mechanism to be well defined.

Condition iii guarantees GSP. If part a is satisfied then whenever coalitions S and T are feasible they will never be chosen. The mechanism will instead choose M. On the other hand, part b guarantees that whenever a mechanism allocates S to the profile u, then any other profile that increases the utilities of agents in S should continue serving S. If after increasing utilities of agents in S we end up with outcome T, then S was not the maximal feasible coalition at u, it was Q.

The cross-monotonic mechanism is a benchmark case. The relation $\succ$ is easy to define, $T \succ S$ if $T \supset S$. If the collection of payments are cross-monotonic, then condition iii will always hold because there are not T and S such that $T \succ S$ and $x^T_L > x^S_L$. Notice that the order $\succ$ is not complete.

As shown above, sequential-dictator mechanisms also belongs to this class of mechanisms. In this case the order $\succ$ over 2^N is complete and transitive. Notice that for a single mechanism there may me more than one representation. For instance, cross-monotonic mechanism have a complete order $\succ$ (from sequential-dictator) and an incomplete order $\succ$ (from above).

Finally, consider the cycle mechanism discussed in section 3.7. There, the order $\succ$ of 2^N is not transitive. $N \succ S$ for all $S \in 2^N$. For all $D, E \in 2^N$ such that $\mid E \mid = n-1$ and $\mid D \mid < n-1$, $E \succ D$. If $F, G \in 2^N$ such that $\mid F \mid, \mid G \mid < n-1$ and $F \subset G$ then $G \succ F$. Finally,

$$N \setminus 1 \succ N \setminus 2 \succ \cdots \succ N \setminus n \succ N \setminus 1$$

We now check this mechanism is generalized cross-monotonic. Condition i is clearly satisfied. For condition ii, notice if Λ contains two coalition of at least $n-1$ agents, then N is feasible maximal. If it contains one coalition of at least $n-1$ agents, then this coalition is the feasible maximal. If it does not contain any coalition of at least $n-1$ agents then it is obvious.

Finally, to check condition iii, if S and T such that $x^T_i > x^S_i$ and $T \succ S$, then S and T contain at least $n-1$ agents. Thus, $M = N$ is feasible maximal. Hence, $M \succ T$ and $M \succ S$.

Proposition 5 *A mechanism is GSP if and only if it is a generalized cross-monotonic mechanisms.*

4.2 Generalized sequential mechanisms

In this section we provide an inductive construction of the class of GSP mechanism.

Definition 13 *Given $(N\setminus i_1; S^+, \varphi^+)$ and $(N\setminus i_1; S^-, \varphi^-)$ arbitrary GSP mechanisms for $N \setminus i_1$ agents in the economy (U_{-i_1}, P_{-i_1}). We say the function $a : 2^{N\setminus i_1} \to P_{i_1}$ is feasible for these mechanisms if:*

For any u such that $j \in S^-(u_{-i_1}) \cap S^+(u_{-i_1})$:

(1) if $\varphi_j^-(u_{-i_1}) < \varphi_j^+(u_{-i_1})$, and furthermore $\varphi_j^-(u_{-i_1})$ and $\varphi_j^+(u_{-i_1})$ are not consecutive, then $a(S^+(u_{-j,i_1}, v_j))$ and $a(S^+(u_{-i_1}))$ are consecutive for all $v_j \in P_j$.

(2) if $\varphi_j^-(u_{-i_1}) > \varphi_j^+(u_{-i_1}) > 0$ then

$$a(S^+(u_{-j,i_1}, \epsilon)) \geq a(S^+(u_{-i_1})),$$

for all $\epsilon > 0$.

Definition 14 *Given an agent i_1, GSP mechanism for $N \setminus i_1$ agents $(N \setminus i_1; S^+, \varphi^+)$ and $(N \setminus i_1; S^-, \varphi^-)$, and a feasible function $a : 2^{N\setminus i_1} \to \mathbb{P}_{i_1}$, we define a generalized sequential mechanism (N, S, φ) as follows:*

- *If $u_{i_1} > a(S^+(u_{-i_1}))$ then $S(u) = i_1 \cup S^+(u_{-i_1})$ and $\varphi(u) = (a(S^+(u_{-i_1})), \varphi^+(u_{-i_1}))$*
- *If $u_{i_1} < a(S^+(u_{-i_1}))$ then $S(u) = S^-(u_{-i_1})$ and $\varphi(u) = (0, \varphi^-(u_{-i_1}))$*

That is, we fix an agent, say i_1, and consider two different GSP mechanism for $N \setminus i_1$ agents. The first mechanism (S^+, φ^+), which we call the upper restriction, will represent the mechanism assuming agent i_1 is assigned a unit of good. The second mechanism (S^-, φ^-), which we call the lower restriction, represents the mechanism assuming agent i_1 is not served.

To construct a mechanism for N agents, we have to associate a payment for agent i_1 on every vector of payments in the upper restriction (this is represented by the function a). Notice if the function a is constant, then it is feasible for the upper and lower restriction.

Given these mechanism and a feasible function a, we implement the mechanism as follows. Agents report the utility profile. With this profile we compute, independent of agent i_1, the allocation in the upper and lower restriction. We offer agent i_1 his payment associated with the outcome of the upper restriction. If he accepts the offer, then the allocation is given by the upper restriction along with i_1 and his payment. If i_1 does not accept it, then he is not served and the outcome is given by the lower restriction.

Condition 1 on a says that whenever agent j is better off with the lower restriction, then the payment of agent i_1 is independent of the utility of agent j (assuming the utility of everyone else is constant). If condition 1 is not met, then the mechanism is not SP, agent j can misreport.

The second condition says that whenever j is better off with the upper restriction, then when everything else is constant but the utility of agent j, agent i_1 is better off with j than without him in the upper restriction. If this condition is not met, then the mechanism is not GSP, agent j can help i_1.

Proposition 6 *A mechanism is GSP if and only if it is a generalized sequential mechanism.*

4.3 Order-invariant mechanisms

Consider the set of payments $A = \{x^T \mid T \subset N, x^T \in P, x^T_{[N \setminus T]} = 0\}$. We will construct a mechanism that have this set of payments.

Consider a sequential tree without payment for the agents in N.

Definition 15 *Given the set of payments and the sequential tree without payments, we construct inductively the mechanism as follows.*

- *If $n = 1$ then the mechanism is obvious. Agent buy at price x^1 if $u_1 > x^1$. He does not buy if $u_1 < x^1$.*
- *For $n > 1$, assume agent i_1 is in the root of the tree. Consider the upper restriction of the payments $A^+ = \{x^T \in A \mid i_1 \in T\}$ and the lower restriction $A^- = \{x^T \in A \mid i_1 \notin T\}$.*

  By induction, a mechanism is defined for agent in $N \setminus i_1$ with payments A^+ and the rightist sequential subtree (call it (φ^+, S^+)). Similarly, a mechanism is defined for agents in $N \setminus i_1$ with payments A^- and the leftist sequential subtree (call it (φ^-, S^-)).

  For the utility report u, offer agent i_1 a unit of good at price $x_{i_1}^{S^+(u_{-i_1}) \cup i_1}$. If he accept is, then the mechanism serves $S^+(u_{-i_1}) \cup i_1$ at prices $x^{S^+(u_{-i_1}) \cup i_1}$. If he reject it, then the mechanism serves $S^-(u_{-i_1})$ at prices $x^{S^-(u_{-i_1})}$.

In other words, this mechanism leaves *inactive* agent i_1 and let other agents compute the payments when i_1 is served and when he is not served. Then they offer to this agent a unit of good at the price decided by the other people. If he accepts they keep this allocation. If he rejects, they keep the allocation without i_1.

Definition 16 *We say the mechanism is order-invariant if this mechanism is independent of the sequential tree.*

Proposition 7 *A mechanism is GSP if and only if it is an order-invariant mechanism.*

Finally, we provide two benchmark two-agents examples that fail GSP. We check they also fail order-invariant.

Example 2 *To illustrate this in the two agents case. Consider a set of feasible payments that does not generate a GSP. For instance* $(10,10),(5,0),(0,15),(0,0)$. *Consider the profile* $u=(7,12)$. *We start first with order 1,2. We assume agent 1 is in, and offer agent 2 a unit at price 10. Agent 2 buys, so we offer agent 1 at price 10. He does not buy, so we offer agent 2 at price 15. He does not buy, hence the allocation is* $(0,0)$. *Now consider the order 2,1. We assume agent 2 is in, so we offer agent 1 at price 10. He does not buy, so we offer agent 2 at price 15. He does not buy, so we offer agent 1 at price 5. He buys and the allocation is* $(5,0)$.

Now consider another non GSP case $(10,10),(5,0),(0,5),(0,0)$. *Let* $u=(7,7)$ *and consider the order 1,2. We assume agent 1 is in, so we offer agent 2 at price 10. He does not buy, so we offer agent 1 at price 5, he buys and the final allocation is* $(5,0)$. *By symmetry, when we consider the order 2,1 the allocation is* $(0,5)$.

5 Proofs

Proof of proposition 1.

Proof of proposition 2.

We ommit this proof as it is similar to the proof of Theorem 1 on chapter 2.

Proof of proposition 3.

We prove it by induction on the size of agents. The base of induction is trivial if $|N| = 1$. Assume this is also valid for any mechanism defined for at most $n-1$ agents. We prove it for a mechanism with $|N| = n$ agents.

By induction hypothesis, of we restrict the mechanism to $U^i = \{u \| u_i\}$ then it is fixed cost. Let x_k^i the payment of agent k in the mechanism restricted to U^i.

Clearly, $x_k^i = x_k^j$ for $i, j, k \in N$ distinct. Indeed by restricting the mechanism to $U^{i,j}$, the mechanism is fixed cost and the payment of agent k must coincide with his payment at U^i and U^j. Let $x_k^* = x_k^i$ for some $i \neq k$.

Assume there is u such that $S(u) = N$. Let $x^N = \varphi(u)$.

If $x_i^* = x_i^N$ for all i, then we finished.

Assume $x_i^* < x_i^N$ for some i. Consider the utility profile such that $u >> x^N$ and $x_j = x_j^N + \epsilon$.

Then every agent is making a profit of ϵ. Then agents $\{i, j\}$ can profit by misreporting. Indeed agent j can help agent i by misreporting $u_j = 0$. Clearly, agent i benefits because he will get a profit $x_i^N - x_i^* > \epsilon$, and agent i can make a transfer of money to agent j equal to ϵ.

Now, assume that $x_k^* \geq x_k^N$ for all k and $x_i^* > x_i^N$ for some i.

Consider $\tilde{u}_{-i,j}$ such that $\tilde{u} >> x_{-i,j}^N$, $\tilde{u}_i > x_i^*$, and $\tilde{x}_j^N - \epsilon < \tilde{u}_j < \tilde{x}_j^N$. Then agents $\{i, j\}$ can profit by misreporting. Indeed, j can help agent i by misreporting $\bar{u}_j > x^N$. $S(\bar{u}_j, u_{-j}) = N$, agent's i profit increase by more than ϵ, thus he can transfer ϵ to agent j.

Proof of propositions 4, 5 and 6.

The next properties are equivalent:

i. The mechanism (S, φ) is GSP.

ii. If $S(u) = S$, then for any $\tilde{u}$ such that $\tilde{u}_{N\setminus S} \leq u_{N\setminus S}$ and $\tilde{u}_S \geq \varphi_S(u)$: $(S(\tilde{u}), \varphi(\tilde{u})) = (S(u), \varphi(u))$.

iii. (S, φ) is a order-invariant mechanism.

iv. (S, φ) is a generalized sequential mechanisms.

v. The mechanism is a generalized cross-monotonic mechanism.

Proof.

The theorem will be proved the next way:

$(i) \Leftrightarrow (ii)$, $(i) \Rightarrow (iii) \Rightarrow (ii)$, $(i) \Rightarrow (iv) \Rightarrow (ii)$ and $(i) \Rightarrow (v) \Rightarrow (ii)$.

$(i) \Rightarrow (ii)$.

Consider a utility profile u as in the statement. Assume $i \notin S(u)$ and $\tilde{u}_i \leq u_i$. Then by GSP $(S, \varphi)(\tilde{u}_i, u_{-i}) = (S, \varphi)(u)$. By repeating this argument to every agent in $N \setminus S : (S, \varphi)(\tilde{u}_{N \setminus S}, u_S) = (S, \varphi)(u)$.

Assume $i \in S(u)$ and $\tilde{u}_i > \varphi_i(u)$. Then by GSP

$$(S, \varphi)(\tilde{u}_{N \setminus S}, \tilde{u}_i, u_{S \setminus i}) = (S, \varphi)(\tilde{u}_{N \setminus S}, u_S) = (S, \varphi)(u).$$

By repeating this argument to every agent, we conclude that $(S, \varphi)(\tilde{u}) = (S, \varphi)(u)$.

$(ii) \Rightarrow (i)$.

Assume the solution is not GSP. Then there is a profile u and a coalition T that misreport with the profile $\tilde{u}_T$. Let $i \in T$. If $i \in S(\tilde{u}_T, u_{-T})$ then by individual rationality $\varphi_i(\tilde{u}_T, u_{-T}) < u_i$ and $\varphi_i(\tilde{u}_T, u_{-T}) < \tilde{u}_i$ Hence by (ii)

$$(S, \varphi)(\tilde{u}_T, u_{-T}) = (S, \varphi)(\tilde{u}_{T \setminus i}, u_i, u_{-T}).$$

By replicating the argument one agent at a time, for $Q = S(\tilde{u}_T, u_{-T}) \cap T$

$$(S, \varphi)(\tilde{u}_T, u_{-T}) = (S, \varphi)(\tilde{u}_{T \setminus Q}, u_Q, u_{-T}).$$

That is, we can assume without loss of generality that only the agents who are not getting a unit are misreporting. Since their current net utility is zero because they are not getting a unit, this means that those agents are helping other agents in Q without getting any benefit for themself. Also, notice those agents did not get a unit of good with $S(u)$ otherwise their net utility will be positive.

Let $\bar{u}_{T \setminus Q} = min(u_{T \setminus Q}, \tilde{u}_{T \setminus Q})$. By (ii),

$$(S, \varphi)(\tilde{u}_{T \setminus Q}, u_Q, u_{-T}) = (S, \varphi)(\bar{u}_{T \setminus Q}, u_Q, u_{-T})$$

and

$$(S, \varphi)(u_{T \setminus Q}, u_Q, u_{-T}) = (S, \varphi)(\bar{u}_{T \setminus Q}, u_Q, u_{-T}).$$

This is clearly a contradiction.

$(iii) \Rightarrow (ii)$

Consider an order invariant mechanism (S, φ). Let $i \in S(u)$, and $\tilde{u}_i$ such that $\varphi_i(u) < \tilde{u}_i$. Consider the order $i, N \setminus i$. Then at u, agents in $N \setminus i$ choose $S(u)$ and agent i gets an offer for a unit of good at price $\varphi_i(u)$. Since $i \in S(u)$, agent i accepts the offer. Similarly, at the profile $(\tilde{u}_i, u_{-i})$ agents in $N \setminus i$ choose $S(u)$ and agent i gets an offer for a unit of good at price $\varphi_i(u)$. Since $\varphi_i(u) < \tilde{u}_i$ then agent i accepts the offer. Hence $(S, \varphi)(u) = (S, \varphi)(\tilde{u}_i, u_{-i})$. By repeating the argument, we prove that $(S, \varphi)(u) = (S, \varphi)(\tilde{u}_{S(u)}, u_{-S(u)})$ for any $\tilde{u}_{S(u)} \geq \varphi_{S(u)}(u)$.

Let $i \notin S(u)$ and $\tilde{u}_i \leq u_i$. Consider the order $i, N \setminus i$. Then at u, agents in $N \setminus i$ choose something and offer agent i a unit of good at price x^i such that $u_i < x^i$. He rejects, and the final allocation is $S(u)$ that does not contain

i. Similarly, at the profile $(\tilde{u}_i, u_{-i})$ we offer agent i a unit at price x^i. Since $\tilde{u}_i \leq u_i < x^i$ then agent i refuses it, and the final allocation is $(S, \varphi)(u)$, that is $(S, \varphi)(\tilde{u}_i, u_{-i}) = (S, \varphi)(u)$. By repeating the same argument with profile $(S, \varphi)(\tilde{u}_i, u_{-i})$ on the remaining agents we prove (ii).

$(i) \Rightarrow (iii)$

We prove the result by induction. If $n = 1$ then it is obvious.

Let (S, φ) a GSP mechanism for $n > 1$ agents and u any utility profile. Let $1 \in N$, and denote by A^+ and A^- the set of payments given by (S, φ) that contain and do not contain 1 respectively (this is well defined by lemma 2 and if coalition T does not appear, then there is some coalition M, such that $T \subset M$ and $x^M_{M \setminus T} = 0$, so make $x^T_{[T]} = x^M_{[T]}$).

Clearly, A^+ and A^- represent mechanism that are GSP for $N \setminus 1$. To see this, notice by choosing the utility of agent 1 large enough, he will always get a unit independent of other people reports. Then we can achieve all payment of A^+ by varying other agents profile u_{-1}. Similarly, if you choose the utility of agent 1 small enough, we get A^-.

Now let's check (S, φ) is a tree mechanism where 1 is on root, $A^- \cup A^+$ the set of payments and the left and right tree are arbitrary. By induction, A^- and A^+ are order-invariant mechanism. By GSP $(S, \varphi)(a^1, u_{-1}) = (S, \varphi)(u)$, where a^1 is large enough so that we are in restriction A^+. Clearly at this profile (a^1, u_{-1}) the tree mechanism A^+ delivers outcome $(S, \varphi)(a^1, u_{-1})$. Similarly for the lower restriction.

$(i) \Rightarrow (iv)$

Assume (φ, S) is GSP. In part (iii) we proved this mechanism can be represented as a mechanism with GSP upper and lower restriction (φ^+, S^+) and (φ^-, S^-), and with such function $a : 2^N \setminus 1 \to \mathbb{Q}_+$. So, we just need to prove a satisfies properties (1) and (2).

To prove (1), let u_{-1} such that for some j, $j \in S^-(u_{-1}) \cap S^+(u_{-1})$ and $\varphi^-_j(u_{-1}) < \varphi^+_j(u_{-1})$. First we consider the case $a(S^+(u_{-j,1}, \frac{\varphi^-_j(u)+\varphi^+_j(u)}{2})) < a(S^+(u_{-1}))$.

Consider the true profile $(u_1, \tilde{u}_j, u_{-1,j})$ such that: $u_{-\{1,j\}}$ is as above. u_1 is such that $a(S^+(u_{-j,1}, \frac{\varphi^-_j(u)+\varphi^+_j(u)}{2})) < u_1 < a(S^+(u_{-1}))$ and $\tilde{u}_j$ is such that $\varphi^-_j(u_{-1}) < \tilde{u}_j < \varphi^+_j(u_{-1})$.

At this profile, we offer agent 1 a unit of good at price $a(S^+(u_{-j,1}, \frac{\varphi^-_j(u_{-1})+\varphi^+_j(u_{-1})}{2}))$ and he accepts it because his utility is larger that this number. On the other hand, notice that $j \notin S^+(u_{-j,1}, \frac{\varphi^-_j(u_{-1})+\varphi^+_j(u_{-1})}{2})$ because $u_j < \varphi^+_j(u_{-1})$. However, agent j can misreport. He can increase his utility to u_j, $u_j > \varphi^+_j(u_{-1})$. Thus we offer agent 1 a utility at price $a(S^+(u_{-1}))$. Since $u_1 < a(S^+(u_{-1}))$, then the outcome is $(S^-(u_{-1}), \varphi^-(u_{-1}))$. Agent j is better off because $\varphi^-_j(u_{-1}) < \tilde{u}_j$.

Consider the opposite case, $a(S^+(u_{-j,1}, \frac{\varphi^-_j(u_{-1})+\varphi^+_j(u_{-1})}{2})) > a(S^+(u_{-1}))$.

Assume u is the true profile, where u_{-1} is as before and u_1 is such that

$$a(S^+(u_{-j,1}, \frac{\varphi_j^-(u) + \varphi_j^+(u)}{2})) > u_1 > a(S^+(u_{-1})).$$

Then we offer agent 1 a unit of good at price $a(S^+(u_{-1}))$. He accepts it and the allocation is $S^+(u_{-1}) \cup 1$. In this case agent j is paying $\varphi_j^+(u_{-1})$. However, he can misreport. He can decrease his utility to $\tilde{u}_j$ such that $\varphi_j^-(u_{-1}) < \tilde{u}_j < \varphi_j^+(u_{-1})$. In this case, agent 1 is offered a unit of good at price $a(S^+(u_{-j,1}, \frac{\varphi_j^-(u)+\varphi_j^+(u)}{2}))$. since his utility is smaller than this number, he does not get a unit. Hence the ultimate outcome is $S^-(u_{-1})$. Agent j pays $\varphi_j^-(u_{-1})$ which is better for him.

To prove (2), let u_{-1} such that for some j, $j \in S^-(u_{-1}) \cap S^+(u_{-1})$ and $\varphi_j^-(u_{-1}) > \varphi_j^+(u_{-1}) > 0$. Also, assume $a(S^+(u_{-j,1}, \frac{\varphi_j^+(u)}{\sqrt{2}})) < a(S^+(u_{-1}))$.

Consider the true profile $(\tilde{u}_j, u_1, u_{-1,j})$ as above, where u_1 is such that $a(S^+(u_{-j,1}, \frac{\varphi_j^+(u)}{\sqrt{(2)}})) < u_1 < a(S^+(u_{-1}))$ and $\tilde{u}_j$ is such that $\varphi_j^+(u_{-1}) < \tilde{u}_j < \varphi_j^-(u_{-1})$.

At this profile, agents 1 and j do not get a unit of good. To see this, notice agent 1 is offered a unit of good at price $a(S^+(u_{-1}))$. He does not buy, hence the ultimate outcome is $S^-(u_{-1,j}, \tilde{u}_j)$. Since $\tilde{u}_j < \varphi_j^-(u_{-1})$ then he does not get a unit of good.

However, at this profile agent j can help agent 1. He can report $\bar{u}_j$ such that $\bar{u}_j = \frac{\varphi_j^+(u)}{\sqrt{2}}$. Here agent 1 is offered a unit at price $a(S^+(u_{-j,1}, \frac{\varphi_j^+(u)}{\sqrt{2}}))$ and he buys it. This is a contradiction.

$(iv) \Rightarrow (ii)$

Let (S, φ) a generalized sequential mechanism with agent i_1 in the top.

Let u a utility profile.

Case 1. $i_1 \notin S(u)$.

Clearly, if $\tilde{u}_{i_1} < u_{i_1}$ then $(S, \varphi)(\tilde{u}_{i_1}, u_{-i_1}) = (S, \varphi)(u)$ because $\tilde{u}_{i_1} \leq u_{i_1} < a(S^+(u_{-i_1}))$.

Let j such that $j \notin S(u)$, $j \neq i_1$. Let $\tilde{u}_j$ such that $\tilde{u}_j < u_j$. If $j \notin S^+(u_{-i_1})$ then by GSP of the upper restriction, $j \notin S^+(\tilde{u}_j, u_{-j,i_1})$. And the outcome does not change.

On the other hand, if $j \in S^+(u_{-i_1})$, then $\varphi_j^+(u) < \varphi_j^-(u)$. If $\varphi_j^+(u) = 0$ then $S^+(\tilde{u}_j, u_{-j,i_1}) = S^+(u_j, u_{-j,i_1})$ so the mechanism does not change. If $0 < \varphi_j^+(u) < \varphi_j^-(u)$ then

$$a(S^+(u_{-j,i_1}, \tilde{u}_j)) = a(S^+(u_{-j,i_1}, \frac{\varphi_j^+(u) + \varphi_j^-(u)}{\sqrt{2}})) \text{ or } a(S^+(u_{-j,i_1}, \tilde{u}_j)) = a(S^+(u_{-i_1})),$$

in any case, since $a(S^+(u_{-j,i_1}, \frac{\varphi_j^+(u)+\varphi_j^-(u)}{\sqrt{2}})) \geq a(S^+(u_{-i_1}))$, then $S(\tilde{u}_j, u_{-j}) = S(u)$.

On the other hand, assume $j \in S(u)$. Let $\tilde{u}_j$ such that $\tilde{u}_j > \varphi_j^-(u)$.

If $\varphi_j^+(u) \leq \varphi_j^-(u)$ then by GSP of the upper restriction $S^+(\tilde{u}_j, u_{-j,i_1}) = S^+(u_{-i_1})$, thus the outcome does not change because the offer the agent i_1 is the same.

On the other hand, if $\varphi_j^+(u) > \varphi_j^-(u)$. Then by (1) $a(S^+(u_{-i_1})) = a(S^+(\tilde{u}_j, u_{-j,i_1}))$. Hence the outcome does not change.

Case 2. $i_1 \in S(u)$.

Clearly, if $\tilde{u}_{i_1} > a(S^+(u_{-i_1}))$ then agent i_1 also buys at price $a(S^+(u_{-i_1}))$, hence the outcome does not change.

Similarly, if $j \in S(u)$, then by GSP on the lower restriction for any $\tilde{u}_j$ such that $\tilde{u}_j \geq \varphi_j(u) = \varphi_j^+(u_{-i_1})$, $S(\tilde{u}_j, u_{-j}) = S(u)$. Hence the outcome does not change.

On the other hand, if $j \notin S(u)$ then $j \notin S^+(u_{i_1})$. Hence for any $\tilde{u}_j$ such that $\tilde{u}_j \leq u_j$ $S^+(\tilde{u}_j, u_{-j,i_1}) = S^+(u_{-i_1})$

$(v) \Rightarrow (ii)$

Consider a mechanism of this type and a profile u such that $S(u) = S$. If $\tilde{u}_{N\setminus S} \leq u_{N\setminus S}$ the clearly $S(\tilde{u}_{N\setminus S}, u_S) = S$ because the set of feasible coalitions did not increase, and S is still feasible.

On the other hand, consider the profile $\tilde{u}_S$ such that $\tilde{u}_S \geq u_S$ and assume $S(\tilde{u}_S, u_{N\setminus S}) = T \neq S$. Then, because S is feasible at $(\tilde{u}_S, u_{N\setminus S})$, $T \succ S$. Since T was not chosen at u, then T was not feasible there, thus $x_L^T > x_L^T$ for some $L \subset S \cap T$. Notice condition (a) is not satisfied, otherwise T will not be chosen at $(\tilde{u}_S, u_{N\setminus S})$. Therefore, condition (b) is satisfied, in particular, such Q satisfies $Q \succ S$. Since Q is feasible at $(\tilde{u}_S, u_{N\setminus S})$, then Q is feasible at u. This is a contradiction, because then S is not maximal at u.

Finally, notice we only proved (ii) when we increased the utility of the agents in S. If we decrease the utilities of some agents in S, bounding below their utilities by x_S^S, then the allocation is S because the set of feasible payments did not increase, and S is still feasible.

Any GSP mechanism is generalized cross-monotonic.

Consider a GSP mechanism (S, φ). By lemma 1, there is a constant set of payments for every coalition. As before, denote by x^T the payment of coalition T.

We say that $T \succ S$ if there is a profile u such that $S(u) = T$ and S is feasible, that is $u \geq x^S$.

First notice this relation is strict. To see this, assume there are profiles u and $\tilde{u}$ such that they are both feasible for S and T but $S(u) = T$ and $S(\tilde{u}) = S$. Let $\bar{u} = min(\tilde{u}, u)$. If $i \in S$ then $\bar{u}_i > x_i^S$ (because $\tilde{u}_i > x_i^S$ and $u_i > x_i^S$). Also, if $i \notin S$ then $\bar{u}_i \leq u_i$. Thus by (ii) $S(\bar{u}) = S$. On the other hand, by a symmetric argument $S(\bar{u}) = T$. This is a contradiction.

(i) is obvious by induction (have to write this little argument).

To prove (ii), consider $\Lambda \subset 2^N$. Let $u = \max_{T\in\Lambda}(x^T) + \epsilon 1_N$, ϵ small. Let $M = S(u)$. Then $M \succ T$ for all $T \succ \Lambda$. By VP, x^M is feasible, thus by choosing small ϵ, $x^M \leq \max_{T\in\Lambda}(x^T)$.

To prove (iii), consider T and S as in the statement. Let $u = max(x^S, x^T) + \epsilon 1_N$. If $S(u) = M$, $M \neq S, T$, then $x^M \leq u$, thus by choosing ϵ small enough, $x^M \leq max(x^S, x^T)$. Hence part (a) is satisfied.

On the other hand, if $S(u) = T$, let $L \subset S \cap T$ maximal such that $x^T_L >> x^S_L$. Consider the profile $\tilde{u}_L = x^S_L + \epsilon$. First notice $(\tilde{u}_L, u_{-L})$ is not feasible at T.

Next, notice $S(\tilde{u}_L, u_{-L}) \neq S$. Indeed, assume $S(\tilde{u}_L, u_{-L}) = S$. Because the mechanism is GSP and because $u_L >> \tilde{u}_L$, $L \subset S$, then $S(u_L, u_{-L}) = S$. This is a contradiction.

Hence $S(\tilde{u}_L, u_{-L}) = Q$, $Q \neq T, S$. Since S is feasible at this profile then $Q \succ S$. Since Q was feasible at u, then $T \succ Q$. By choosing small ϵ, $x^Q_S \leq x^S_S$, $x^Q_{T \setminus S} \leq x^T_{T \setminus S}$ and $x^Q_{(S \cup T)^c} = 0$. Hence part (b) is satisfied.

■

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